

# Universal Design Description of Cubic Cyclic Fields and an Algorithm for Computing of Minimal Polynomials of Generating Elements

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**Abstract**—This paper presents an effective description of the universal design cubic cyclic fields and using an algorithm to compute the minimum polynomials generating elements. The calculations are carried out with the use of the properties of Gauss sums corresponding primitive Dirichlet character. Built fundamental and integral basis normal cubic cyclic fields. Necessary and sufficient criterion of element reversibility of totally solid cyclic fields is given here. The different examples of computational exercises of minimal polynomials have been mentioned here. Constructing the model Abelian field, Kronecker-Weber theorem was taken into account; it states that any Abelian extension of the field of rational numbers is in some cyclotomic field. This theorem gives a classification of Abelian fields, and also determines the laws of decomposition, makes it possible to determine the structure of their discriminants and to obtain explicit expressions for the number of ideal classes. The results of the work describe Abelian extensions of a given field. Obtained theoretical data may be used in subsequent studies of algebraic theories, namely, the problem of absolutely Abelian fields.

**Keywords**—gauss sums; integer basis; dirichlet characters; absolutely abelian fields

## I. INTRODUCTION

One of the central problems in algebraic number theory is a specific description of Abelian extensions of a given field (Azizi, Zekhnini, & Taous, 2016; Carrozza, Oriti, & Rivasseau, 2012). This problem has a deep connection with the inverse problem of Galois. Even with the construction of the particular examples given Galois group applies to complex tasks of this problem (Bartel, 2013; Bystritskaya, Burkhanova, Voronin, Ivanova, & Grigoryeva, 2016; Horton & Panasuk, 2012). A specific description of Abelian extensions remains poorly studied, with the exception of some fields of special structure, cyclotomic fields, quadratic fields. In numerous papers B.M.Urazbaeva (Urazbaev, 1972), and his students studied completely Abelian field with a given discriminant in the case of non-critical and promoted count fields, but specific description of these fields is left unexplored. An effective tool for the specific structural description and study of Abelian extensions of the special role played by the study of the character of modules leading Dirichlet and Gauss sums of Dirichlet characters (Borevich & Shafarevich, 1985; Gauss, 1959; Ireland & Rosen, 1986). The Kronecker-Weber theorem (Bolen, 2008; G. Hasse, 1953; H. Hasse, 1923), states that any

Abelian field (a.a.f.) is a subfield of, cyclotomic field  $Q(\zeta_f)$ , with one  $f$ , where  $\zeta_f = \cos \frac{2\pi}{f} + i \sin \frac{2\pi}{f} \in C^*$ , and  $Q$  - is a field of rational numbers. The leading a.a.p. divisor  $K$  is called the smallest natural number  $F(K) = f$  such that  $Q(\zeta_f)$ , is the smallest field of circle division, which contains  $K$ . Here are some of the necessary definitions and approvals from works [8.9.10], which provides a multi-purpose constructive description of the totally cyclic fields of primary degree  $p^l$ , where  $p$  is a prime number. Let  $X_{p^l}$  is a pattern of all the primitive characters of the primary order  $p^l$ , and  $X_{p^l} / \sim$  is a factor of patterns, the pattern  $X_{p^l}$  of all the primitive characters of the primary order  $p^l$  on the equivalence of  $\sim$ ,  $P_{p^l}$  is the set of all totally cyclic fields of primary degree  $p^l$ . Normal extension  $F = Q(\theta)$  of the 3<sup>rd</sup> degree of the field of rational numbers  $Q$  is identified as totally cubic cyclic field. Let assume that  $Z/(m)$  is a residue ring modulo  $m$ ,  $U(m)$  is a multiplicative group of invertible elements of the ring  $Z/(m)$ , and  $X(m)$  is a character group of Abelian group  $U(m)$ .

## II. METHODS

The theoretical basis of the research is composed of scientific methods of analysis and synthesis, induction and deduction, structural-functional and system analysis, as well as the general logical methods. We generalized the scientific expertise of scientists, who were investigating this problematics.

In order to achieve the purpose of the paper, Dirichlet characters and Gauss sum were studied, as the Kronecker-Weber theorem was considered.

## III. DATA, ANALYSIS, AND RESULTS

**Definition 1.** The Dirichlet character modulo  $m$  is called the complex-valued function  $\chi^*: Z \rightarrow C$ , defined in set of all integer values  $Z$ , through the characters  $\chi \in X(m)$  by the formula

$$\chi^*(n) = \begin{cases} \chi(\bar{n}), & \text{if } (n, m) = 1, \bar{n} \in U(m) \\ 0 & \text{if } (n, m) > 1 \end{cases}$$

If  $\chi_1^* \in X^*(m)$  is the Dirichlet character modulo  $m$ , and  $\chi_2^* \in X^*(k)$  is the Dirichlet character modulo  $m$ , then intersection determined by the formula:  $\chi_1^* \chi_2^*(n) = \chi_1^*(n) \chi_2^*(n)$ , is revealed the character  $\chi_1^* \cdot \chi_2^* \in X([m, k])$  modulo  $[m, k]$ , where  $[m, k]$ , is the lowest common multiple of the integers  $m$  и  $k$ . The intrinsic aping  $f: X(m) \rightarrow X^*(m)$ ,  $f(\chi) = \chi^*$ ,  $\chi \in X(m)$ ,  $\chi^* \in X^*(m)$ , is the isomorphic mapping of the group  $X(m)$  for the group  $X^*(m)$  of the Dirichlet characters modulo  $m$ . Noting this intrinsic isomorphic mapping, then let us label the Dirichlet characters modulo  $m$ , by  $X(m)$ . Note that the mapping  $In_{d,m}: X(d) \rightarrow X(m)$  determined by the formula  $In_{d,m}(\chi) = \chi \cdot \varepsilon_m$   $\chi \in X(m)$  is revealed isomorphic embedding of  $X(d)$  group to  $X(m)$  group, where  $\varepsilon_m \in X(m)$ , is an identity character of  $X(m)$  group. The Dirichlet character  $In_{d,m}(\chi) \in X(m)$  is called the induced Dirichlet character  $\chi \in X(d)$ . If for the Dirichlet character  $\lambda \in X(m)$  modulo  $m$ , there is that proper divisor  $d$  of the integer  $m$ ,  $m:d$ ,  $0 < d < m$ , and that Dirichlet character  $\chi \in X(d)$  modulo  $d$ , which induced  $\lambda \in X(m)$ , then that character  $\lambda \in X(m)$  is identify not primitive, it is named the primitive in the contrary case. Each Dirichlet character  $\lambda \in X(m)$  modulo  $m$ , is induced by the unambiguous primitive Dirichlet character  $\chi \in X(f)$ , on uniquely determined leading module  $F(\lambda) = F(\chi) = f$ . If the positive integer  $m \in \mathbb{Z}^+$  has the canonical decomposition  $m = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$ , then each Dirichlet character  $\chi \in X(m)$  modulo  $m$  has Dirichlet character of the form  $\chi(n) = \chi_1(n) \chi_2(n) \dots \chi_s(n)$ ,  $n \in \mathbb{Z}$ ,  $\chi_i \in X(p_i^{k_i})$ ,  $i = 1, 2, \dots, s$

here  $F(\chi) = F(\chi_1) F(\chi_2) \dots F(\chi_s)$

**Proposition 1.** Let us assume that  $p^l$ , is the degree of uneven prime integer  $p$ . The primitive Dirichlet character, where  $\chi \in X(f)$  modulo  $f$ , with order of  $|\chi| = p^l$ , exists if and only if when  $f$  has the canonical decomposition of the form  $f = p_1 p_2 \dots p_s$  or  $f = p^{l_0+1} p_1 p_2 \dots p_s$  where  $p_i \equiv 1 \pmod{p^{l_i}}$ ,  $l_i \in \mathbb{Z}^+$ ,  $l = \max\{l_1, l_2, \dots, l_s\}$ , or  $l = \max\{l_0, l_1, l_2, \dots, l_s\}$

**Definition 2.** Two Dirichlet primitive characters  $\chi_1, \chi_2 \in X(f)$ , modulo  $f$ , are identified as indistinguishable in the language and assigned as  $\chi_1 \sim \chi_2$ , if their leading modules are the same  $F(\chi_1) = F(\chi_2) = f$ , and provide the same subgroup of the group  $X(f)$ , viz  $(\chi_1) = (\chi_2)$ .

**Proposition 2.** Let us assume that  $p > 2$  is the prime integer, and the positive integer  $f$  has the canonical decomposition of the form  $f = p_1 p_2 \dots p_s$  or  $f = p^{l_0+1} p_1 p_2 \dots p_s$  where  $p_i \equiv 1 \pmod{p^{l_i}}$ ,  $l_i \in \mathbb{Z}^+$ ,  $l = \max\{l_1, l_2, \dots, l_s\}$ , or  $l = \max\{l_0, l_1, l_2, \dots, l_s\}$ . Then, the integer  $N_{p^l}(f, l_0, l_1, l_2, \dots, l_s)$  of the Dirichlet primitive characters  $\chi \in X(f)$  modulo  $f$ , with order of  $|\chi| = p^l$ , which has the canonical decomposition of the form  $\chi = \chi_0 \chi_1 \chi_2 \dots \chi_s$ , where  $\chi_i \in X(p_i)$ ,  $F(\chi_i) = p_i$ ,  $|\chi_i| = p^{l_i}$ ,  $i = 1, 2, \dots, s$ ,  $\chi_0 \in X(p^{l_0+1})$ ,  $F(\chi_0) = p^{l_0+1}$ , is expressed by the formula:  $N_{p^l}(f, l_1, l_2, \dots, l_s) = \prod_{i=1}^l \varphi(p^{l_i})$ , with  $f = p_1 p_2 \dots p_s$ ;  $N_{p^l}(f, l_0, l_1, l_2, \dots, l_s) = \prod_{i=0}^l \varphi(p^{l_i})$ , with  $f = p^{l_0+1} p_1 p_2 \dots p_s$ .

**Theorem.** The mapping  $\psi: X_{p^l} / \sim \rightarrow P_{p^l}$ , determined by the formula  $\psi(\chi) = Q(\theta(\chi))$ ,  $\chi \in X_{p^l}$ ,  $F(\chi) = f$ , The primitive Dirichlet character modulo  $f$ , with order of  $|\chi| = p^l$ ,  $\theta(\chi) = \sum_{t \in \text{Ker } \chi} \zeta_f^t \in Q(\zeta_f)$  is revealed the bijective mapping, with  $F(\chi) = F(Q(\theta(\chi)))$ ,  $|Q(\theta(\chi))| = |\chi|$  then the sum total  $p^l$  of adjoined numbers  $g^j(\theta(\chi))$ ,  $|\overline{g}| = p^l$ ,  $U(f)/\ker \chi = (\overline{g}) = \{g^i \cdot \ker \chi \mid 0 \leq i < p^l\}$ ,  $\chi(g) = \zeta_{p^l}$ , is associated with Gauss sums  $\tau(\chi^i) = \sum_{t \in U(f)} \chi^i(t) \zeta_f^t$  with the

relators  $g^j(\theta(\chi)) = \frac{1}{p^l} \sum_{i=1}^{p^l-1} \zeta_{p^l}^{-ij} \tau(\chi^i)$ , If  $f = p_1 p_2 \dots p_s$ ,  $p_i \equiv 1 \pmod{p^{l_i}}$ ,  $l_i \in \mathbb{Z}^+$ ,  $l = \max\{l_1, \dots, l_s\}$ ,

$\chi = \chi_1 \dots \chi_s \in X(f)$ ,  $\chi_i \in X(p_i)$ ,  $|\chi_i| = p^{l_i}$  then the sum total  $p^l$  of adjoined numbers  $g^j(\theta(\chi))$  compose normal integer basis of the field  $Q(\theta(\chi))$ , and the field amount with the

discriminant  $D = \prod_{i=1}^s p_i^{p^l - p^{l-l_i}}$  is equal

to  $N_{p^l}(D, l_1, l_2, \dots, l_s) = \varphi(p^l)^{-1} \prod_{i=1}^l \varphi(p^{l_i})$ . If  $p$  is odd prime,

$f = p^{l_0+1} p_1 p_2 \dots p_s$ ,  $p_i \equiv 1 \pmod{p^{l_i}}$ ,  $l_i \in \mathbb{Z}^+$ ,  $l = \max\{l_0, l_1, \dots, l_s\}$ ,  $\chi = \chi_0 \chi_1 \dots \chi_s \in X(f)$ ,

$|\chi_i| = p^{l_i}$ ,  $\chi_i \in X(p_i)$ ,  $p_0 = p^{l_0+1}$  then the sum total of  $p^l$  - integers  $1, \theta_1, \theta_2, \dots, \theta_{p^l-1}$  is the integer basis of the field

$Q(\theta(\chi))$ , and the field amount with the discriminant

$$D = p^{p^{l-l_1}[(l_0+1)p^{l_0} - \frac{p^{l_0}-1}{p-1}]} \prod_{i=1}^s p_i^{p^l - p^{l-l_i}} \quad \text{is equal to}$$

$$N_{p^l}(D, l_1, l_2, \dots, l_l) = \varphi(p^l)^{-1} \prod_{i=0}^l \varphi(p^i)^{l_i}$$

Keep in mind that the declared theorem is the synthesis of well-known fact that: «between the pattern of quadratic fields and the pattern of all primitive Dirichlet quadratic characters there exists an unambiguous correspondence», this proved that the Gaussian sums are an effective tool to describe and examine not only a class of quadratic fields, but to describe and study the class of all absolutely cyclic fields of primary degree  $p^l$ . For more effectiveness of this multi-purpose constructive description of totally cubic cyclic fields let us describe the calculation algorithm of minimal polynomials of their generating elements with the use of sums of properties of Gauss and Jacobi for primitive characters of order 3, (Borevich & Shafarevich, 1985; Gauss, 1959; Ireland & Rosen, 1986; Venkov, 1937). Calculation of minimal polynomials of totally cubic cyclic fields is reduced to calculation of Jacobi sum

$$A = \sum_{i=1}^2 J(\chi^i, \chi^i) \quad \text{corresponding to primitive Dirichlet characters for simple modulo } p \text{ with order } 3.$$

**Proposition 3.** The mapping,  $\psi: X_3/\sim \rightarrow P_3$  determined by the formula  $\psi(\chi) = Q(\theta(\chi))$ , where  $\chi \in X_3$ ,

$$\theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3} [\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)],$$

$F(\chi) = f = 3^k p_1 p_2 \dots p_s$ ,  $k=0,2$ ,  $p_i \equiv 1(3)$ , Canonical decomposition of the leading module of the character  $\chi \in X_3$ , is a bijective mapping, where the leading module of the character  $\chi \in X_3$ , is the same as for the conductor of the field  $Q(\theta(\chi))$ ,  $F(\chi) = f = F(Q(\theta(\chi)))$  and the discriminant of the field  $Q(\theta(\chi))$  is equal to  $D = f^2$ . The minimal polynomial  $\theta(\chi)$  is expressed by the formula

$$p(x) = x^3 - \mu(f)x^2 + \frac{1}{3}[\mu(f)^2 - f]x - \frac{1}{27}[\tau(\chi)^3 + \tau(\chi^2)^3 + (1-3f)\mu(f)]$$

If  $f = p_1 p_2 \dots p_s$ ,  $p_i \equiv 1(3)$ ,  $p_i \equiv 1(3)$ ,  $\chi = \chi_1 \dots \chi_s \in X(f)$ ,  $\chi_i \in X(p_i)$ ,  $|\chi_i| = 3$ ,  $U(f)/\ker \chi = (\bar{g})$ ,  $|\bar{g}| = 3$ ,  $\gamma(g) = \zeta_3$ , then the total of adjoined numbers  $\theta_0 = g^0(\theta(\chi))$ ,  $\theta_1 = g(\theta(\chi))$ ,  $\theta_2 = g^2(\theta(\chi))$  is normal integer basis of the field  $Q(\theta(\chi))$ , of degree 3, where  $2^s$  the primitive characters Dirichlet modulo  $f$ , of the form  $\chi_0^{i_0} \chi_1^{i_1} \dots \chi_s^{i_s} \in X(f)$ ,  $(i_1 \dots i_s) \in I_2^s$ ,  $I_2 = [1,2]$ , all primitive characters are completed modulo  $f$ , with order 3 and the number of all cyclic fields of degree 3, with the discriminant

$D = f^2$  is equal to  $N_3(f^2) = 2^{s-1}$ . If  $\chi \in X_3$ , is the primitive character Dirichlet, modulo  $f = 3^2 p_1 p_2 \dots p_s$ ,  $p_i \equiv 1(3)$ ,  $\chi = \chi_0 \chi_1 \dots \chi_s \in X(f)$ ,  $\chi_i \in X(p_i)$ ,  $|\chi_i| = 3$ ,  $|\chi| = 3$ , then the normal basis of the field  $Q(\theta(\chi))$ , does not exist, however  $1, \theta_0, \theta_1$  is a substantial field basis  $Q(\theta(\chi))$ , and  $2^{s+1}$  the primitive characters Dirichlet modulo  $f$  of the form  $\chi_0^{i_0} \chi_1^{i_1} \dots \chi_s^{i_s} \in X(f)$ ,  $(i_0, i_1 \dots i_s) \in I_2^{s+1}$ ,  $I_2 = [1,2]$  so, all primitive characters are completed modulo  $f$  and the number of all cyclic fields of degree 3, with the discriminant  $D = f^2 = (3^2 p_1 p_2 \dots p_s)^2$  is equal to  $N_3(f^2) = 2^s$ .

**Proof.** Let us assume, that  $Q(\theta)$  is the solid cyclic field, with leading divisor  $f$ . The Galois group of the field  $Q(\theta)$  is  $U(f)/\ker \chi = \{g^i \ker \chi | 0 \leq i < 2\}$  where  $\chi \in X(f)$  is a primitive character Dirichlet modulo  $f$ , with order of  $|\chi| = 3$  where

$$\theta = \theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3} [\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)] \quad \theta = \theta(\chi),$$

$$F(\chi) = f = F(Q(\theta(\chi))) \quad \text{As far as}$$

$$[3\theta]^3 = [\mu(f) + \tau(\chi) + \tau(\chi^2)]^3 = \mu(f)\theta^2 - 9[\mu(f)^2 - f]\theta - \frac{1}{3}[\tau(\chi)^3 + \tau(\chi^2)^3 + (1-3f)\mu(f)],$$

Then the minimal polynomial of integer is  $\theta = \theta(\chi)$ ,

$$p(x) = x^3 - \mu(f)x^2 + \frac{1}{3}[\mu(f)^2 - f]x - \frac{1}{27}[\tau(\chi)^3 + \tau(\chi^2)^3 + (1-3f)\mu(f)]$$

In case if  $f = p$ , and the prime integer is  $p \equiv 1(3)$ ,

$U(p) = (g)$ ,  $X(p) = (\lambda_p)$ ,  $\lambda_p(g) = \zeta_{\varphi(p)}$ , there is in existence accurate to equivalency an unambiguous primitive character Dirichlet modulo  $p$ ,  $\chi = \lambda_p^{\frac{p-1}{3}}$ ,  $\gamma(g) = \zeta_3$ , with order of  $|\chi| = 3$ , and an unambiguous cyclic field

$$Q(\theta(\chi)) \quad \theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3} [\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)] \quad \text{with the}$$

discriminant  $D = p^2$  is available. The minimal polynomial of the generator  $\theta(\chi)$  is expressed by the formula:

$$p(x) = x^3 + x^2 + \frac{1-p}{3}x - \frac{pA+3p-1}{27} \quad A = \sum_{i=1}^2 J(\chi^i, \chi^i)$$

Because of  $\text{Ker}\chi = \text{Ker}\lambda_p^{\frac{p-1}{3}} = \{g^0, g^3, g^{2 \cdot 3}, \dots, g^{\frac{p-4}{3}}\}$ ,

$$U(f)/\ker \chi = (\bar{g}) \quad , \quad |\bar{g}| = 3 \quad , \quad \text{then}$$

$$\theta(\chi) = \zeta_p + \zeta_p^{g^3} + \zeta_p^{g^6} + \dots + \zeta_p^{g^{\frac{p-4}{3}}} \quad \text{is a } \frac{p-1}{3} \text{ - termed}$$

Gaussian period, and the plurality of adjoined numbers  $\theta_0 = g^0(\theta(\chi))$ ,  $\theta_1 = g(\theta(\chi))$ ,  $\theta_2 = g^2(\theta(\chi))$  is the Gaussian

integer basis of the field,  $Q(\theta(\chi))$  of degree 3, with the discriminant  $D = p^2$ . In the case of  $f = p$ , and a prime number is  $p \equiv 1(3)$ , we shall find a number of different ways of evaluation of integer A. If  $p = 3m + 1$ ,  $J(\chi, \chi)_1 = a + b\zeta_3$  then  $a \equiv -1(\text{mod}3)$ ,  $A = 2a - b \equiv 1(\text{mod}3)$ . The integer A is determined as univalent, just as absolutely least residue of the integer  $-C_{2b}^b = -\frac{2m(2m-1)\dots(m+1)}{1 \cdot 2 \cdot 3 \dots m}$ , modulo p. The integer A is determined as univalent, from condition  $A \equiv 1(\text{mod}3)$ ,  $4p = A^2 + 27B^2$ .

Then the integer  $b = 3B$  is determined as univalent, up to sign, that is  $2a = A + b$ ,  $J(\chi, \chi)_1 = a + b\zeta_3$ ,  $J(\chi^2, \chi^2)_1 = a - b - b\zeta_3$ . Also note that the integer A is determined as univalent, with condition that  $\tau(\chi)^3 + \tau(\chi^2)^3 = Ap$ . If  $f = 3^2 p_1 p_2 \dots p_s$ ,  $p_i \equiv 1(3)$ ,  $\chi = \chi_0 \chi_1 \dots \chi_s \in X(f)$ ,  $\chi_i \in X(p_i)$ ,  $|\chi_i| = 3$ ,  $|\chi| = 3$  then  $\tau(\chi)^3 + \tau(\chi^2)^3 = \tau(\chi_0)^3 \tau(\chi_1)^3 \dots \tau(\chi_s)^3 + \tau(\chi_0^2)^3 \tau(\chi_1^2)^3 \dots \tau(\chi_s^2)^3 =$   

$$= f \left[ \prod_{i=0}^s J(\chi_i, \chi_i) + \prod_{i=0}^s J(\chi_i^2, \chi_i^2) \right]$$
  

$$J(\chi_i, \chi_s) + J(\chi_i^2, \chi_s^2) = [(a_i + b_i \zeta_3^2) + (a_{i_i} + b_i \zeta_3)] = A_i p_i$$
  

$$\prod_{i=0}^s J(\chi_i, \chi_i) + \prod_{i=0}^s J(\chi_i^2, \chi_i^2) = A$$
, the sum total of integers  $1, \theta_0, \theta_1$  is revealed the integer basis of the field  $Q(\theta(\chi))$ , and  $2^{s+1}$  the primitive characters of Dirichlet modulo f, in the form of  $\chi_0^{i_0} \chi_1^{i_1} \dots \chi_s^{i_s} \in X(f)$ ,  $(i_0, i_1 \dots i_s) \in I_2^{s+1}$ ,  $I_2 = [1, 2]$ , and there are completed all primitive characters modulo f. If  $f = p_1 p_2 \dots p_s$ ,  $p_i \equiv 1(3)$ ,  $\chi = \chi_1 \dots \chi_s \in X(f)$ ,  $\chi_i \in X(p_i)$ ,  $|\chi_i| = 3$  and Galois group  $Q(\theta(\chi))$  is a polycyclic group  $U(f)/\ker \chi = (\bar{g})$ ,  $|\bar{g}| = 3$ , then the sum total of adjoined numbers  $\theta_0 = g^0(\theta(\chi))$ ,  $\theta_1 = g(\theta(\chi))$ ,  $\theta_2 = g^2(\theta(\chi))$  is a normal integer basis of the field  $Q(\theta(\chi))$  of degree 3, with the discriminant

$$D = \Delta(\theta_0, \theta_1, \theta_2) = \begin{vmatrix} \theta_0 & \theta_1 & \theta_2 \\ \theta_1 & \theta_2 & \theta_0 \\ \theta_2 & \theta_0 & \theta_1 \end{vmatrix} =$$

$$= \prod_{k=0}^2 (\theta_0 + \zeta_3^k \theta_1 + \zeta_3^{2k} \theta_2)^2 = \mu(f)^2 \tau(\chi)^2 \tau(\chi^2)^2 = f^2$$

and the integer of all cyclic fields of degree 3, with the discriminant  $D = f^2$  is equal to  $N_3(f^2) = 2^{s-1}$ . If  $f = 3^2 p_1 p_2 \dots p_s$ ,  $p_i \equiv 1(3)$ ,  $\chi = \chi_0 \chi_1 \dots \chi_s \in X(f)$ ,

$\chi_i \in X(p_i)$ ,  $|\chi_i| = 3$ ,  $|\chi| = 3$  then there is not normal basis of the field  $Q(\theta(\chi))$ , except that  $1, \theta_0, \theta_1$  is revealed a substantial basis of the field  $Q(\theta(\chi))$ , and a number of all totally cyclic fields of degree 3, with the discriminant

$$\Delta(1, \theta_0, \theta_1) = \begin{vmatrix} 1 & \theta_0 & \theta_1 \\ 1 & \theta_1 & \theta_2 \\ 1 & \theta_2 & \theta_0 \end{vmatrix}^2 = f^2$$

$$D = f^2 = (3^2 p_1 p_2 \dots p_s)^2 \quad \text{is } \varepsilon \theta \nu \alpha \lambda \quad \tau \theta \quad N_3(f^2) = 2^s.$$

**Proposition 4.** There is in existence accurate to equivalency an unambiguous primitive character  $\chi \in X(3^{l+1})$  with order of  $|\chi| = 3^l$  and an unambiguous cyclic field  $Q(\theta(\chi))$  of degree  $|Q(\theta(\chi)) : Q| = 3^l$  and with the discriminant  $D = 3^{(l+1)3^l - \frac{3^l-1}{2}}$  is available. For  $f_{3^l}(x)$ , minimal polynomial  $\theta(\chi_{3^l}) \in Q(\zeta_{3^{l+1}})$ , the recurrence formulas are correct,  $f_{3^l}(x) = f_{3^{l-1}}(x^3 - 3x)$ ,  $f_3(x) = x^3 - 3x + 1$ .

**Proof.** Let us assume that  $U(3^{l+1}) = (\bar{2})$ ,  $X(3^{l+1}) = (\lambda_{3^{l+1}})$ ,  $\lambda_{3^{l+1}}(2) = \zeta_{\varphi(3^{l+1})}$  with any  $l \in \mathbb{Z}^+$ . Then there is in existence accurate to equivalency modulo  $3^{l+1}$  an unambiguous primitive character,  

$$\chi_{3^l} = \lambda_{3^{l+1}}^{\frac{\varphi(3^{l+1})}{3^l}} = \lambda_{3^{l+1}}^2$$
,  
 $\chi_{3^l}(2) = \lambda_{3^{l+1}}^2(2) = \zeta_{3^l}$ ,  $F(\chi_{3^l}) = 3^{l+1}$ , with order of  $|\chi_{3^l}| = 3^l$ ,  $\chi_{3^l}(2^t) = \zeta_{3^l}^t$ ,  $\text{Ker} \chi_{3^l} = (2^{3^l}) = \{1, -1\}$ ,  
 $\theta(\chi_{3^l}) = \zeta_{3^{l+1}} + \zeta_{3^{l+1}}^{-1}$ . The Galois group of the field  $Q(\theta(\chi_{3^l}))$  is  $U(3^{l+1})/\ker \chi_{3^l} = \{2^i \ker \chi_{3^l} \mid 0 \leq i < 3^l\}$  and an unambiguous cyclic field  $Q(\theta(\chi_{3^l}))$  degree of

$|Q(\theta(\chi_{3^l})) : Q| = 3^l$  with the discriminant  $D = 3^{(l+1)3^l - \frac{3^l-1}{2}}$ . Let us set by  $f_{3^l}(x)$  the minimal polynomial of integer is  $Q(\theta(\chi_{3^l}))$ . Because of  

$$\theta(\chi_{3^l})^3 = (\zeta_{3^{l+1}} + \zeta_{3^{l+1}}^{-1})^3 = \zeta_{3^l} + \zeta_{3^l}^{-1} +$$

$$+ 3(\zeta_{3^{l+1}} + \zeta_{3^{l+1}}^{-1}) = \theta(\chi_{3^{l-1}}) + 3\theta(\chi_{3^l})$$

Then the recurrence formulas are correct,  $f_{3^l}(x) = f_{3^{l-1}}(x^3 - 3x)$ .  $U(9) = (2)$ ,  $X(9) = (\lambda)$ ,

$\lambda(2) = \zeta_6$  ,  $\chi = \lambda^2$  ,  $\theta_0 = \theta(\chi) = \zeta_9 + \zeta_9^{-1} = 2\cos\frac{2\pi}{9}$  ,  
 $\theta_1 = 2^1(\theta(\chi)) = \zeta_9^2 + \zeta_9^{-2} = 2\cos\frac{4\pi}{9}$  ,  
 $\theta_2 = 2^2(\theta(\chi)) = \zeta_9^4 + \zeta_9^{-4} = 2\cos\frac{8\pi}{9}$  .  $1, \theta_0, \theta_1$  ; is the  
 integer basis of the field  $Q(\theta_0)$  , with the discriminant  
 $D = \Delta(1, \theta_0, \theta_1) = 9^2 = 81$  . The minimal polynomial  
 $f_{3^2}(x) = f_3(x^3 - 3x) =$   
 $Q(\theta_0)$  is,  $x^9 - 9x^7 + 27x^5 - 30x^3 + 9x - 1$  , and etc.

**Proposition 5.** In order that  
 $\alpha = x + y\theta_0 + z\theta_0^2 \in Z(F)$  ,  $\theta_i = g^i(\theta(\chi))$  could be a  
 reversible ring element of algebraic integers  $Z(F)$  , the solid  
 cyclic field

$$F = Q(\theta(\chi))$$

$\theta_0 = \theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3}[\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)]$  is necessary

And sufficient for norm function  
 $\alpha = x + y\theta_0 + z\theta_0^2 \in Z(F)$  to be  
 $N_{F/Q}(x + y\theta_0 + z\theta_0^2) = \pm 1$  , or so that solution of  
 Diophantine equation could be available  
 $N_{F/Q}(\alpha) = x^3 + \sigma_3 y^3 + \sigma_3^2 z^3 + \sigma_1 x^2 y +$   
 $(\sigma_1^2 - 2\sigma_2)x^2 z + \sigma_2 xy^2 + (\sigma_2^2 - 2\sigma_1\sigma_3)xz^2 +$   
 $+ \sigma_1\sigma_3 y^2 z + \sigma_2\sigma_3 yz^2 + (\sigma_1\sigma_2 - 3\sigma_3)xyz = \pm 1$

where

$$\begin{aligned}
 \sigma_1 &= \theta_0 + \theta_1 + \theta_2 = \mu(f) \\
 \sigma_2 &= \theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2 = \frac{1}{3}(\mu(f)^2 - f) \\
 \sigma_3 &= \theta_0\theta_1\theta_2 = \frac{1}{27}[\tau(\chi)^2 + \tau(\chi^2)^2 + (1-3f)\mu(f)].
 \end{aligned}$$

**Necessity.** If  $\alpha \in U(F)$  is a reversible element of the  
 ring  $Z(F)$  , then there is available such an element,  
 as  $\beta \in Z(F)$  , that  $\alpha\beta = 1$  , or  
 $N_{F/Q}(\alpha) \cdot N_{F/Q}(\beta) = N_{F/Q}(1) = 1$

It means that  $N_{F/Q}(\alpha) = \pm 1$  .

**Sufficiency.** If  $\alpha \in Z(F)$  and  $N_{F/Q}(\alpha) = \pm 1$   
 then  $N_{F/Q}(\alpha) = \alpha \cdot g(\alpha) \cdot g^2(\alpha) = \pm 1$  . As far  
 as  $g(\alpha)g^2(\alpha) \in Z(F)$  , then  $\alpha \in U(F)$  . Thus there exist such  
 elements, as  $x, y, z \in Z$  ,  
 that

$$N_{F/Q}(\alpha) = (x + y\theta_0 + z\theta_0^2)(x + y\theta_1 + z\theta_1^2)(x + y\theta_2 + z\theta_2^2) = \pm 1$$

it means that  
 $N_{F/Q}(\alpha) = x^3 + \theta_0\theta_1\theta_2 y^3 + \theta_0^2\theta_1^2\theta_2^2 z^3 +$   
 $+ (\theta_0 + \theta_1 + \theta_2)x^2 y + (\theta_0^2 + \theta_1^2 + \theta_2^2)x^2 z + (\theta_0\theta_1 +$   
 $+ \theta_0\theta_2 + \theta_1\theta_2)xy^2 + (\theta_0^2\theta_1^2 + \theta_0^2\theta_2^2 + \theta_1^2\theta_2^2)xz^2 +$   
 $+ (\theta_0^2\theta_1\theta_2 + \theta_0\theta_1^2\theta_2 + \theta_0\theta_1\theta_2^2)y^2 z + (\theta_0^2\theta_1^2\theta_2 +$   
 $+ \theta_0^2\theta_1\theta_2^2 + \theta_0\theta_1^2\theta_2^2)yz^2 + (\theta_0^2\theta_1 + \theta_0\theta_1^2 +$   
 $+ \theta_0^2\theta_2 + \theta_0\theta_2^2 + \theta_1\theta_2^2 + \theta_1^2\theta_2)xyz$

Or there exist such elements as  $x, y, z \in Z$  , that

$$\begin{aligned}
 N_{F/Q}(\alpha) &= x^3 + \sigma_3 y^3 + \sigma_3^2 z^3 + \sigma_1 x^2 y + \\
 &+ (\sigma_1^2 - 2\sigma_2)x^2 z + \sigma_2 xy^2 + (\sigma_2^2 - 2\sigma_1\sigma_3)xz^2 + \\
 &+ \sigma_1\sigma_3 y^2 z + \sigma_2\sigma_3 yz^2 + (\sigma_1\sigma_2 - 3\sigma_3)xyz = \pm 1
 \end{aligned}$$

#### IV. EXAMPLES.

**1.** There is in existence accurate to equivalency an  
 unambiguous primitive

Dirichlet character  $\chi \in X(7)$  modulo  $f = 7$  ,  $U(7) = (\bar{3})$  ,  
 $X(7) = (\lambda_7)$  ,  $\lambda_7(3) = \zeta_6$  , with order of 3 and an unambiguous  
 solid cyclic field  $F = Q(\theta(\chi))$

$\theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3}[\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)]$  of degree 3,  
 with a leading divisor  $f = 7$  and the discriminant  $D = 49$  .

The number of  $A = \sum_{i=1}^2 J(\chi^i, \chi^i)$  is defined identically,

$A \equiv -C_6^3 \equiv -20 \equiv 1(3)$  ,  $A = 1$  ,  $27B^2 = 28 - A^2 = 27$  .  
 $b = 3B = \pm 3$  . If  $b = 3$  , then  $2a = A + b = 4$  ,  $a = 2$  ,  
 $J(\chi, \chi)_1 = a + b\zeta_3 = 2 + 3\zeta_3$  , If  $b = -3$  then  
 $2a = A + b = -2$  ,  $a = -1$  ,  $J(\chi^2, \chi^2)_1 = -1 - 3\zeta_3 = 2 + 3\zeta_3^2$  .  
 $\{1, \theta(\chi), \theta(\chi)^2\}$  is the integer basis of the field  $Q(\theta(\chi))$  . Note  
 that  $J(\lambda_7^2, \lambda_7^2) = -1 - 3\zeta_3$  . The minimal polynomial  $\theta(\chi)$   
 is  $p(x) = x^3 + x^2 - 2x - 1$  .  $\{1, \theta(\chi), \theta(\chi)^2\}$  is the integer basis  
 of the field  $Q(\theta(\chi))$  . Note that  $J(\lambda_7^2, \lambda_7^2) = -1 - 3\zeta_3$  ,  
 $\tau(\chi)^3 + \tau(\chi^2)^3 = 7[J(\chi, \chi) + J(\chi^2, \chi^2)] = 14 + 21\zeta_3$  ,  
 $\text{Ker}\lambda_7^2 = \{\bar{1}, \bar{-1}\}$

$$\begin{aligned}
 \theta_0 &= 3^0(\theta(\chi)) = \zeta_7 + \zeta_7^{-1} = 2\cos\frac{2\pi}{7} \\
 \theta_1 &= 3^1(\theta(\chi)) = \zeta_7^3 + \zeta_7^{-3} = 2\cos\frac{6\pi}{7} \\
 \theta_2 &= 3^2(\theta(\chi)) = \zeta_7^2 + \zeta_7^{-2} = 2\cos\frac{4\pi}{7} .
 \end{aligned}$$



integer basis of the field  $Q(\theta(\chi))$ . The reversible elements of the ring  $Z(F)$  are complied with solution of Diophantine equation

$$N_{F/Q}(\alpha) = x^3 + y^3 + z^3 - x^2y + 5x^2z - 2xy^2 + 6xz^2 - y^2z - 2yz^2 - xyz = \pm 1$$

For example, solution  $x = y = 1, z = 0$ , is corresponding to reversible element  $\alpha = 1 + \theta_0 \in U(F)$ ,  $N_{F/Q}(1 + \theta_0) = (1 + \theta_0)(1 + \theta_1)(1 + \theta_2) = -1$ , of the ring  $Z(F)$ .

**2.** There is in existence accurate to equivalency an unambiguous primitive

Dirichlet character  $\chi = \lambda_9^2$  modulo  $f = 3^2$ ,  $U(9) = (2)$ ,  $X(9) = (\lambda_9)$ ,  $\lambda_9(2) = \zeta_6$ ,  $\chi = \lambda_9^2$ , with order of 3 and an unambiguous solid cyclic field  $Q(\theta(\chi))$ ,

$$\theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_9^t = \frac{1}{3}[\tau(\lambda_9^2) + \tau(\lambda_9^4)] = \zeta_9 + \zeta_9^{-1},$$

$\ker \lambda_9^2 = \{1, 2^3 = -1\}$ , of degree 3, with a leading divisor  $f = 9$  and the discriminant  $D = 81$ .  $J(\lambda_9^2, \lambda_9^2) = \zeta_3$ ,

$$\tau(\lambda_9^2)^3 = J(\lambda_9^2, \lambda_9^2)9 = 27\zeta_3,$$

$$\tau(\lambda_9^4)^3 = J(\lambda_9^4, \lambda_9^4)9 = 27\zeta_3^2.$$

$$A = J(\lambda_9^2, \lambda_9^2) + J(\lambda_9^4, \lambda_9^4) = 3\zeta_3 + 3\zeta_3^2 = -3$$

$$\theta_0 = \theta(\chi) = \zeta_9 + \zeta_9^{-1} = 2\cos\frac{2\pi}{9}$$

$\theta_1 = 2^1(\theta(\chi)) = \zeta_9^2 + \zeta_9^{-2} = 2\cos\frac{4\pi}{9}$ ,  $1, \theta_0, \theta_1$ ; is the integer basis of the field  $Q(\theta_0)$ . The minimal polynomial  $\theta_0$

is  $p(x) = x^3 - 3x + 1$ . The reversible elements of the ring  $Z(F)$  are complied with solution of Diophantine equation  $N_{F/Q}(\alpha) = x^3 - y^3 + z^3 + 6x^2z - 3xy^2 + 9xz^2 + 3yz^2 + 9xyz = \pm 1$ .

For example, solution  $x = 1, y = -1, z = 0$ , is corresponding to reversible element  $\alpha = 1 - \theta_0 \in U(F)$ ,  $N_{F/Q}(1 - \theta_0) = (1 - \theta_0)(1 - \theta_1)(1 - \theta_2) = -1$ , of the ring  $Z(F)$ .

**3.** There is in existence accurate to equivalency an unambiguous primitive

Dirichlet character  $\chi \in X(13)$  modulo  $f = 13$ ,  $U(13) = \{1, 2, 2^2, 2^3 = -5, 2^4 = 3, 2^5 = 6, 2^6 = -1, \dots\}$ ,

$X(13) = (\lambda_{13})$ ,  $\lambda_{13}(2) = \zeta_{12}$ , with order of 3 and an unambiguous solid cyclic field

$$Q(\theta(\chi)), \theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3}[\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)], \text{ of}$$

degree 3, with a leading divisor  $f = 13$  and the discriminant

$D = 169$ . The number of  $A = \sum_{i=1}^2 J(\chi^i, \chi^i)$  is defined

identically  $A \equiv -C_8^4 \equiv -70 \equiv -5(3)$ ,  $A = -5$ ,  $27B^2 = 52 - A^2 = 27$ .  $b = 3B = \pm 3$ . If  $b = 3$ , then  $2a = A + b = -2$ ,  $a = -1$ ,  $J(\chi, \chi)_1 = a + b\zeta_3 = -1 + 3\zeta_3$ . If  $b = -3$  then  $2a = A + b = -8$ ,  $a = -4$ ,  $J(\chi^2, \chi^2)_1 = -4 - 3\zeta_3 = -1 + 3\zeta_3^2$ . The minimal polynomial  $\theta(\chi)$  is  $p(x) = x^3 + x^2 - 4x - 1$ .

$\{1, \theta(\chi), \theta(\chi)^2\}$  is the integer basis of the field  $Q(\theta(\chi))$ . Note

that  $J(\lambda_{13}^4, \lambda_{13}^4) = -4 - 3\zeta_3$ ,  $\text{Ker}\lambda_{13}^4 = \{1, -1, 5, -5\}$ ,

$$\theta(\lambda_{13}^4) = \zeta_{13} + \zeta_{13}^{-1} + \zeta_{13}^5 + \zeta_{13}^{-5}$$

$$\theta_0 = \theta(\lambda_{13}^4) = 2(\cos\frac{2\pi}{13} + \cos\frac{10\pi}{13}).$$

$$4. \quad U(19) = \{1, 2, 2^2, 2^3, 2^4 = -3, 2^5 = -6, 2^6 = 7, 2^7 = -5, 2^8 = 9, 2^9 = -1, \dots\}$$

There is in existence accurate to equivalency an unambiguous primitive Dirichlet character  $\chi \in X(19)$  modulo  $f = 19$ ,  $X(19) = (\lambda_{19})$ ,  $\lambda_{19}(2) = \zeta_{18}$  with order of 3 and an unambiguous solid cyclic

$$Q(\theta(\chi)) \quad \theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3}[\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)] \text{ of}$$

degree 3, with a leading divisor  $f = 19$  and the discriminant

$D = 361$ . The number of  $A = \sum_{i=1}^2 J(\chi^i, \chi^i)$  is defined

identically  $A \equiv -C_{12}^6 \equiv 7(19)$ ,  $A = 7$ ,  $27B^2 = 76 - A^2 = 27$ .  $b = 3B = \pm 3$ . If  $b = 3$ , then  $2a = A + b = 10$ ,  $a = 5$ ,  $J(\chi, \chi)_1 = a + b\zeta_3 = 5 + 3\zeta_3$ . If  $b = -3$  then  $2a = A + b = 4$ ,  $a = 2$ ,  $J(\chi^2, \chi^2)_1 = 2 - 3\zeta_3 = 5 + 3\zeta_3^2$ .

The minimal polynomial  $\theta(\chi)$  is  $p(x) = x^3 + x^2 - 6x - 7$ .  $\{1, \theta(\chi), \theta(\chi)^2\}$  is the integer basis of the field  $Q(\theta(\chi))$ . Note

that,  $J(\lambda_{19}^6, \lambda_{19}^6) = 5 - 3\zeta_3^2$ ,  $\text{Ker}\lambda_{19}^6 = \{1, -1, 7, -7, 8, -8\}$ ,

$$\theta_0 = \theta(\lambda_{19}^6) = 2(\cos\frac{2\pi}{19} + \cos\frac{14\pi}{19} + \cos\frac{16\pi}{19}).$$

**5.** There is in existence accurate to equivalency an unambiguous primitive Dirichlet character  $\chi \in X(31)$

modulo  $f = 31$ ,  $X(31) = (\lambda_{31})$ ,  $\lambda_{31}(3) = \zeta_{30}$ ,

$$U(31) = \{1, 3, 3^2, 3^3 = -4, 3^4 = -12, 3^5 = -5, 3^6 = 15, 3^7 = -14, 3^8 = -11, 3^9 = -2, 3^{10} = -6,$$

$$3^{11} = 13, 3^{12} = 8, 3^{13} = -7, 3^{14} = 10, 3^{15} = -1, \dots\}$$

order of 3 and an unambiguous solid cyclic field of

$$Q(\theta(\chi)), \theta(\chi) = \sum_{t \in \text{Ker}\chi} \zeta_f^t = \frac{1}{3}[\tau(\chi^0) + \tau(\chi) + \tau(\chi^2)] \text{ of}$$

degree 3, with a leading divisor  $f = 31$  and the discriminant

$D=961$  The number of  $A = \sum_{i=1}^2 J(\chi^i, \chi^i)$  is defined

identically  $A \equiv -C_{20}^{10} \equiv 4(31)$ ,  $A = 4$ ,  $27B^2 = 124 - A^2 = 108$ .  
 $b = 3B = \pm 6$  If  $b = 6$ , then  $2a = A + b = 10$ ,  $a = 5$ ,  
 $J(\chi, \chi)_1 = a + b\zeta_3 = 5 + 6\zeta_3$ , If  $b = -6$ , then  
 $2a = A + b = -2$ ,  $a = -1$ ,  
 $J(\chi^2, \chi^2)_1 = -1 - 6\zeta_3 = 5 + 6\zeta_3^2$ . The minimal

polynomial  $\theta(\chi)$  is  $p(x) = x^3 + x^2 - 10x - 8$ .

$\{1, \theta(\chi), \theta(\chi)^2\}$  is the integer basis of the field  $\mathbb{Q}(\theta(\chi))$ . Note

that  $\text{Ker}\lambda_{19}^6 = \{\bar{1}, -\bar{1}, \bar{2}, -\bar{2}, \bar{4}, -\bar{4}, \bar{8}, -\bar{8}, \bar{15}, -\bar{15}\}$ ,

$$\theta(\lambda_{31}^{10}) = \zeta_{31} + \zeta_{31}^{-1} + \zeta_{31}^2 + \zeta_{31}^{-2} + \zeta_{31}^4 + \zeta_{31}^{-4} + \zeta_{31}^8 + \zeta_{31}^{-8} + \zeta_{31}^{15} + \zeta_{31}^{-15}.$$

## V. CONCLUSION

There are in existence accurate to equivalency 4 primitive characters

$\lambda_9^2 \lambda_7^2 \lambda_{13}^4, \lambda_9^2 \lambda_7^2 \lambda_{13}^8, \lambda_9^2 \lambda_7^4 \lambda_{13}^4, \lambda_9^2 \lambda_7^4 \lambda_{13}^8 \in X(819)$  modulo

$f = 3^2 \cdot 7 \cdot 13 = 819$ ,  $X(63) = (\lambda_9)(\lambda_7)(\lambda_{13})$  with order of 3

$J(\lambda_9^2, \lambda_9^2) = \zeta_3$ ,  $J(\lambda_7^2, \lambda_7^2) = -1 - 3\zeta_3$ ,

$J(\lambda_{13}^4, \lambda_{13}^4) = -4 - 3\zeta_3$ ,

$J(\lambda_9^2, \lambda_9^2)J(\lambda_7^2, \lambda_7^2)J(\lambda_{13}^4, \lambda_{13}^4) = -18 - 33\zeta_3$

The minimal polynomial  $\theta(\lambda_9^2 \lambda_7^2 \lambda_{13}^4)$  is

$$p(x) = x^3 - \mu(f)x^2 + \frac{1}{3}[\mu(f)^2 - f]x - \frac{1}{27}[\tau(\chi)^3 + \tau(\chi^2)^3 + (1-3f)\mu(f)]$$

$$p(x) = x^3 - 273x + 91.$$

$J(\lambda_9^2, \lambda_9^2)J(\lambda_7^2, \lambda_7^2)J(\lambda_{13}^8, \lambda_{13}^8) = -27 + 3\zeta_3$ . The

minimal polynomial  $\theta(\lambda_9^2 \lambda_7^2 \lambda_{13}^8)$  is

$$p(x) = x^3 - 273x + 1729.$$

$J(\lambda_9^2, \lambda_9^2)J(\lambda_7^4, \lambda_7^4)J(\lambda_{13}^4, \lambda_{13}^4) = 27 + 30\zeta_3$ . The

minimal polynomial  $\theta(\lambda_9^2 \lambda_7^4 \lambda_{13}^4)$  is

$$p(x) = x^3 - 273x - 728.$$

$J(\lambda_9^2, \lambda_9^2)J(\lambda_7^4, \lambda_7^4)J(\lambda_{13}^8, \lambda_{13}^8) = -33 - 18\zeta_3$ . The

minimal polynomial  $\theta(\lambda_9^2 \lambda_7^4 \lambda_{13}^8)$  is

$$p(x) = x^3 - 273x + 1456.$$

Thus, the article describes Abelian extensions. Consideration of given function would help in solving the inverse Galois problem, because of their close relationship. Solving this problem, Dirichlet characters and Gauss sum for a Dirichlet character were investigated.

This work can serve a theoretical basis for further researches on absolute Abelian fields.

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