Optimal Error Bound and A Semi-discrete Difference Scheme for the Cauchy Problem of the Modified Helmholtz Equation

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Abstract. The Cauchy problem of Helmholtz equation is severely ill-posed problem. In this paper, we consider the Cauchy problem for the Helmholtz equation where the Cauchy data is given at $x = 0$ and the solution is sought in the interval $0 < x < 1$. A semi-discrete difference schemes together with a choice of regularization parameter is presented and error estimate is obtained. The numerical example shows the effectiveness of this method.

Introduction

The modified Helmholtz equation appears in many areas of engineering and science[1], such as in implicit marching schemes for the heat equation, in Debye-Huckel theory, in the linearization of the Poisson-Boltzmann equation and so on [5, 8, 16]. The direct problems for the modified Helmholtz equation have been studied extensively in the past century [19, 7]. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points, giving rise to an inverse problem [6]. The Cauchy problem for the modified Helmholtz equation is an inverse problem and is typically ill-posed. It means that a small change in the Cauchy data may result a dramatic change in the solution. Therefore, it is necessary to study different highly efficient algorithms for solving it. In recent years, there are many special numerical methods to deal with this problem, such as the boundary element method[9], the method of fundamental solutions[10, 17], the conjugate gradient method[11], the Landweber method[9], quasi-reversibility and truncation method[14], quasi-boundary and Tikhonov type regularization method[13, 18], the Fourier regularization method [3, 4] and so on[15]. In the present paper we will consider the following problem with inhomogeneous Dirichlet data in a strip domain:

$$\begin{aligned} u_{xx} + u_{yy} - k^2 u &= 0, 0 < x < 1, -\infty < y < \infty \\ u(0, y) &= \phi(y), -\infty < y < \infty \\ u_x(0, y) &= 0, -\infty < y < \infty \end{aligned} \quad (1.1)$$

Where $k > 0$ is a real number. We shall use a semi-discrete difference scheme to construct stable solution of the problem (1.1) and then obtain error estimate.

Regularization for a Cauchy Problem of the Modified Helmholtz Equation

Let $\hat{g}(\xi)$ denote the Fourier transform of function $g(x)$, which is defined by

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x) dx \quad (2.1)$$

The function $\phi(y), \phi^\delta(y) \in L^2(R)$ are the exact and measured data for problem (1.1) respectively, which satisfies

$$\|\phi^\delta(\cdot) - \phi(\cdot)\| \leq \delta \quad (2.2)$$

Where $\|\cdot\|$ denotes the L^2 -norm and constant $\delta > 0$ is the noise level. Assume $u(x, \cdot) \in L^2(R)$ for all $0 \leq x < 1$ and there holds the following a priori bound,

$$\|u(1, \cdot)\| \leq E, \quad (2.3)$$

Where E is a positive constant. It is easy to know that for problem (1.1), there holds

$$\hat{u}(x, \xi) = \cosh(\sqrt{\xi^2 + k^2} x) \hat{g}(\xi), \quad (2.4)$$

and equivalently,

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} \cosh(x\sqrt{|\xi|^2 + k^2}) \hat{\phi}(\xi) d\xi \quad (2.5)$$

Note that the fact $\cosh(x\sqrt{|\xi|^2 + k^2})$ increases exponentially for $0 < x < 1$ as $|\xi| \rightarrow \infty$, the small disturbance for the data $\phi(x)$ will be amplified infinitely by this factor and lead to the integral (2.5) blow-up. Therefore, recovering the value $u(x, y)$ from the measured data $\phi^\delta(x)$ is severely ill-posed. To deal with this ill-posed problem, we introduce an regularization method. By using the central difference with step length h to approximate the second derivative u_{yy} , we can get the following problem

$$\begin{aligned} v_{xx} + \frac{v(x, y+h) - 2v(x, y) + v(x, y-h)}{h^2} - k^2 u &= 0, 0 < x < 1, -\infty < y < \infty \\ v(0, y) &= g_\delta(y), -\infty < y < \infty \\ v_x(0, y) &= 0, -\infty < y < \infty. \end{aligned} \quad (2.6)$$

Taking Fourier transformation in the frequency space for problem (1.1), we can get

$$\hat{v}(x, \xi) = \cosh\left[\frac{x}{h} \sqrt{(k^2 h^2 + 2) - 2 \cos(h\xi)}\right] \hat{g}_\delta(\xi) \quad (2.7)$$

Error Estimates

Lemma 2.1 Let $p(\xi) = \frac{\cosh(\sqrt{\xi^2 + k^2} x)}{\cosh \sqrt{\xi^2 + k^2}}$ ($\xi \geq 0$), then

- (1) $p(\xi)$ is decrease as $\xi \geq 0$;
- (2) $e^{-(1-x)\sqrt{\xi^2 + k^2}} \leq p(\xi) \leq 2e^{-(1-x)\sqrt{\xi^2 + k^2}}$;

$$(3) \quad e^{\frac{x}{h}\sqrt{k^2h^2+4\sin^2\frac{h\xi}{2}}-\sqrt{\xi^2+k^2}x} \leq \frac{e^{\frac{x}{h}\sqrt{k^2h^2+4\sin^2\frac{h\xi}{2}}} + e^{-\frac{x}{h}\sqrt{k^2h^2+4\sin^2\frac{h\xi}{2}}}{e^{\sqrt{\xi^2+k^2}x} + e^{-\sqrt{\xi^2+k^2}x}} \leq e^{\frac{x}{h}\sqrt{k^2h^2+4\sin^2\frac{h\xi}{2}}-\sqrt{\xi^2+k^2}x}$$

Theorem 2.2 Suppose $v(x, y)$ is regularization solution for problem (1.1), whose Fourier Transformation is given by (2.7), $u(x, y)$ is exact solution for problem (1.1), whose Fourier transformation

$$h = \frac{2}{\operatorname{arccosh} h(\frac{E}{\delta})}$$

is given by (2.4), if , then we have the following error estimate

$$(1) \text{ If } \frac{E}{\delta} < \cosh(3), \text{ then } \|u(x, \cdot) - v(x, \cdot)\| \leq 2E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)). \quad (3.1)$$

$$(2) \text{ If } \frac{E}{\delta} > \cosh(3), \text{ then}$$

$$(2.1) \text{ for } 0 < x < 1 - \frac{3k}{2},$$

$$\|u(x, \cdot) - v(x, \cdot)\| \leq E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) + \max \left\{ \frac{9}{e^3(1-x)^3} \frac{E}{[\operatorname{arccosh} h(\frac{E}{\delta})]^2}, E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) \right\} \quad (3.2)$$

$$(2.2) \text{ for } 1 - \frac{3k}{2} \leq x < 1,$$

$$\|u(x, \cdot) - v(x, \cdot)\| \leq E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) + \max \left\{ \frac{x \operatorname{arccosh} h(\frac{E}{\delta})}{3} e^{-(1-x) \operatorname{arccosh} h(\frac{E}{\delta})}, E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) \right\}.$$

Proof. According Parseval formula and (2.2), (2.3), we can get

$$\begin{aligned} \|u(x, \cdot) - v(x, \cdot)\| &= \|\hat{u}(x, \cdot) - \hat{v}(x, \cdot)\| = \left\| \cosh(\sqrt{\xi^2 + k^2}x) \hat{g}(\xi) - \hat{g}_\delta(\xi) \cosh\left[\frac{x}{h} \sqrt{(k^2h^2 + 2) - 2\cos(h\xi)}\right] \right\| \\ &\leq \left\| \frac{\cosh(\sqrt{\xi^2 + k^2}x) - \cosh\left[\frac{x}{h} \sqrt{(k^2h^2 + 2) - 2\cos(h\xi)}\right]}{\cosh\sqrt{\xi^2 + k^2}} \hat{f}(\xi) \right\| + \left\| \cosh\left[\frac{x}{h} \sqrt{(k^2h^2 + 2) - 2\cos(h\xi)}\right] (\hat{g}(\xi) - \hat{g}_\delta(\xi)) \right\| \\ &\leq \sup_{\xi \in R} A(\xi) E + \sup_{\xi \in R} B(\xi) \delta. \end{aligned}$$

where

$$A(\xi) = \frac{\cosh(\sqrt{\xi^2 + k^2}x) - \cosh\left[\frac{x}{h} \sqrt{(k^2h^2 + 2) - 2\cos(h\xi)}\right]}{\cosh\sqrt{\xi^2 + k^2}} \quad (3.3)$$

$$B(\xi) = \cosh\left[\frac{x}{h} \sqrt{(k^2h^2 + 2) - 2\cos(h\xi)}\right] \quad (3.4)$$

Firstly we can estimate $B(\xi)$ as

$$B(\xi) \leq \cosh\left(\frac{x}{h}\sqrt{k^2 h^2 + 4}\right) \leq \cosh\left(\frac{2x}{h}\right) \quad (3.5)$$

It is easy to know that the asymptotic expression of $B(\xi)\delta$ is

$$B(\xi)\delta \leq \cosh\left(\frac{2x}{h}\right)\delta = E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)), \delta \rightarrow 0. \quad (3.6)$$

Because $A(\xi)$ is a even function, then

Case 1. If $\sqrt{\xi^2 + k^2} \geq \xi_0 = \frac{2}{k} = \operatorname{arccosh}\left(\frac{E}{\delta}\right)$, from $\sqrt{\xi^2 + k^2} x \leq \frac{x}{h}\sqrt{k^2 h^2 + 4} \sin^2 \frac{h\xi}{2}$, we have

$$\frac{e^{\frac{x}{h}\sqrt{k^2 h^2 + 4} \sin^2 \frac{h\xi}{2}} + e^{-\frac{x}{h}\sqrt{k^2 h^2 + 4} \sin^2 \frac{h\xi}{2}}}{e^{\sqrt{\xi^2 + k^2} x} + e^{-\sqrt{\xi^2 + k^2} x}} \leq 1 \quad (3.7)$$

from Lemma 2.1, we have

$$\begin{aligned} A(\xi)E &\leq \frac{\cosh(\sqrt{\xi^2 + k^2} x)}{\cosh \sqrt{\xi^2 + k^2}} E \leq \frac{\cosh\left(\frac{2x}{h}\right)}{\cosh\left(\frac{2}{h}\right)} E \\ &= \delta \cosh\left(x \operatorname{arccosh}\left(\frac{E}{\delta}\right)\right) = E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)), \delta \rightarrow 0. \end{aligned} \quad (3.8)$$

Case 2. if $\sqrt{\xi^2 + k^2} < \xi_0 = \frac{2}{h} = \operatorname{arccosh}\left(\frac{E}{\delta}\right)$, from Lemma 2.1, we have

$$A(\xi)E \leq 2e^{-(1-x)\sqrt{\xi^2 + k^2}} \left(1 - e^{\frac{x}{h}\sqrt{k^2 h^2 + 4} \sin^2 \frac{h\xi}{2} - \sqrt{\xi^2 + k^2} x}\right) E. \quad (3.9)$$

By using the inequality $1 - y \leq e^{-y}$ ($y > 0$), we get

$$A(\xi)E \leq E \frac{2x}{h} e^{-(1-x)\sqrt{\xi^2 + k^2}} \left[h\sqrt{\xi^2 + k^2} - 2 \left| \sin \frac{h\sqrt{\xi^2 + k^2}}{2} \right| \right] \quad (3.11)$$

Let $r = \frac{h\sqrt{\xi^2 + k^2}}{2}$, for $h\sqrt{\xi^2 + k^2} < 2$, we have

$$A(\xi)E \leq E \frac{4x}{h} e^{\frac{2(x-1)r}{h}} (r - |\sin r|) \quad (3.12)$$

From $|\sin r| \geq r - \frac{r^3}{6}$ ($r > 0$), we have

$$A(\xi)E \leq E \frac{4x}{h} e^{\frac{2(x-1)r}{h}} \frac{r^3}{6} = \frac{2x}{3h} m(r, x), \quad (3.13)$$

Where $m(r, x) = e^{\frac{2(x-1)r}{h}} r^3$, from $\frac{\partial m}{\partial r} = e^{\frac{2r(x-1)}{h}} r^2 (3 - \frac{2r(1-x)}{h})$, we know that $r_0 = \frac{3h}{2(1-x)}$ is the extreme point of function $m(r, x)$. If $r_0 \leq 1$, then

$$m(r, x) \leq m(r_0, x) = e^{\frac{-2r_0(1-x)}{h}} r_0^3 = e^{-3} \left(\frac{3h}{2(1-x)} \right)^3, \quad (3.14)$$

$$A(\xi)E \leq \frac{2x}{3h} e^{\frac{-2r_0(1-x)}{h}} r_0^3 E = \frac{9}{e^3(1-x)^3} \frac{E}{[\operatorname{arccosh}(\frac{E}{\delta})]^2} \rightarrow 0$$

So

(3.15)

If $r_0 > 1$, then the extreme point of function $m(r, x)$ is $r = 1$, so

$$A(\xi)E \leq \frac{2x}{3h} m(1, x)E = \frac{x \operatorname{arccosh}(\frac{E}{\delta})}{3} e^{-(1-x) \operatorname{arccosh}(\frac{E}{\delta})} E \rightarrow 0. \quad (3.16)$$

We can conclude that the following hold,

(1) if $\frac{E}{\delta} < \cosh(3)$, then for $x \in (0, 1)$, $r_0 = \frac{3h}{2(1-x)} > 1$, combining (3.8) and (3.16), we have

$$A(\xi)E \leq \max \left\{ \frac{x \operatorname{arccosh}(\frac{E}{\delta})}{3} e^{-(1-x) \operatorname{arccosh}(\frac{E}{\delta})} E, E^x \left(\frac{\delta}{2} \right)^{1-x} (1 + o(1)) \right\}. \quad (3.17)$$

Because $\operatorname{arccosh}(x) = \ln(x + \sqrt{x^2 - 1})$, we have the following asymptotic expression

$$e^{-(1-x) \operatorname{arccosh}(\frac{E}{\delta})} E = e^{-(1-x)(\ln(\frac{E}{\delta}) + \sqrt{(\frac{E}{\delta})^2 - 1})} = E^x \left(\frac{\delta}{2} \right)^{1-x} (1 + o(1)), \delta \rightarrow 0. \quad (3.18)$$

$$\text{So } \|u(x, \cdot) - v(x, \cdot)\| \leq 2E^x \left(\frac{\delta}{2} \right)^{1-x} (1 + o(1)). \quad (3.19)$$

(2) if $\frac{E}{\delta} > \cosh(3)$ then

(2.1) if $0 < x < 1 - \frac{3h}{2}$, then $r_0 < 1$, so for $h\sqrt{k^2 + \xi^2} < 2$, combining (3.15) and (3.8), we have

$$A(\xi)E \leq \max \left\{ \frac{9}{e^3(1-x)^3} \frac{x E}{[\operatorname{arccosh}(\frac{E}{\delta})]^2}, E^x \left(\frac{\delta}{2} \right)^{1-x} (1 + o(1)) \right\}. \quad (3.20)$$

Similarly, we have

$$\|u(x, \cdot) - v(x, \cdot)\| \leq E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) + \max \left\{ \frac{9}{e^3 (1-x)^3} \frac{E}{[\operatorname{arccosh}(\frac{E}{\delta})]^2}, E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) \right\}. \quad (3.21)$$

(2.2) if $1 - \frac{3h}{2} \leq x < 1$, then $r_0 \geq 1$, so for $h\sqrt{k^2 + \xi^2} < 2$, combining (3.15) and (3.8), we have the following error estimate

$$\|u(x, \cdot) - v(x, \cdot)\| \leq E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) + \max \left\{ \frac{x \operatorname{arccosh}(\frac{E}{\delta})}{3} e^{-(1-x) \operatorname{arccosh}(\frac{E}{\delta})}, E^x \left(\frac{\delta}{2}\right)^{1-x} (1 + o(1)) \right\}$$

Remark. It is easy to see that the space step length h is the regularization parameter of this problem. In the conclusion, we give a rule for choosing the parameter, which is very important for the study of ill-posed problems.

Numerical Test

In this section, a simple numerical example is devised to verify the validity of the proposed method.

We consider the following Cauchy problem:

$$\begin{aligned} \Delta u(x, y) - k^2 u(x, y) &= 0, 0 < x < \pi, 0 < y < 1, \\ u(x, 0) &= \phi(x) = \sum_{n=1}^{\infty} c_n \sin nx, 0 \leq x \leq \pi, \\ u_y(x, 0) &= h(x) = 0, 0 \leq x \leq \pi, \\ u(0, y) &= u(\pi, y) = 0, 0 \leq y \leq 1. \end{aligned} \quad (4.1)$$

Take $u(x, y) = \sin x \cosh(\sqrt{1+k^2} y) - 2 \sin(2kx) \cosh(\sqrt{5}ky)$ as the exact solution for the Cauchy problem (4.1). The Cauchy data $\phi(x) = \sin x - \sin 2kx$. Suppose the vector Φ represents samples from the function ϕ . The noisy discrete data function is generated by adding random errors to the exact data function, at every grid point, i.e.,

$$\Phi_{\delta}(n) = \Phi(n) + \varepsilon_n, |\varepsilon_n| \leq \delta, n = 0, 1, \dots, N_t,$$

Where the ε_n 's are independent random Gaussian variables with variance $\sigma^2 = \varepsilon^2$. In the following, the capital letter is used to denote the samples of the corresponding function. We implement the algorithm in Matlab 7.10. The relative weighted l^2 errors for the recovered function are calculated as

$$\frac{\left[\frac{1}{N+1} \sum_{n=0}^N |U_{\alpha}^{\delta}(n, y) - U(n, y)|^2 \right]^{\frac{1}{2}}}{\left[\frac{1}{N+1} \sum_{n=0}^N |U(n, y)|^2 \right]^{\frac{1}{2}}}$$

Where N is the number of grid in the interval $[0, \pi]$. Figure 1-3 demonstrate the computation effects at $y = 1, 0.5, 0.1$, respectively. From the results in figure 1-3, we see that it is better when y is closer to $y = 0$. For large noise level $\delta = 0.1$, the results are given as in Fig.4, which also displays good approximation.

Figure1:(a): $y = 1, \delta = 0.01, \alpha = 0.13, k = 1$. (b): $y = 1, \delta = 0.001, \alpha = 0.13, k = 1$.

Figure2:(a): $y = 0.5, \delta = 0.01, \alpha = 0.04, k = 1$. (b): $y = 0.5, \delta = 0.001, \alpha = 0.04, k = 1$.

Figure3:(a): $y = 0.1, \delta = 0.01, \alpha = 0.002, k = 1$. (b): $y = 0.1, \delta = 0.001, \alpha = 0.002, k = 1$.

Figure4:(a): $y = 0.5, \delta = 0.10, \alpha = 0.10, k = 1$. (b): $y = 1, \delta = 0.1, \alpha = 0.10, k = 1$.

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