Lax Pair, Hirota Bilinear Form and Soliton Solutions of the Seventh-Order Kaup-Kupershmidt Equation in Plasma Physics

Yuping Zhang [a], Wenguang Zhang [b]

School of Computer Science and Engineering, Beihang University, Beijing 100191, China

[a]zhyuping@buaa.edu.cn, [b]zhangwg1990@126.com

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Abstract. In this paper, the seventh-order Kaup–Kupershmidt (KK) equation arising in fluids and plasmas is considered. The pseudopotentials of the equation are derived, based on the pseudopotentials, the Lax pair of the equation is get, which shows that the equation has the Lax integrability. It is a very important property in researching nonlinear evolution equations (NLEEs). By virtue of the symbolic computation, the equation is transformed into its bilinear form and soliton solutions of the equation are obtained.

1. Introduction

Integrability plays an important role in studying of NLEEs [1]. Lax pair, which is a kind of integrability of NLEEs, can be regarded as a predictor of the equation's complete integrability.

In nonlinear theory, the Korteweg–de Vries (KdV) type equations are very important since they can model many physical phenomena such as wave propagation in the ocean, stratified internal waves, ion-acoustic waves and so on [2]. The KK equations, as a kind of KdV type equations, are of great value to research. In this paper, we will investigate the Seven-order KK equation,

$$u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{2x} + 504u^2u_{3x} + 252u_{2x}u_{3x} + 147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0, \tag{1}$$

Where u is a real function of x and t. With another group of parameters, there is the Sawada–Kotera–Ito equation,

$$u_t + 252u^3u_x + 63u_x^3 + 378uu_xu_{2x} + 126u^2u_{3x} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0. \tag{2}$$

N-soliton solutions, bilinear form and Lax pair for equation (2) have been given [3], and the Bell-polynomial approach is extended to the equation.

Different from the research method to equation (2), the poseudopotential procedure will be, hereby introduced to deal with equation (1). In section 2, the Lax pair of equation (1) will be constructed from poseudopotentials. Then in section 3 base on the Lax pair, the bilinear form of equation (1) will be derived and the soliton solutions will be given.

2. Poseudopotential and Lax pair

If we can find the Lax pair of an NLEE, then we can say that the equation is complete integrable. It is very useful for further research of the equation.

In this section, we will construct the Lax pair of equation (1) by means of the method developed by Nucci [4]. Nucci considered the Riccati-type pseudopotentials of NLEEs, and extended them to derive Lax pairs, auto-Bäcklund transformations, and singularity manifold equations. Afterwards this method was extended to investigate many NLEEs.

For equation (1), we assume there exists a pseudopotential with the form,

$$q_x = kq^2 + F_1(u)q + F_0(u), \tag{3}$$

$$q_t = G(q, u, u_x, u_{2x}, u_{3x}, u_{4x}, u_{5x}, u_{6x}, u_{7x}, u_{8x}), \tag{4}$$

Where G is a second-order function in q, and k is a constant. If we can get the pseudopotential of equation (1), we can construct the Lax pairs and the auto-Bäcklund transformation.

Based on the compatible condition $q_{xt} = q_{tx}$, we obtain the poseudopotential of equation (1)

$$q_x = \frac{\gamma^2}{4k} + \gamma q + kq^2 + \frac{3}{2}u, \tag{5}$$

$$q_t = -\frac{3}{2k}[(96u^3 + 18u_x^2 + 36uu_{2x} + u_{4x})(\gamma + 2kq) + 288u^2u_x + 72u_xu_{2x} + 36uu_{3x} + u_{5x}]_x, \tag{6}$$

Where γ is a spectral parameter. Then applying the transformation $q = -(\ln\psi)_x$, the pseudopotential system (5) and (6) can be linearized to the following system,

$$\psi_{xx} = -\frac{1}{4}(\gamma^2 + 6u)\psi + \gamma\psi_x, \tag{7}$$

$$\begin{aligned}\psi_t = {} & \frac{3\psi}{2k}[96\gamma u^3 + 288u^2u_x + 18\gamma u_x^2 + 72u_xu_{2x} + 36u(\gamma u_{2x} + u_{3x}) + \gamma u_{4x} + u_{5x}] \\ & -\frac{3\psi_x}{k}(96u^3 + 18u_x^2 + 36uu_{2x} + u_{4x}).\end{aligned} \tag{8}$$

With the transformation $q = \frac{\phi_1}{\phi_2}$ and $\gamma = 2\lambda$, we can get the Lax pair of equation (1) in the Ablowitz-Kaup-Newell-Segur (AKNS) form,

$$\Phi_x = U\Phi = \begin{pmatrix} \lambda & \lambda^2 + \frac{3u}{2k} \\ -k & -\lambda \end{pmatrix}\Phi, \tag{9}$$

$$\Phi_t = V\Phi = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & -A(x,t,\lambda) \end{pmatrix}\Phi, \tag{10}$$

where

$$A(x,t,\lambda) = -[(576u^3 + 108u_x^2 - 216uu_{2x} - 6u_{4x})\lambda + 864u^2u_x + 216u_xu_{2x} + 108uu_{3x} - 3u_{5x}]/2, \tag{11}$$

$$\begin{aligned}B(x,t,\lambda) = {} & -[(288u^3 + 54u_x^2 + 108uu_{2x} - 3u_{4x})\lambda^2 + (864u^2u_x + 216u_xu_{2x} + 108uu_{3x} + 3u_{5x})\lambda \\ & + 432u^4 + 945uu_x^2 + 594u^2u_{2x} + 108u_{2x}^2 + 162u_xu_{3x} + 117uu_{4x}/2 + 3u_{6x}]/k,\end{aligned} \tag{12}$$

$$C(x,t,\lambda) = 288ku^3 + 54ku_x^2 + 108kuu_{2x} + 3ku_{4x}. \tag{13}$$

It can be verified that the compatibility condition $q_{xt} = q_{tx}$ with equation (11) to equation (13) leads to equation (1), which ensured the correction of the Lax pair we obtained.

3. Bilinear form and soliton solutions

In this section, we first derive the bilinear representation of equation (1), and then construct it's multi-soliton solutions using Hirota's bilinear method [5].

Integrate equation (1) with respect to *x*, we have

$$\int u_t \mathrm{d}x + 504u^4 + 630uu_x^2 + 504u^2u_{2x} + \frac{147}{2}u_{2x}^2 + 105u_xu_{3x} + 42uu_{4x} + u_{6x} = 0. \tag{14}$$

Based on Painlevé expansion, the transformation can be written as

$$u = \frac{1}{2}(\ln f)_{2x} + u_0, \tag{15}$$

Where u_0 is a solution of equation (1) and *f* is a function of *x* and *t*. Here we take the trivial solution $u_0 = 0$. Substituting transformation (15) into equation (1), we have

$$
\begin{aligned}
&\frac{f_{xt}f-f_xf_t}{2f^2}+\frac{1}{8f^2}(4f_{8x}f-32f_xf_{7x}-28f_{2x}f_{5x}-14f_{3x}f_{5x}+7f_{4x}^2)\\
&\quad+\frac{7}{4f^3}(3f_{2x}^2f_{4x}-5f_{2c}f_{3x}^2+15f_xf_{2x}f_{5x}+f_xf_{3x}f_{4x}+10f_x^2f_{6x})\\
&\quad+\frac{105}{4f^6}(3f_x^4f_{2x}^2-4f_x^5f_{3x})++\frac{105}{4f^5}(4f_x^4f_{4x}-3f_x^2f_{2x}^3+2f_x^3f_{2x}f_{3x})\\
&\quad+\frac{21}{8f^4}(3f_{2x}^4+12f_xf_{2x}^3f_{3x}+2f_x^2f_{3x}^2-30f_x^2f_{2x}f_{4x}-20f_x^3f_{5x})=0.
\end{aligned}
\tag{16}
$$

Introducing an auxiliary function g defined as

$$\frac{g}{f}=\frac{D_x^4f\cdot f}{64f^2}, \tag{17}$$

We can obtain the bilinear form of equation (1) as bellow

$$[D_xD_t-\frac{1}{64}D_x^8]f\cdot f-65D_x^4f\cdot g=0, \tag{18}$$

$$D_x^4f\cdot f+64f\cdot g=0, \tag{19}$$

Where the bilinear operator D is defined by

$$D_x^mD_t^nf(x,t)\cdot g(x,t)=\frac{\partial^m}{\partial r^m}\frac{\partial^n}{\partial s^n}\times f(x+r,t+s)g(x-r,t-s)|_{r=0,s=0},\quad m,n=1,2,\cdots \tag{20}$$

To find the one-soliton solution, we make the ansatz

$$
\begin{aligned}
&f=1+\epsilon e^{\eta_1}+\epsilon^2f_2,\\
&\eta_1=k_1x+\omega_1t+\xi_1^0,\\
&g=\epsilon g_1.
\end{aligned}
\tag{21}
$$

Substituting (21) into (18) and (19), we have

$$
\begin{aligned}
&\omega_1=-k_1^7,\\
&g_1=-\frac{k_1^4e^{\eta_1}}{32},\\
&f_2=\frac{e^{2\eta_1}}{16},
\end{aligned}
\tag{22}
$$

And with some symbolic computation, we get the one-soliton solution

$$u=\frac{8k_1^2e^{\eta_1}(16+4e^{\eta_1}+e^{2\eta_1})}{(16+16e^{\eta_1}+e^{2\eta_1})^2}, \tag{23}$$

Where $\eta_1=k_1x-k_1^7t+\xi_1^0$, k_1 and ξ_1^0 are arbitrary real constants.

To get the two-soliton solutions, we assume

$$
\begin{aligned}
&f=1+\epsilon e^{\eta_1+\eta_2}+\epsilon^2f_2+\epsilon^3f_3+\epsilon^4f_4,\\
&g=\epsilon g_1+\epsilon^2g_2+\epsilon^3g_3,
\end{aligned}
\tag{24}
$$

Where $\eta_i=k_ix+\omega_it+\xi_i^0\ (i=1,2)$. Substituting equation (24) into the bilinear form equation (18) and equation (19), we obtain

$$
\begin{aligned}
&\omega_1 = -k_1^7,\\
&\omega_2 = -k_2^7,\\
&f_2 = \frac{1}{16}e^{2\eta_1} + \frac{1}{16}e^{2\eta_2} + \mu_{12}e^{\eta_1+\eta_2},\\
&f_3 = \sigma_{12}(e^{2\eta_1+\eta_2} + e^{\eta_1+2\eta_2}),\\
&f_4 = \sigma_{12}e^{2\eta_1+2\eta_2},
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
&\mu_{12} = \frac{2k_1^4 + 2k_2^4 - k_1^2k_2^2}{2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)},\\
&\sigma_{12} = \frac{(k_1-k_2)^2(k_1^2+k_2^2-k_1k_2)}{16(k_1+k_2)^2(k_1^2+k_2^2+k_1k_2)}.
\end{aligned} \tag{26}
$$

In the same way, we get the two-soliton solution as

$$
u = \frac{1}{2}(\ln f)_{2x}, \tag{27}
$$

where f and η_j is given by

$$
\begin{aligned}
f = {} & 1 + e^{\eta_1} + e^{\eta_2} + \frac{e^{2\eta_1}}{16}\\
& + \frac{e^{2\eta_2}}{16} + \mu_{12}e^{\eta_1+\eta_2}\\
& + \sigma_{12}(e^{2\eta_1+\eta_2} + e^{\eta_1+2\eta_2}) + \sigma_{12}^2 e^{2\eta_1+2\eta_2},\\
\eta_j = {} & k_j x - k_j^7 t + \xi_j^0,\ j = 1,\ 2.
\end{aligned} \tag{28}
$$

4. Conclusion

Seventh-order Kaup–Kupershmidt (KK) equation arising in fluids and plasmas was considered in this paper. With symbol computation, Lax pair in AKNS form are given by poseudopotential. The bilinear form of the equation is found by Lax pair. Then based on the bilinear, the soliton solutions are obtained.

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