On Some New Results of Poverty Orderings and Their Applications

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Abstract

The paper proposes to derive some new poverty indices which depend on aging classes. We also give some properties of it and show the connection between economic measure and new poverty measures these based on the concept of reversed residual incomes. In addition, the characterization of Pareto distribution based on new poverty functions is obtained. Furthermore, the stochastic orderings of new poverty indices are studied and their properties. In addition, the weighted poverty gap indices and stochastic dominance which involve the concept of inactivity incomes and its features are studied.

Keywords: Poverty gap; the severity of poverty; poverty ordering; weighed functions; lorenz curve; the reversed proportional failure rate.

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1. Introduction

Let $X \geq 0$ be the random variable expresses the income of individuals with a density function $f_X(\theta)$, a distribution function ($F_X(\theta)$), a reversed hazard rate $\tilde{r}_X(\theta) = f_X(\theta)/F_X(\theta)$ (*RH*) and a poverty line θ, where $\theta \in \Re^+$ are considered; such that $F_X(\theta)$ represents the proportion of the poor. The poverty is then quantified in term of the proportion of the poor people and their income inequality (see Atkinson [1], Foster and Shorrocks [2,3] and Belzunce et al. [4]).

The difference between the poverty line θ and individual income y_i is defined as the poverty gap, P_i, where $P_i = \theta - y_i$. The traditional poverty gap index is introduced by Sen [5] as the percentage of individuals below θ. Almost the poverty papers consider the poverty gap index as

$$\Delta = \sum_{i \in \aleph} P_i / (m\theta), \quad \aleph \text{ is the set of poor individuals.}$$

it depends on uniform distribution. In many cases, the distribution of income is not uniform distribution.

There are some related aging classes which related to $\tilde{r}_X(\theta)$ such as mean reversed hazard lifetime. It should be noted that the $\tilde{r}_X(\theta)$ function and its related functions are less intuitive functions.

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The first goal of this study is to introduce new poverty indices which depend on distribution function of income in general cases, i.e., the distribution of income may be not uniform such as exponential or Pareto distribution and so on. Moreover, these indices consider as right truncated distribution. In addition, the second goal of this paper is spread out the usefulness of $\tilde{r}_X(\theta)$ by relating it to new poverty gap indices.

The study of the poverty measures (poverty gap and the poverty severity) based on the concept of reversed residual incomes are defined in Section 2. We also give some properties of it and show the relationship between Lorenz curve and new poverty measures these based on the concept of reversed residual incomes. In addition, the characterization of Pareto distribution based on new poverty functions is obtained. Section 3 consists of a through the study of the stochastic dominance of new poverty measures. The last section provides the weighted poverty gap function and stochastic dominance which involve the concept of inactivity incomes and its features are studied.

2. The Poverty Measures Functions

Let $X \geq 0$ be the random variable expresses the income of individuals with an absolutely continuous distribution function $F_X(.)$, and a density function $f_X(.)$. We consider $F_X(\theta)$ for a poverty line θ as a right truncated income distribution at θ, then we interesting to study the random variable:

$$X_r(\theta) = \{X | X \leq \theta\},$$

and its distribution function is given by

$$F_{X_r(\theta)}(x) = \frac{F_X(x)}{F_X(\theta)} \quad \text{for } x \leq \theta.$$

Then, the mean of $X_r(\theta)$ (the average income below the poverty line θ) can define as follows:

$$\psi(\theta) = E(X | X \leq \theta) = \int_0^\theta s dF_X(s) / F_X(\theta), \qquad \theta \in \Re^+. \tag{2.1}$$

It is satisfies the following properties:

- $\psi(\theta) \leq \theta$, for all $\theta \in \Re^+$,
- $\lim_{\theta \to \infty} \psi(\theta) = E(\theta)$.

and its distribution function is given by
The following definition is essential for this work:

Definition 2.1: Let the random variable X expresses the income of individuals, with distribution function $F_X(.)$ and a density function $f_X(.)$. Then, the reversed proportional failure rate (RPF), $\tilde{p}_X(\theta)$, is defined as follows:

$$\tilde{p}_F(\theta) = \theta f_X / F_X, \quad \text{for all } \theta \in \Re^+. \tag{2.2}$$

The reversed proportional failure rate function has been studied by Block et al. [6], Chandra and Roy [7], Finkelstein [8, 9, 10] and Gupta et al. [11].

Then, by using (2.1), we can obtain the poverty gap (PG) index as the following formula

$$\beta_X(\theta) = 1 - \psi(\theta)/\theta, \quad \text{for all } \theta \in \Re^+. \tag{2.3}$$

Moreover, from (2.2) and (2.3), we obtain:

$$\partial \beta_X(\theta) / \partial \theta = (1/\theta)[1 - \tilde{p}_X(\theta)\beta_X(\theta)], \quad \text{for all } \theta \in \Re^+,$$

where $\tilde{p}_X(\theta)$ is the reversed proportional failure rate function at a poverty line θ.

Moreover, the relationship between $\psi(\theta)$ and $\beta_X(\theta)$ is given by

$$\psi(\theta) = \theta[1 - \beta_X(\theta)],$$

and

$$\partial\psi(\theta)/\partial\theta = \tilde{p}_X(\theta)\beta_X(\theta), \quad \text{for all } \theta \in \Re^+.$$

In the current investigation, we study the properties of new poverty index in term of axioms for a good index of poverty.

Theorem 2.1: *The poverty gap index of (2.3) satisfying axioms:*

- *Normalization (*N*),*
- *Increasing in subsistence income (ISI),*
- *Monotonicity (M).*

Proof. It is easy to proof that if all incomes have zero. Therefore, $\beta_X(\theta)$ takes the value unity, it is means that $\beta_X(\theta)$ is satisfying *N*. For proof that $\beta_X(\theta)$ is increasing in θ, we should be achieve the following relation

$$\tilde{p}_X(\theta)\beta_X(\theta) < 1,$$

since $E(X|X \le \theta) < \theta \Rightarrow E(X|X \le \theta)/\theta < 1$ and $\tilde{p}_X(\theta)$ is probability value, then $\tilde{p}_X(\theta)\beta_X(\theta) < 1$. This means that $\beta_X(\theta)$ is satisfying *ISI*. When we reduction in an income with value *a* below θ, the poverty gap index can be shown as

$$\beta_{X-a}(\theta) = 1 - \left(\int_0^\theta x dF_X(x)/F_X(\theta) - a\right)/\theta, \text{ for all } \theta \in \Re^+$$

$$= 1 - (\psi(\theta)/\theta) + (a/\theta) = \beta_X(\theta) + (a/\theta),$$

since *a* and θ are positive numbers, then

$$\partial\beta_{X-a}(\theta)/\partial\theta \ge \partial\beta_X(\theta)/\partial\theta, \text{ for all } \theta \in \Re^+,$$

it is complete proof that $\beta_X(\theta)$ is satisfying *M*. □

To compare the distribution of income of different countries at the same time we may be used lorenz curve that is introduced by Lorenz [12], and it is also, defined by Gastwirth [13] as following definition:

Definition 2.2: Let $W \ge 0$ be a random variable with cumulative distribution function $F_W(\theta)$ and a density function $f_W(\theta)$, with a finite mean μ. The Lorenz curve $L_W(p)$ of *W* is defined in terms of two parametric equations in *x* namely

$$p = F_W(\theta) = \int_0^\theta dF_W(t),$$

and

$$L_w(\theta) = F_1(\theta) = \frac{1}{\mu}\int_0^\theta t f_W(t)dt, \tag{2.4}$$

where $F_W(\theta)$ can interpret as the proportion of the poor below the level θ. $F_1(\theta)$ can be viewed as the proportional share of the total income of poor below the level θ.

It follows from (2.4) that the Lorenz curve is the first moment distribution function of $F_W(\theta)$. It may be noticed that both $F_W(\theta)$ and $F_1(\theta)$ lies between zero and one. In addition, the Lorenz curve being the plot of the points ($F_W(\theta), F_1(\theta)$) is represented in the unit square. $L_W(p)$ can be interpreted as the proportion of the total wealth possessed by the poorest p^{th} fraction of the population. The Lorenz curve defined by (2.4) acquires the following properties:

- *L(0) = 0*, *L(1) = 1*, and $L_W(p)$ is continuous and strictly increasing on *(0, 1)*, as

$$\partial L_W(p)/\partial\theta = \theta(F)/\mu, \quad \text{for all } \theta \in \Re^+.$$

where

$$L_W(p)=\int_0^p x(F_1)dF_1 / \int_0^1 x(F_1)dF_1 \text{ and } x(F_1)=\inf\{y: F(y)\geq F_1\}.$$

- $L_W(p)$ is twice differentiable and is strictly convex on (0,1) as

$$\partial^2 L_W(p)/\partial\theta^2 = 1/\mu\, f(\theta(F)w) > 0.$$

where the inverse of the distribution function can interpret as *x(F)*.

For $W \geq 0$ be any random variable with distribution function F_W and a finite mean μ, the Lorenz curve $L_W(p)$ is defined as

$$L_w(p)=\frac{1}{\mu}\int_0^p F_W^{-1}(t)dt, \quad \text{for all } p\in[0,1], \tag{2.5}$$

where $F_\theta^{-1}(t)=\inf\{\theta : F_W(\theta)\geq t\}$ is the left continuous inverse of F_W (also known as the quantile function).

Thompson [14] has proved the following properties for the Lorenz curve defined by (2.5).

We can use the following characterizations theorem for the Pareto distribution (Pareto(.)) by using new measures.

Theorem 2.2: *Suppose W be a non-negative random variable. Then we get*

1.The poverty gap function is

$$\beta_W(\theta)=1-\frac{\alpha^\kappa \kappa\theta^{1-\kappa}}{(1-\kappa)p} \;. \tag{2.6}$$

2. If $r_W(\theta)=f_W(\theta)/(1-F_W(\theta))$ *represents the hazard rate and* $\beta_W(\theta)$ *the poverty gap function, therefore, the relationship*

$$\beta_W(\theta)=1-\frac{\theta^2}{(1-\kappa)}r_W(\theta). \tag{2.7}$$

3. If W have the Lorenz curve $L_W(p)$, *the poverty gap* $\beta_W(\theta)$ *and the finite mean* μ, *then the relationship*

$$\beta_W(\theta)=1-g_1(\Theta)L_W(\theta)/\mu; \quad g_1(x)=1/p \tag{2.8}$$

4.We have the following relationship

$$\beta_W(\theta)=1+\frac{\theta(1-L_W(p))}{pL^{\backslash}{}_W(p)},$$

where $L^{\backslash}{}_W(p)=\partial L_W(p)/\partial p$. *Will hold for all real* $p\geq 0$ *if and only if W follows* $Pareto(\kappa,\alpha)$ *with distribution function as follows:*

$$F_W(\theta)=1-(\alpha/\theta)^\kappa,\;\; \theta\geq\alpha \text{ and } \kappa>1. \tag{2.9}$$

Proof. Since

$$\psi(\theta)=\frac{\alpha^\kappa\kappa\theta^{1-\kappa}}{(1-\kappa)(1-\alpha^\kappa\theta^{-\kappa})}=\frac{\alpha^\kappa\kappa\theta^{1-\kappa}}{(1-\kappa)p}, \text{ for all } \theta\in\Re^+; \kappa\neq 0.$$

Hence by (2.3) the required result (1) follows.

In addition, when $r_W(\theta)$holds. Hence, we can calculate $\beta_W(\theta)$ directly by using (2.6) as in Eq. (2.7).

Furthermore,

$$L_W(\theta)=\frac{1}{\mu}\int_0^\theta t dF_W(t)dt=\alpha^{\kappa-1}\frac{\theta^{1-\kappa}}{(1-\kappa)} \text{ and } \mu=\kappa/\theta.$$

Then the required result (3) follows. □

Assume Eq. (2.9) holds, and according to Arnold [15], we obtain:

$$L_W(s) = 1 - (1-s)^{-1/\kappa+1}.$$

and

$$\partial L_W(s)/\partial s = (\kappa - 1)/\left(\kappa(1-s)^{1/\kappa}\right), \text{ for all } \kappa > 1.$$

Hence, (2.6) gives the needed result (3).

Next, we consider a new measure of the poverty that takes the variations in the distribution of welfare amongst the poor into account. It is defined as the severity of poverty.

Let $X \geq 0$ be a random variable denote an income below level θ with cumulative distribution function $F_X(.)$. In many economic problems, there are attention to the following random variable:

$$X^2{}_r(w) = \{X^2 | X \leq \theta\}.$$

The severity of the poverty (SP) of the poor people is defined as

$$\rho_X(\theta) = E\left(\left(\frac{\theta - X}{\theta}\right)^2 | X \leq \theta\right).$$

By using integration by parts, we can conclude that:

$$\rho_X(\theta) = \frac{2}{\theta^2 F_X(\theta)} \int_0^\theta \int_0^y F_X(x) dx dy = \frac{2}{\theta}[\theta\beta_X(\theta) - \Phi(\theta)], \text{ for all } \theta \in \Re^+,$$

where

$$\Phi(\theta) = \int_0^\theta x F_X(x) dx / \theta F_X(\theta).$$

Now, we can define new class of the severity as follows:

A random variable X is said to have increasing (decreasing) the poverty severity, IPS (DPS), if

$\rho_X(\theta)$ is increasing (decreasing) in θ,

or, equivalently, $F_X \in IPS\,(DPS)$ iff

$$\beta_X(\theta) \geq (\leq)\, \Phi(\theta).$$

3. Stochastic Dominance of The Poverty Measures Functions

Let $X \geq 0$ and $Y \geq 0$ have the distribution functions F and G, and the reversed proportional failure rate (*RPF*) functions $\tilde{r}$ and $\tilde{q}$ respectively. Then X is said to be smaller than or equal (≤) Y in *RPF* order (denoted as $X \leq_{rp} Y$) if,

$$\tilde{r}(\theta) \leq \tilde{q}(\theta), \text{ for all } \theta \in \Re^+.$$

To compare the poverty gap function of two group poor people, let X and Y be two nonnegative random variables (incomes of two poor groups), having distribution functions F_X and G_X, and the poverty gap functions $\beta_X(\theta)$ and $\beta_Y(\theta)$ respectively. The following definition of the poverty gap ordering is essential for this section:

Definition 3.1: X is said to be smaller than or equal to Y in the poverty gap function ordering ($X \leq_{pg} Y$) if

$$\beta_X(\theta) \geq \beta_Y(\theta), \quad \text{for all } \theta \geq 0,$$

where θ is a poverty line. It can be written as