

# Complete Convergence for Arrays of Rowwise Negatively Dependent Random Variables

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**Abstract.** In this article, we establish a complete convergence theorem for arrays of rowwise m-negatively dependent (m-ND) random variables. Our results generalize those on complete convergence theorem previously obtained by Hu et al. (1998) and Sung et al. (2005) from independent distributed case to m-ND arrays.

## 1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947). Hu et al. (1998) proposed the following general complete convergence of rowwise independent arrays of random variables:

**Theorem A** Let  $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$  be an array of rowwise independent random variables and  $\{c_n\}$  be a sequence of positive real numbers. Suppose that for every  $\varepsilon > 0$  and some  $\delta > 0$ :

- (i)  $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} < \infty$ ,
- (ii) there exists  $j \geq 1$  such that

$$\sum_{n=1}^{\infty} c_n \left( \sum_{i=1}^{k_n} E X_{ni}^2 I\{|X_{ni}| \leq \delta\} \right)^j < \infty,$$

- (iii)  $\sum_{i=1}^{k_n} E X_{ni} I\{|X_{ni}| \leq \delta\} \rightarrow 0$  as  $n \rightarrow \infty$ .

Then

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

In this paper we let  $\{k_n, n \geq 1\}$  be a sequence of positive integers such that  $\lim_{n \rightarrow \infty} k_n = \infty$ .

The proof of Hu et al. (1998) is mistakenly based on the fact that the assumptions of Theorem A imply convergence in probability of the corresponding partial sums. Hu and Volodin (2000) and Hu et al. (2003) presented counterexamples to this proof. They mentioned that whether Theorem A is true has remained open. Since then many authors attempted to solve this problem. Hu et al. (2003) and Kuczmazewskagave (2004) gave partial solution to this question. Sung et al. (2005) completely solved this problem by using a symmetrization procedure and Kruglov et al. (2006) obtained the complete convergence for maximum partial sums by using a submartingale approach.

Recently, Chen et al. (2008) extended Theorem A to the case of arrays of rowwise negatively associated random variables and obtained the complete convergence for maximum partial sums. Hu et al. (2009) obtained complete convergence for maximum partial sums similar to Theorem A for arrays of rowwise m-negatively associated random variables. Qiu et al. (2011) obtained similar result for arrays of rowwise negatively dependent random variables.

Lehmann (1996) introduced the concept of negatively dependent random variables as follows.

**Definition 1.** A sequence of random variables  $\{X_n; n \geq 1\}$  is said to be negatively dependent (abbreviated to ND in the following) if for any  $n \geq 2$ , the following two inequalities hold:

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) \leq \prod_{i=1}^n P(X_i \leq x_i)$$

$$P(X_1 > x_1, \dots, X_n > x_n) \leq \prod_{i=1}^n P(X_i > x_i)$$

for every sequence  $\{x_1, \dots, x_n\}$  of real numbers.

**Definition 2.** Let  $m \geq 1$  be a fixed integer. A sequence of random variables  $\{X_n; n \geq 1\}$  is said to be  $m$ -negatively dependent (abbreviated to  $m$ -ND in the following) if for any  $n \geq 2$  and  $i_1, \dots, i_n$  such that  $|i_k - i_j| \geq m$  for all  $1 \leq k \neq j \leq n$ , we have that  $X_{i_1}, \dots, X_{i_n}$  are negatively dependent.

The concept of  $m$ -ND random variables is a natural extension from ND random variables (wherein  $m = 1$ ).

The main purpose of this article is to generalize Theorem A to the case of arrays of rowwise  $m$ -ND random variables.

## 2. Main results

**Theorem 1.** Let  $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$  be an array of rowwise  $m$ -ND random variables and  $\{c_n\}$  be a sequence of positive real numbers. Assume that for every  $\varepsilon > 0$  and some  $\delta > 0$ :

- (i)  $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$ ,
- (ii) there exists  $j \geq 1$  such that

$$\sum_{n=1}^{\infty} c_n \left( \sum_{i=1}^{k_n} E X_{ni}^2 I\{|X_{ni}| \leq \delta\} \right)^j < \infty,$$

Then

$$\sum_{n=1}^{\infty} c_n P \left( \left| \sum_{i=1}^{k_n} (X_{ni} - E X_{ni} I\{|X_{ni}| \leq \delta\}) \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

## 3. Proofs of main results

In order to prove our results, we need the following lemmas.

**Lemma 1.** (see Qiu et al. (2011), Lemma 2) Let  $\{X_n; n \geq 1\}$  be a sequence of negatively dependent random variables with  $EX_n = 0$  and  $EX_n^2 < \infty$ ,  $n \geq 1$ . Let  $S_n = \sum_{i=1}^n X_i$ ,  $B_n = \sum_{i=1}^n EX_i^2$ . Then for all  $x > 0$ ,  $a > 0$

$$P(S_n > x) \leq P\left(\max_{1 \leq k \leq n} |X_k| > a\right) + \exp \left\{ \frac{x}{a} - \frac{x}{a} \ln \left(1 + \frac{xa}{B_n}\right) \right\}.$$

$$P(|S_n| > x) \leq 2P\left(\max_{1 \leq k \leq n} |X_k| > a\right) + 2 \exp \left\{ \frac{x}{a} - \frac{x}{a} \ln \left(1 + \frac{xa}{B_n}\right) \right\}.$$

**Lemma 2.** Let  $\{X_n; n \geq 1\}$  be a sequence of  $m$ -ND random variables with  $EX_n = 0$  and  $EX_n^2 < \infty$ ,  $n \geq 1$ . Let  $S_n = \sum_{i=1}^n X_i$ ,  $B_n = \sum_{i=1}^n EX_i^2$ . Then for all  $n \geq m$ ,  $x > 0$ ,  $a > 0$

$$P(S_n > x) \leq mP\left(\max_{1 \leq k \leq n} |X_k| > a\right) + m \exp \left\{ \frac{x}{ma} - \frac{x}{ma} \ln \left(1 + \frac{xa}{mB_n}\right) \right\}$$

and

$$P(|S_n| > x) \leq 2mP\left(\max_{1 \leq k \leq n} |X_k| > a\right) + 2m \exp \left\{ \frac{x}{ma} - \frac{x}{ma} \ln \left(1 + \frac{xa}{mB_n}\right) \right\}.$$

**Proof.** Given any  $1 \leq k \leq n$ , Let  $r = \lfloor \frac{n}{m} \rfloor$ . Set

$$Y_i = \begin{cases} X_i & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i > n \end{cases}$$

and  $S'_{mk+j} = \sum_{i=0}^k Y_{mi+j}$  for  $1 \leq j \leq m$ .

Since  $\{S_n > x\} \subset \{S'_{mr+1} > \frac{x}{m}\} \cup \dots \cup \{S'_{mr+m} > \frac{x}{m}\}$ ,

By Lemma 3.1, we have

$$\begin{aligned} P(S_n > x) &\leq \sum_{j=1}^m P\left(S'_{mr+j} > \frac{x}{m}\right) \\ &\leq \sum_{j=1}^m P\left(\max_{0 \leq i \leq r} |Y_{mi+j}| > a\right) + \sum_{j=1}^m \exp\left\{\frac{x}{ma} - \frac{x}{ma} \ln\left(1 + \frac{xa}{m \sum_{i=0}^r E Y_{mi+j}^2}\right)\right\} \\ &\leq mP\left(\max_{1 \leq k \leq n} |X_k| > a\right) + m \exp\left\{\frac{x}{ma} - \frac{x}{ma} \ln\left(1 + \frac{xa}{mB_n}\right)\right\}. \end{aligned}$$

If we consider  $-X_n$  instead of  $X_n$  in the arguments above, by a similar way we get

$$P(-S_n > x) \leq mP\left(\max_{1 \leq k \leq n} |X_k| > a\right) + m \exp\left\{\frac{x}{ma} - \frac{x}{ma} \ln\left(1 + \frac{xa}{mB_n}\right)\right\}.$$

Therefore,

$$P(|S_n| > x) \leq 2mP\left(\max_{1 \leq k \leq n} |X_k| > a\right) + 2m \exp\left\{\frac{x}{ma} - \frac{x}{ma} \ln\left(1 + \frac{xa}{mB_n}\right)\right\}.$$

Lemma 3. Let  $Y_{ni} = \delta I\{X_{ni} > \delta\} + X_{ni} I\{|X_{ni}| \leq \delta\} - \delta I\{X_{ni} < -\delta\}$ . Under the conditions of Theorem 2.1, we have

$$\max_{1 \leq i \leq k_n} |EY_{ni}| \leq \left(\sum_{i=1}^{k_n} E X_{ni}^2 I\{|X_{ni}| \leq \delta\}\right)^{\frac{1}{2}} + \delta \sum_{i=1}^{k_n} P(|X_{ni}| > \delta).$$

Proof.  $\{Y_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  is an array of rowwise m-ND random variables.

$$\begin{aligned} \max_{1 \leq i \leq k_n} |EY_{ni}| &\leq \max_{1 \leq i \leq k_n} E|Y_{ni}| = \max_{1 \leq i \leq k_n} E|X_{ni} I\{|X_{ni}| \leq \delta\} + \delta I\{X_{ni} > \delta\} - \delta I\{X_{ni} < -\delta\}| \\ &\leq \max_{1 \leq i \leq k_n} (E|X_{ni} I\{|X_{ni}| \leq \delta\}| + \delta P(|X_{ni}| > \delta)) \\ &\leq \max_{1 \leq i \leq k_n} (EX_{ni}^2 I\{|X_{ni}| \leq \delta\})^{\frac{1}{2}} + \delta \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) \\ &\leq \left(\sum_{i=1}^{k_n} E X_{ni}^2 I\{|X_{ni}| \leq \delta\}\right)^{\frac{1}{2}} + \delta \sum_{i=1}^{k_n} P(|X_{ni}| > \delta). \end{aligned}$$

Proof. Proof of Theorem 2.1 Let

$$Y_{ni} = \delta I\{X_{ni} > \delta\} + X_{ni} I\{|X_{ni}| \leq \delta\} - \delta I\{X_{ni} < -\delta\} \quad \text{and}$$

$$Y'_{ni} = \delta I\{X_{ni} > \delta\} - \delta I\{X_{ni} < -\delta\} \quad \text{and} \quad 1 \leq i \leq k_n, n \geq 1.$$

$\{Y_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  is an array of rowwise m-ND random variables. Note that

$$\begin{aligned} &P\left(\left|\sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I\{|X_{ni}| \leq \delta\})\right| > \varepsilon\right) \\ &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + P\left(\left|\sum_{i=1}^{k_n} (X_{ni} I\{|X_{ni}| \leq \delta\} - EX_{ni} I\{|X_{ni}| \leq \delta\})\right| > \varepsilon\right) \\ &= \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + P\left(\left|\sum_{i=1}^{k_n} (Y_{ni} - EY_{ni} - Y'_{ni} + EY'_{ni})\right| > \varepsilon\right) \\ &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + P\left(\left|\sum_{i=1}^{k_n} (Y'_{ni} - EY'_{ni})\right| > \varepsilon/2\right) + P\left(\left|\sum_{i=1}^{k_n} (Y_{ni} - EY_{ni})\right| > \varepsilon/2\right) \\ &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + C \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + P\left(\left|\sum_{i=1}^{k_n} (Y_{ni} - EY_{ni})\right| > \varepsilon/2\right) \end{aligned}$$

Hence, by conditions (i), it is sufficient to prove that

$$\sum_{n=1}^{\infty} c_n P \left( \left| \sum_{i=1}^{k_n} (Y_{ni} - EY_{ni}) \right| > \varepsilon/2 \right) < \infty.$$

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