

# Auxiliary Principle and Iterative Algorithm for a System of Generalized Nonlinear Mixed Quasi-variational-like Inequalities

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**Abstract**—In this paper, the auxiliary principle technique is extended to study a system of generalized nonlinear mixed quasi-variational-like inequalities in Hilbert spaces. First, we establish the existence of solutions of the corresponding system of auxiliary generalized nonlinear mixed quasi-variational-like inequalities. Then based on the existence result, we construct a new iterative algorithm. Finally, both the existence of solutions of the original problem and the convergence of iterative sequences generated by the algorithm are proved. Our results improve and extend some known results.

**Keywords**—auxiliary principle; existence; iterative algorithm; system of generalized nonlinear mixed quasi-variational-like inequalities; convergence

## I. INTRODUCTION AND PRELIMINARIES

Throughout the paper, let  $I = \{1, 2\}$  be an index set, for each  $i \in I$ , let  $H_i$  be a real Hilbert spaces with inner product  $\langle \cdot, \cdot \rangle_i$  and norm  $\|\cdot\|_i$ , and  $2^{H_i}$  be the family of all nonempty subset of  $H_i$ . Let  $K_i : H_i \rightarrow 2^{H_i}$  be a set-valued mapping such that for each  $x_i \in H_i$ ,  $K_i(x_i)$  is a nonempty closed convex subset of  $H_i$ . Let  $N_i, \eta_i : H_i \times H_i \rightarrow H_i$ ,

$$M_i : H_1 \times H_2 \rightarrow H_i, A_i, B_i : H_i \rightarrow H_i, C_i : H_1 \rightarrow H_1, D_i : H_2 \rightarrow H_2$$

be nonlinear single-valued mappings. We consider the following system of generalized nonlinear mixed quasi-variational-like inequalities (for short, denoted by SGNMQVLI): Find  $(u, v) \in K_1(u) \times K_2(v)$  such that

$$\begin{cases} \langle N_1(A_1 x_1, B_1 x_1) + M_1(C_1 x_1, D_1 y), \eta_1(u, x) \rangle_1 + b_1(x, u) - b_1(x, u) \geq 0, \forall u \in K_1(x), \\ \langle N_2(A_2 y, B_2 y) + M_2(C_2 x, D_2 y), \eta_2(v, y) \rangle_2 + b_2(y, v) - b_2(y, y) \geq 0, \forall v \in K_2(y). \end{cases} \quad (1)$$

where  $b_i : H_i \times H_i \rightarrow R$  is a bifunction, which has the following properties:

- (i)  $b_i(\cdot, \cdot)$  is linear in the first argument;
- (ii)  $b_i(\cdot, \cdot)$  is convex in the second argument;

(iii)  $b_i(\cdot, \cdot)$  is bounded, that is, there exists a constant  $C_i > 0$  such that  $b_i(u_i, v_i) \leq C_i \|u_i\| \|v_i\|, \forall u_i, v_i \in H_i$ ;

(iv)  $b_i(u_i, v_i) - b_i(u_i, w_i) \leq b_i(u_i, v_i - w_i), \forall u_i, v_i, w_i \in H_i$ .

In many important applications,  $K_1(u)$  and  $K_2(v)$  have the following forms [1, 3]:

$$K_1(u) = m_1(u) + K_1, \forall u \in H_1, K_2(v) = m_2(v) + K_2, \forall v \in H_2, \quad (2)$$

where  $m_i : H_i \rightarrow H_i$  is a single-valued mapping and  $K_i$  is a nonempty closed convex subset of  $H_i$ .

Noting that if  $H_1 = R^n, H_2 = R^m, K_1(x) = K_1, K_2(y) = K_2, N_1 = N_2 = b_1 = b_2 = 0, C_1 = C_2 = I_{H_1}, D_1 = D_2 = I_{H_2}, \eta_1 = u - x, \eta_2 = v - y, \forall x, u \in H_1, y, v \in H_2$ , where  $I_{H_1}$  and  $I_{H_2}$  are identity mappings on  $H_1$  and  $H_2$ , respectively, then SGNMQVLI(1) reduces to the following system of variational inequalities (for short, denoted by SVI): Find  $(u, v) \in K_1 \times K_2$  such that

$$\begin{cases} \langle M_1(u, v), w - u \rangle_1 \geq 0, \forall w \in K_1, \\ \langle M_2(u, v), z - v \rangle_2 \geq 0, \forall z \in K_2. \end{cases} \quad (3)$$

SVI(3) was introduced and studied by Zhao et al. [3], in which they employed the Brouwer fixed point theorem to obtain some existence results for SVI(3), moreover, by projection technique, they established the existence and uniqueness theorem for SVI(3) and suggested an iterative algorithm and analysed convergence of the algorithm. But in SGNMQVLI(1), the two convex sets depend on the solutions implicitly or explicitly, and  $b_i$  is a nonlinear mapping, so the projection technique cannot be applied to it. This fact motivated many authors to develop the auxiliary principle technique to study the existence of solutions of generalized mixed type quasi-variational inequalities and also to develop a large number of numerical methods for solving various variational inequalities,

complementarity problems and optimization problems. The auxiliary principle technique was first introduced by Glowinski et al. [4]. Recently, auxiliary principle technique has especially attracted the attention of scholars in the area of variational inequality theory, for details, see [5-8] and the references therein.

Motivated and inspired by the above research work, in this paper, we introduce and study SGNMQVLI(1) in Hilbert spaces. By applying the auxiliary principle technique, we show the existence and uniqueness theorem of solution for the corresponding auxiliary principle relative to SGNMQVLI (1) by minimizing sequence method. For finding the approximate solutions of SGNMQVLI (1), we suggest an iterative algorithm by the auxiliary problem. Under certain conditions, we obtain the existence result of solution for SGNMQVLI (1) and prove the convergence of iterative sequences generated by the iterative algorithm. Our results improve and generalized many known results.

In order to obtain our main results, we first recall some concept and assumption.

**Definition 1.1** Let  $H$  be a real Hilbert space,  $A, B: H \rightarrow H$ ,  $N, \eta: H \times H \rightarrow H$ , be single-valued mappings.

(1)  $\eta$  is said to be  $\delta$ -Lipschitz continuous, if there exists a constant  $\delta > 0$  such that

$$\|\eta(x, y)\| \leq \delta \|x - y\|, \forall x, y \in H;$$

(2)  $A$  is said to be  $\omega$ -Lipschitz continuous, if there exists a constant  $\omega > 0$  such that

$$\|Ax - Ay\| \leq \omega \|x - y\|, \forall x, y \in H;$$

(3)  $N$  is said to be  $(\mu, \nu)$ -Lipschitz continuous, if there exists a pair of constants  $\mu, \nu > 0$  such that

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leq \mu \|x_1 - x_2\| + \nu \|y_1 - y_2\|,$$

$$\forall x_1, x_2, y_1, y_2 \in H;$$

(4)  $N$  is said to be  $\xi$ -relaxed Lipschitz with respect to  $A$  and  $B$  in the first argument and second argument, if there exists a constants  $\xi > 0$  such that

$$\langle N(Ax, Bx) - N(Ay, By), x - y \rangle \leq -\xi \|x - y\|^2, \forall x, y \in H;$$

(5)  $N$  is said to be  $\alpha$ -Lipschitz continuous in the first argument, if there exists a constant  $\alpha > 0$  such that

$$\|N(x, z) - N(y, z)\| \leq \alpha \|x - y\|, \forall x, y, z \in H;$$

In a similar way, we can define the Lipschitz continuity of  $N$  in the second argument.

**Assumption1.1** The mapping  $\eta_i, N_i: H_i \times H_i \rightarrow H_i$ ,  $A_i, B_i: H_i \rightarrow H_i, C_i: H_1 \rightarrow H_1, D_i: H_2 \rightarrow H_2$  satisfies the following conditions:

$$(1) \eta_i(u_i, v_i) = \eta_i(u_i, z_i) + \eta_i(z_i, v_i), \forall u_i, v_i, z_i \in H_i;$$

$$(2) \eta_i(u_i + v_i, w_i) = -\eta_i(w_i - u_i, v_i), \forall u_i, v_i, w_i \in H_i;$$

(3) The functions

$$\begin{aligned} u &\rightarrow \langle N_1(A_1x, B_1x) + M_1(C_1x, D_1y), \eta_1(u, x) \rangle_1, \\ v &\rightarrow \langle N_2(A_2y, B_2y) + M_2(C_2x, D_2y), \eta_2(v, y) \rangle_2 \end{aligned}$$

are both continuous and linear for all  $(u, v) \in H_1 \times H_2$ .

**Remark1.1** It follows from Assumption1.1 (1) that

$$\eta_i(u_i, u_i) = 0, \eta_i(u_i, v_i) = -\eta_i(v_i, u_i), \text{ for } \alpha, \lambda, u_i, v_i \in H_i.$$

## II. AUXILIARY PROBLEM AND ITERATIVE ALGORITHM

For each  $i \in I$ , given  $(u, v) \in K_1(u) \times K_2(v)$ , we consider the following problem:

find  $(p, q) \in K_1(u) \times K_2(v)$  such that

$$\begin{aligned} \langle p, u - p \rangle_1 &\geq \langle x, u - p \rangle_1 - \rho_1 \langle N_1(A_1x, B_1x) + M_1(C_1x, D_1y), \eta_1(u, p) \rangle_1 \\ &\quad + \rho_1 [b_1(x, p) - b_2(x, u)], \forall u \in K_1(x), \end{aligned} \quad (4)$$

$$\begin{aligned} \langle q, v - q \rangle_2 &\geq \langle y, v - q \rangle_2 - \rho_2 \langle N_2(A_2y, B_2y) + M_2(C_2x, D_2y), \eta_2(v, q) \rangle_2 \\ &\quad + \rho_2 [b_2(y, q) - b_2(y, v)], \forall v \in K_2(y), \end{aligned}$$

where  $\rho_1, \rho_2 > 0$  are constants. This problem is called the system of auxiliary generalized nonlinear mixed quasi-variational-like inequalities, for short, denoted by SAGNMQVLI (4) related to SGNQVLI (1).

**Theorem 2.1** For each  $i \in I$ , let  $K_i: H_i \rightarrow 2^{H_i}$  be set-valued mapping such that for each  $x_i \in H_i, K_i(x_i)$  is a nonempty closed convex subset of  $H_i$ .  $N_i, \eta_i: H_i \times H_i \rightarrow H_i, M_i: H_1 \times H_2 \rightarrow H_i$ ,  $A_i, B_i: H_i \rightarrow H_i, C_i: H_1 \rightarrow H_1, D_i: H_2 \rightarrow H_2$  be nonlinear single-valued mappings, and  $b_i: H_i \times H_i \rightarrow R$  be a bifunction such that for each given  $(x, y) \in H_1 \times H_2$ , the functions  $u \mapsto b_1(x, u)$  and

$v \mapsto b_2(y, v)$  are proper convex and lower semicontinuous. If Assumption1.1 holds, then for any given  $(x, y) \in H_1 \times H_2$ , define the functions  $J_1: K_1(x) \rightarrow R$  and  $J_2: K_2(y) \rightarrow R$  as follows:

$$J_1(u) = \frac{1}{2} \langle u, u \rangle_1 + j_1(u), \quad J_2(v) = \frac{1}{2} \langle v, v \rangle_2 + j_2(v),$$

Where

$$\begin{aligned} j_1(u) &= \rho_1 \langle N_1(A_1x, B_1x) + M_1(C_1x, D_1y), \eta_1(u, x) \rangle_1 + \rho_1 b_1(x, u) - \langle x, u \rangle_1, \\ j_2(v) &= \rho_2 \langle N_2(A_2y, B_2y) + M_2(C_2x, D_2y), \eta_2(v, y) \rangle_2 + \rho_2 b_2(y, v) - \langle y, v \rangle_2. \end{aligned}$$

Then we have:

(i)  $J_1$  has a unique minimum point  $p \in K_1(x)$ , and  $J_2$  has a unique minimum point  $q \in K_2(y)$ .

(ii)  $J_1$  and  $J_2$  have unique minimum points  $p \in K_1(x)$  and  $q \in K_2(y)$ , respectively, if and only if  $(p, q)$  is a unique solution of SAGNMQVLI (4).

**Proof** Similarly argument in Theorem 2.1[8], the conclusions are immediately obtained, so are omitted, completing the proof.

Based on Theorem 2.1, we suggest an iterative algorithm for solving SGNQVLI (1).

**Algorithm 2.1** For given  $(u_0, v_0) \in H_1 \times H_2$ , let the sequence  $\{(u_n, v_n)\} \in K_1(u_n) \times K_2(v_n)$  satisfies the following conditions:

$$\begin{aligned} \langle x_{n+1}, u - x_{n+1} \rangle_1 &\geq \langle x_n, u - x_{n+1} \rangle_1 \\ &- \rho_1 \langle N_1(A_1x_n, B_1x_n) + M_1(C_1x_n, D_1y_n), \eta_1(u, x_{n+1}) \rangle_1 \\ &+ \rho_1 [b_1(x_n, x_{n+1}) - b_1(x_n, u)], \forall u \in K_1(x_{n+1}); \end{aligned} \quad (5)$$

$$\begin{aligned} \langle y_{n+1}, v - y_{n+1} \rangle_2 &\geq \langle y_n, v - y_{n+1} \rangle_2 \\ &- \rho_2 \langle N_2(A_2y_n, B_2y_n) + M_2(C_1x_n, D_2y_n), \eta_2(v, y_{n+1}) \rangle_2 \\ &+ \rho_2 [b_2(y_n, y_{n+1}) - b_2(y_n, v)], \forall v \in K_2(y_{n+1}), \end{aligned} \quad (6)$$

for every  $n = 0, 1, 2, 3, \dots$ , where  $\rho_1, \rho_2 > 0$  are constants.

### III. EXISTENCE AND CONVERGENCE THEOREM

**Theorem 3.1** For each  $i \in I$ , let  $H_i$  be a Hilbert space, and  $K_i : H_i \rightarrow 2^{H_i}$  be set-valued mapping such that for each  $x_i \in H_i$ ,  $K_i(x_i)$  is a nonempty closed convex subset of  $H_i$ . Let  $N_i, \eta_i : H_i \times H_i \rightarrow H_i, M_i : H_1 \times H_2 \rightarrow H_i, A_i, B_i : H_i \rightarrow H_i, C_i : H_1 \rightarrow H_1, D_i : H_2 \rightarrow H_2$  be nonlinear single-valued mappings. Let  $m_i : H_i \rightarrow H_i$  satisfy (2), and  $b_i : H_i \times H_i \rightarrow R$  be a real-valued functional satisfying the properties in Theorem 2.1 and properties (i)-(iv). Assume that following conditions are satisfied:

(1)  $m_i$  is  $\tau_i$ -Lipschitz continuous;

(2)  $N_i$  is  $\xi_i$ -relaxed Lipschitz with respect to  $A_i$  and  $B_i$  in the first and second arguments, and  $N_i$  is  $\alpha_i$ -Lipschitz continuous in the first argument and is  $\beta_i$ -Lipschitz continuous in the second argument, respectively;

(3)  $M_i$  is  $(\mu_i, \nu_i)$ -Lipschitz continuous and  $M_i$  is  $k_i$ -Lipschitz continuous in the first argument and is  $l_i$ -Lipschitz continuous in the second argument, respectively;

(4)  $A_i$  is  $\omega_i$ -Lipschitz continuous;

(5)  $B_i$  is  $\gamma_i$ -Lipschitz continuous;

(6)  $C_i$  is  $\lambda_i$ -Lipschitz continuous;

(7)  $D_i$  is  $\sigma_i$ -Lipschitz continuous;

(8)  $\eta_i$  is  $\delta_i$ -Lipschitz continuous.

If Assumption 1.1 holds and there exist constants  $\rho_1, \rho_2 > 0$  such that

$$\begin{aligned} \frac{1}{1-2\tau_1} \left[ 1 + \rho_1 C_1 + \delta_1 C_1 + \sqrt{1-2\rho_1 \xi_1 + \rho_1^2 (\alpha_1 \omega_1 + \beta_1 \gamma_1)^2} + \rho_1 \mu_1 \lambda_1 \right] + \frac{\rho_2 \delta_2 \nu_2 \sigma_2}{1-2\tau_2} < 1 \\ \frac{1}{1-2\tau_2} \left[ 1 + \rho_2 C_2 + \delta_2 C_2 + \sqrt{1-2\rho_2 \xi_2 + \rho_2^2 (\alpha_2 \omega_2 + \beta_2 \gamma_2)^2} + \rho_2 \mu_2 \lambda_2 \right] + \frac{\rho_1 \delta_1 \nu_1 \sigma_1}{1-2\tau_1} < 1 \end{aligned} \quad (7)$$

Then there exists  $(u, v) \in K_1(u) \times K_2(v)$  is a solution of SGNQVLI (1), and the sequence  $\{(u_n, v_n)\}$  generated by Algorithm 2.1 strongly converges to  $(u, v)$ .

**Proof** First, it follows from (5) in Algorithm 2.1 that, for any  $w \in K_1(u_n)$ ,

$$\begin{aligned} \langle x_n, u - x_n \rangle_1 &\geq \langle x_{n-1}, u - x_n \rangle_1 - \rho_1 \langle N_1(A_1x_{n-1}, B_1x_{n-1}) + M_1(C_1x_{n-1}, D_1y_{n-1}), \eta_1(u, x_n) \rangle_1 \\ &+ \rho_1 [b_1(x_{n-1}, x_n) - b_1(x_{n-1}, u)]. \end{aligned} \quad (8)$$

and for any  $z \in K_1(u_{n+1})$ ,

$$\begin{aligned} \langle x_{n+1}, u - x_{n+1} \rangle_1 &\geq \langle x_n, u - x_{n+1} \rangle_1 - \rho_1 \langle N_1(A_1x_n, B_1x_n) + M_1(C_1x_n, D_1y_n), \eta_1(u, x_{n+1}) \rangle_1 \\ &+ \rho_1 [b_1(x_n, x_{n+1}) - b_1(x_n, u)]. \end{aligned} \quad (9)$$

Adding  $\langle -m_1(u_n), w - u_n \rangle_1$  to the two sides of inequality (8) and then taking  $w = m_1(u_n) + u_{n+1} - m_1(u_{n+1}) \in K_1(u_n)$ , we get

$$\begin{aligned} &\langle x_n - m_1(x_n), m_1(x_n) + x_{n+1} - m_1(x_{n+1}) - x_n \rangle_1 \\ &\geq \langle x_{n-1} - m_1(x_n), m_1(x_n) + x_{n+1} - m_1(x_{n+1}) - x_n \rangle_1 \\ &- \rho_1 \langle N_1(A_1x_{n-1}, B_1x_{n-1}) + M_1(C_1x_{n-1}, D_1y_{n-1}), \eta_1(m_1(x_n) + x_{n+1} - m_1(x_{n+1}), x_n) \rangle_1 \\ &+ \rho_1 [b_1(x_{n-1}, x_n) - b_1(x_{n-1}, m_1(x_n) + x_{n+1} - m_1(x_{n+1}))]. \end{aligned} \quad (10)$$

Adding  $\langle -m_1(u_{n+1}), w - u_{n+1} \rangle_1$  to the two sides of inequality (9) and then taking  $w = m_1(u_{n+1}) + u_n - m_1(u_n) \in K_1(u_{n+1})$ , we get

$$\begin{aligned}
 & \langle x_{n+1} - m_1(x_{n+1}), m_1(x_{n+1}) + x_n - m_1(x_n) - x_{n+1} \rangle_1 \\
 & \geq \langle x_n - m_1(x_{n+1}), m_1(x_{n+1}) + x_n - m_1(x_n) - x_{n+1} \rangle_1 \\
 & \quad - \rho_1 \langle N_1(A_1x_n, B_1x_n) + M_1(C_1x_n, D_1y_n), \eta_1(m_1(x_{n+1}) + x_n - m_1(x_n), x_{n+1}) \rangle_1 \\
 & \quad + \rho_1 [b_1(x_n, x_{n+1}) - b_1(x_n, m_1(x_{n+1}) + x_n - m_1(x_n))].
 \end{aligned} \tag{11}$$

Adding (10) and (11), by properties (i) and (iii) of  $b_i$  and Assumption 1.1 (2), we have

$$\begin{aligned}
 & \langle u_n - u_{n+1} - m_1(u_n) + m_1(u_{n+1}), u_n - u_{n+1} - m_1(u_n) + m_1(u_{n+1}) \rangle_1 \\
 & \leq \langle u_{n-1} - u_n - m_1(u_n) + m_1(u_{n+1}), u_n - u_{n+1} - m_1(u_n) + m_1(u_{n+1}) \rangle_1 \\
 & \quad + \rho_1 \langle N_1(A_1u_{n-1}, B_1u_{n-1}) - N_1(A_1u_n, B_1u_n), \\
 & \quad \eta_1(m_1(u_n) + u_{n+1} - m_1(u_{n+1}), u_n) \rangle_1 \\
 & \quad + \rho_1 \langle M_1(C_1u_{n-1}, D_1v_{n-1}) - M_1(C_1u_n, D_1v_n), \\
 & \quad \eta_1(m_1(u_n) + u_{n+1} - m_1(u_{n+1}), u_n) \rangle_1 \\
 & \quad - \rho_1 [b_1(u_{n-1}, u_n) - b_1(u_{n-1}, m_1(u_n) + u_{n+1} - m_1(u_{n+1}))] \\
 & \quad - \rho_1 [b_1(u_n, u_{n+1}) - b_1(u_n, m_1(u_{n+1}) + u_{n+1} - m_1(u_n))] \\
 & \leq \langle u_{n-1} - u_n - m_1(u_n) + m_1(u_{n+1}), u_n - u_{n+1} - m_1(u_n) + m_1(u_{n+1}) \rangle_1 \\
 & \quad + \rho_1 \langle N_1(A_1u_{n-1}, B_1u_{n-1}) - N_1(A_1u_n, B_1u_n), \\
 & \quad \eta_1(m_1(u_n) + u_{n+1} - m_1(u_{n+1}), u_n) \rangle_1 \\
 & \quad + \rho_1 \langle M_1(C_1u_{n-1}, D_1v_{n-1}) - M_1(C_1u_n, D_1v_n), \\
 & \quad \eta_1(m_1(u_n) + u_{n+1} - m_1(u_{n+1}), u_n) \rangle_1 \\
 & \quad + \rho_1 b_1(u_n - u_{n-1}, u_n - m_1(u_n) - u_{n+1} + m_1(u_{n+1})).
 \end{aligned} \tag{12}$$

By property (iii) of  $b_i$ , (12) implies that

$$\begin{aligned}
 & \|x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\|_1^2 \\
 & \leq \|x_{n-1} - x_n - m_1(x_n) + m_1(x_{n+1})\|_1 \cdot \|x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\|_1 \\
 & \quad + [\|x_{n-1} - x_n + \rho_1 (N_1(A_1x_{n-1}, B_1x_{n-1}) - N_1(A_1x_n, B_1x_n))\|_1 + \\
 & \quad \|x_{n-1} - x_n - \rho_1 (M_1(C_1x_{n-1}, D_1y_{n-1}) - M_1(C_1x_n, D_1y_n))\|_1] \cdot \\
 & \quad \|\eta_1(m_1(x_n) + x_{n+1} - m_1(x_{n+1}), x_n)\|_1 \\
 & \quad + \rho_1 l_1 \|x_n - x_{n+1}\|_1 \|x_n - m_1(x_n) - x_{n+1} + m_1(x_{n+1})\|_1.
 \end{aligned} \tag{13}$$

It follows from Condition (6) and (13), we have

$$\begin{aligned}
 & \|u_n - u_{n+1} - m_1(u_n) + m_1(u_{n+1})\|_1^2 \\
 & \leq \|u_{n-1} - u_n - m_1(u_n) + m_1(u_{n+1})\|_1 \cdot \|u_n - u_{n+1} - m_1(u_n) + m_1(u_{n+1})\|_1 \\
 & \quad + [\|u_{n-1} - u_n + \rho_1 (N_1(A_1u_{n-1}, B_1u_{n-1}) - N_1(A_1u_n, B_1u_n))\|_1 + \\
 & \quad \|u_{n-1} - u_n - \rho_1 (M_1(C_1u_{n-1}, D_1v_{n-1}) - M_1(C_1u_n, D_1v_n))\|_1] \cdot \\
 & \quad \|\eta_1(m_1(u_n) + u_{n+1} - m_1(u_{n+1}), u_n)\|_1 \\
 & \quad + \rho_1 l_1 \|u_n - u_{n+1}\|_1 \|u_n - m_1(u_n) - u_{n+1} + m_1(u_{n+1})\|_1.
 \end{aligned}$$

$$\begin{aligned}
 & \|u_n - u_{n+1}\|_1 \\
 & \leq \|u_{n-1} - u_n\|_1 + 2 \|m_1(u_n) - m_1(u_{n+1})\|_1 \\
 & \quad + \delta_1 [\|u_{n-1} - u_n + \rho_1 (N_1(A_1u_{n-1}, B_1u_{n-1}) - N_1(A_1u_n, B_1u_n))\|_1 \\
 & \quad + \|u_{n-1} - u_n\|_1 + \rho_1 \|M_1(C_1u_{n-1}, D_1v_{n-1}) - M_1(C_1u_n, D_1v_n)\|_1] \\
 & \quad + \rho_1 C_1 \|u_{n-1} - u_n\|_1.
 \end{aligned} \tag{14}$$

By Conditions (2), (4) and (5), we have

$$\begin{aligned}
 & \|x_{n-1} - x_n + \rho_1 (N_1(A_1x_{n-1}, B_1x_{n-1}) - N_1(A_1x_n, B_1x_n))\|_1^2 \\
 & \leq [1 - 2\rho_1\xi_1 + \rho_1^2(\alpha_1\omega_1 + \beta_1\gamma_1)^2] \|x_{n-1} - x_n\|_1^2.
 \end{aligned} \tag{15}$$

By Conditions (1), (3), (6) and (7), it follows from (14) and (15) that

$$\begin{aligned}
 \|x_n - x_{n+1}\|_1 & \leq \frac{1}{1-2\tau_1} \{ [1 + \rho_1 C_1 + \delta_1 (1 + \sqrt{1 - 2\rho_1\xi_1 + \rho_1^2(\alpha_1\omega_1 + \beta_1\gamma_1)^2}) + \rho_1\mu_1\lambda_1] \cdot \\
 & \quad \|x_{n-1} - x_n\|_1 + \rho_1\delta_1 v_1\sigma_1 \|y_{n-1} - y_n\|_2 \}.
 \end{aligned} \tag{16}$$

And it follows from (6), for any  $v \in K_2(y_n)$ , we have

$$\begin{aligned}
 \langle y_n, v - y_n \rangle_2 & \geq \langle y_{n-1}, v - y_n \rangle_2 \\
 & \quad - \rho_2 \langle N_2(A_2y_{n-1}, B_2y_{n-1}) + M_2(C_2x_{n-1}, D_2y_{n-1}), \eta_2(v, y_n) \rangle_2 \\
 & \quad + \rho_2 [b_2(y_{n-1}, y_n) - b_2(y_{n-1}, v)].
 \end{aligned} \tag{17}$$

And, for any  $z \in K_2(v_{n+1})$ ,

$$\begin{aligned}
 \langle y_{n+1}, v - y_{n+1} \rangle_2 & \geq \langle y_n, v - y_{n+1} \rangle_2 \\
 & \quad - \rho_2 \langle N_2(A_2y_n, B_2y_n) + M_2(C_2x_n, D_2y_n), \eta_2(v, y_{n+1}) \rangle_2 \\
 & \quad + \rho_2 [b_2(y_n, y_{n+1}) - b_2(y_n, v)].
 \end{aligned} \tag{18}$$

Adding  $\langle -m_2(v_n), h_2 - v_n \rangle_2$  to the two sides of inequality (17) and then taking  $z = m_2(v_n) + v_{n+1} - m_2(v_{n+1}) \in K_2(v_n)$ , we get

$$\begin{aligned}
 & \langle y_n - m_2(y_n), m_2(y_n) + y_{n+1} - m_2(y_{n+1}) - y_n \rangle_2 \\
 & \geq \langle y_{n-1} - m_2(y_n), m_2(y_n) + y_{n+1} - m_2(y_{n+1}) - y_n \rangle_2 \\
 & \quad - \rho_2 \langle N_2(A_2 y_{n-1}, B_2 y_{n-1}) + M_2(C_2 x_{n-1}, D_2 y_{n-1}), \eta_2(m_2(y_n) + y_{n+1} - m_2(y_{n+1}), y_n) \rangle_2 \\
 & \quad + \rho_2 [b_2(y_{n-1}, y_n) - b_2(y_{n-1}, m_2(y_n) + y_{n+1} - m_2(y_{n+1}))].
 \end{aligned} \tag{19}$$

Adding  $\langle -m_2(y_{n+1}), z - y_{n+1} \rangle_2$  to the two sides of inequality (18) and then taking  $z = m_2(y_{n+1}) + y_n - m_2(y_n) \in K_2(y_{n+1})$ , we get

$$\begin{aligned}
 & \langle y_{n+1} - m_2(y_{n+1}), m_2(y_{n+1}) + y_n - m_2(y_n) - y_{n+1} \rangle_2 \\
 & \geq \langle y_n - m_2(y_{n+1}), m_2(y_{n+1}) + y_n - m_2(y_n) - y_{n+1} \rangle_2 \\
 & \quad - \rho_2 \langle N_2(A_2 y_n, B_2 y_n) + M_2(C_2 x_n, D_2 y_n), \eta_2(m_2(y_{n+1}) + y_n - m_2(y_n), y_{n+1}) \rangle_2 \\
 & \quad + \rho_2 [b_2(y_n, y_{n+1}) - b_2(y_n, m_2(y_{n+1}) + y_n - m_2(y_n))].
 \end{aligned} \tag{20}$$

Then repeating the method, we have

$$\begin{aligned}
 \|y_n - y_{n+1}\|_2 & \leq \frac{1}{1-2\tau_2} \left\{ [1 + \rho_2 C_2 + \delta_2 (1 + \sqrt{1 - 2\rho_2 \xi_2 + \rho_2^2 (\alpha_2 \omega_2 + \beta_2 \gamma_2)^2} + \rho_2 \mu_2 \lambda_2)] \right. \\
 & \quad \left. \|y_{n-1} - y_n\|_1 + \rho_2 \delta_2 v_2 \sigma_2 \|x_{n-1} - x_n\|_1 \right\}.
 \end{aligned} \tag{21}$$

From (16) and (21), we have

$$\begin{aligned}
 & \|x_n - x_{n+1}\|_1 + \|y_n - y_{n+1}\|_2 \\
 & \leq \left\{ \frac{1}{1-2\tau_1} [1 + \rho_1 C_1 + \delta_1 (1 + \sqrt{1 - 2\rho_1 \xi_1 + \rho_1^2 (\alpha_1 \omega_1 + \beta_1 \gamma_1)^2} + \rho_1 \mu_1 \lambda_1)] + \frac{\rho_2 \delta_2 v_2 \sigma_2}{1-2\tau_2} \right\} \\
 & \quad \|x_{n-1} - x_n\|_1 \\
 & + \left\{ \frac{1}{1-2\tau_2} [1 + \rho_2 C_2 + \delta_2 (1 + \sqrt{1 - 2\rho_2 \xi_2 + \rho_2^2 (\alpha_2 \omega_2 + \beta_2 \gamma_2)^2} + \rho_2 \mu_2 \lambda_2)] + \frac{\rho_1 \delta_1 v_1 \sigma_1}{1-2\tau_1} \right\} \\
 & \quad \|y_{n-1} - y_n\|_2 \\
 & \leq \max \{ \theta_1, \theta_2 \} (\|x_{n-1} - x_n\|_1 + \|y_{n-1} - y_n\|_2).
 \end{aligned} \tag{22}$$

Where  $\theta = \max \{ \theta_1, \theta_2 \}$ ,

$$\begin{aligned}
 \theta_1 & = \frac{1}{1-2\tau_1} [1 + \rho_1 C_1 + \delta_1 (1 + \sqrt{1 - 2\rho_1 \xi_1 + \rho_1^2 (\alpha_1 \omega_1 + \beta_1 \gamma_1)^2} + \rho_1 \mu_1 \lambda_1)] + \frac{\rho_2 \delta_2 v_2 \sigma_2}{1-2\tau_2}, \\
 \theta_2 & = \frac{1}{1-2\tau_2} [1 + \rho_2 C_2 + \delta_2 (1 + \sqrt{1 - 2\rho_2 \xi_2 + \rho_2^2 (\alpha_2 \omega_2 + \beta_2 \gamma_2)^2} + \rho_2 \mu_2 \lambda_2)] + \frac{\rho_1 \delta_1 v_1 \sigma_1}{1-2\tau_1}.
 \end{aligned}$$

By Condition (7), we have  $\theta < 1$ . Therefore  $\{(u_n, v_n)\}$  is a Cauchy sequence in  $H_1 \times H_2$ . Let  $(u_n, v_n) \rightarrow (u, v) \in H_1 \times H_2$  as  $n \rightarrow \infty$ . Next, we claim that  $(u, v) \in H_1 \times H_2$  is a solution of SGNQVLI (1). In fact, by Theorem 2.1, we may assume that  $(p, q) \in K_1(u) \times K_2(v)$  is the unique solution of SAGNQVLI (4), that is,

$$\begin{aligned}
 \langle p, u - p \rangle_1 & \geq \langle x, u - p \rangle_1 - \rho_1 \langle N_1(A_1 x, B_1 x) + M_1(C_1 x, D_1 y), \eta_1(u, p) \rangle_1 \\
 & \quad + \rho_1 [b_1(x, p) - b_2(x, u)], \forall u \in K_1(x).
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 \langle q, v - q \rangle_2 & \geq \langle y, v - q \rangle_2 - \rho_2 \langle N_2(A_2 y, B_2 y) + M_2(C_2 x, D_2 y), \eta_2(v, q) \rangle_2 \\
 & \quad + \rho_2 [b_2(y, q) - b_2(y, v)], \forall v \in K_2(y).
 \end{aligned} \tag{24}$$

Similar argument as in proving (22), we have  $\|x_n - p\|_1 + \|y_n - q\|_2 \leq \theta^n (\|x_0 - p\|_1 + \|y_0 - q\|_2)$ , where  $\theta = \max \{ \theta_1, \theta_2 \}$ ,  $\theta_1$  and  $\theta_2$  is as above. It follows from Condition (7) that  $\theta < 1$ , so  $\|u_n - p\|_1 + \|v_n - q\|_2 \rightarrow 0 (n \rightarrow \infty)$ , this is,  $p = u, q = v$ .

Taking them into (23) and (24), we have

$$\begin{aligned}
 & \langle N_1(A_1 x, B_1 x) + M_1(C_1 x, D_1 y), \eta_1(u, x) \rangle_1 - b_1(x, x) + b_1(x, u) \geq 0, \forall u \in K_1(x), \\
 & \langle N_2(A_2 y, B_2 y) + M_2(C_2 x, D_2 y), \eta_2(v, y) \rangle_2 - b_2(y, y) + b_2(y, v) \geq 0, \forall v \in K_2(y).
 \end{aligned}$$

That is,  $(u, v) \in K_1(u) \times K_2(v)$  is a solution of SGNQVLI (1). This completes the proof.

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