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Characterizations of the Generalized Beta-generated family of distributions

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Several characterizations of Generalized Beta-generated family of distributions, introduced by Alexander et al. (2012), are presented, giving special emphasis to the Kumaraswamy skew-normal distribution and the Beta skew-normal distribution, special cases of this family.
These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) a single function of the random variable; (iv) truncated moments of a single function of the $n-$th order statistic.

Keywords: Generalized Beta-generated family; Beta skew-normal; Kumaraswamy skew-normal.

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1. Introduction

Characterizations of distributions are important to many researchers in the applied fields. Various characterizations of distributions have been established in many different directions in the literature. In this short note, several characterizations of the Generalized Beta-generated family of distributions, introduced by [1] are presented, giving special emphasis to two special cases of this class which are called the Kumaraswamy skew-normal (*KwSN*) ([15]) and the Beta skew-normal distributions (*BSN*) ([16]).
These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) a single function of the random variable; (iv) truncated moments of a single function of the $n-$th order statistic.
The Generalized Beta-generated class has been introduced by [1] to generalize the class of the Beta-generated family of distributions ([8]) and the class of the Kumaraswamy generated distributions ([3]).

Given a cumulative distribution function (cdf), $P_\tau(x)$ with parameter vector τ and associated density function (pdf) $p_\tau(x)$, [1] represented the cumulative distribution function of the Generalized Beta-generated (GBG) class as

$$G_{P_\tau(x)}(x;a,b,c) = I(P_\tau(x)^a;c,b), \text{ with } x \in \mathbb{R}, \tag{1.1}$$

where $I(\cdot;c,b)$ denotes the incomplete Beta ratio, and a, b and c are positive real scalars.
The density function correspondent to (1.1) is

$$g_{P_\tau(x)}(x;a,b,c) = \frac{a}{B(c,b)} p_\tau(x) P_\tau(x)^{ac-1}(1-P_\tau(x)^a)^{b-1}, \text{ with } x \in \mathbb{R}. \tag{1.2}$$

Setting $c = 1$ we obtain the Kumaraswamy generated family ([3]) with parameters a and b; while setting $a = 1$ we recover the Beta-generated family ([8]) with parameters c and b. We refer the reader to [1] for a general treatment of the generalized Beta-generated class.

The paper unfolds as follows: in Subsection 2.1, we present our characterization results for the Generalized Beta-generated class based on truncated moments. Subsection 2.2 is devoted to characterization of the GBG family in terms of the hazard function. In Subsection 2.3 some characterizations of the GBG class are given in terms of a single function of the random variable. Subsection 2.4 deals with a characterization of the GBG class based on truncated moments of the $n-$th order statistic. Some concluding remarks are given in Section 3.

2. Characterizations

The problem of characterizing a distribution is an important problem in various fields which has recently attracted the attention of many researchers. These characterizations have been established in many different directions. The present work deals with the characterizations of the Generalized Beta-generated family of distributions along the directions outlined before.

2.1. *Characterizations based on two truncated moments*

In this Subsection we present characterizations of Generalized Beta-generated family of distributions in terms of a simple relationship between two truncated moments. Other works dealing with characterizations of distributions along this direction are [4], [6], [5], [7] and [9, 10]. The first of our characterization results borrows from the following theorem due to [4].

Theorem 2.1 ([4]). *Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a given probability space and let $H = [d,e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with distribution function F and let g and h be two real functions defined on H such that*

$$\mathbf{E}[g(X) \mid X \geq x] = \mathbf{E}[h(X) \mid X \geq x]\,\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $g,\ h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)\, h(u) - g(u)} \right| \exp(-s(u))\; du\,,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, h}{\eta\, h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

We refer the interested reader to [4] for a proof of the above Theorem. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in [6], this characterization is stable in the sense of weak convergence.

Proposition 2.1. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $h(x) = [1-(P_\tau(x))^a]^{1-b}$ and $g(x) = h(x)\,(P_\tau(x))^{ac}$ for $x \in \mathbb{R}$. The random variable X belongs to the Generalized Beta-generated family (1.2) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\,[1 + (P_\tau(x))^{ac}]\,, \quad x \in \mathbb{R}. \tag{2.1}$$

Proof. Let X be a random variable with density (1.2), then

$$\left(1 - G_{P_\tau(x)}(x;a,b,c)\right)\; \mathbf{E}\,[h(X) \mid X \geq x] = \frac{1}{cB(c,b)}\,[1-(P_\tau(x))^{ac}]\,, \quad x \in \mathbb{R},$$

and

$$\left(1 - G_{P_\tau(x)}(x;a,b,c)\right)\; \mathbf{E}\,[g(X) \mid X \geq x] = \frac{1}{2cB(c,b)}\left[1-(P_\tau(x))^{2ac}\right], \quad x \in \mathbb{R},$$

and finally

$$\eta(x)\, h(x) - g(x) = \frac{1}{2}\,[1-(P_\tau(x))^a]^{1-b}\,[1-(P_\tau(x))^{ac}] > 0 \quad \text{for} \quad x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, h(x)}{\eta(x)\, h(x) - g(x)} = \frac{ac\, p_\tau(x)\,(P_\tau(x))^{ac-1}}{1-(P_\tau(x))^{ac}}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = -\ln\{(1-P_\tau(x))^{ac}\}\,, \quad x \in \mathbb{R}.$$

Now, in view of Theorem 2.1, X has density (1.2). □

Corollary 2.1. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $h(x)$ be as in Proposition 2.1. The pdf of X is (1.2) if and only if there exist functions g and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,h(x)}{\eta(x)\,h(x)-g(x)}=\frac{acp_{\tau}(x)\,(P_{\tau}(x))^{ac-1}}{1-(P_{\tau}(x))^{ac}},\quad x\in\mathbb{R}. \tag{2.2}$$

The general solution of the differential equation in Corollary 2.1 is

$$\eta(x)=(1-(P_{\tau}(x)^{ac}))^{-1}\left[-\int acP_{\tau}(x)\,(P_{\tau}(x))^{ac-1}\,(h(x))^{-1}\,g(x)\,dx+D\right], \tag{2.3}$$

where D is a constant. Note that a set of functions satisfying the differential equation is given in Proposition 2.1 with $D=\frac{1}{2}$. However, it should be also noted that there are other triplets (h,g,η) satisfying the conditions of Theorem 2.1.

Now we consider the special case of the Beta skew-normal distribution, which is obtained by replacing the distribution function of the skew-normal $\Phi_{\lambda}(x)$ in (1.2) and setting $a=1$. The distribution function is

$$G_{\Phi_{\lambda}(x)}(x;\lambda,c,b)=\frac{1}{B(c,b)}\int_{0}^{\Phi_{\lambda}(x)}z^{c-1}(1-z)^{b-1}dz \tag{2.4}$$

and the probability density function is

$$g_{\Phi_{\lambda}(x)}(x;\lambda,c,b)=\frac{1}{B(c,b)}(\Phi_{\lambda}(x))^{c-1}(1-\Phi_{\lambda}(x))^{b-1}\phi_{\lambda}(x), \tag{2.5}$$

with $\lambda\in\mathbb{R}$, $b,\ c>0$, where $\phi_{\lambda}(x)$, $\Phi_{\lambda}(x)$ denote the pdf, and the cdf of skew-normal distribution [2], respectively.

Proposition 2.2. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $h(x)=[1-(\Phi_{\lambda}(x))]^{1-b}$ and $g(x)=h(x)(\Phi_{\lambda}(x))^{c}$ for $x\in\mathbb{R}$. The pdf of X is (2.5) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left[1+(\Phi_{\lambda}(x))^{c}\right],\quad x\in\mathbb{R}.$$

2.2. *Characterizations based on hazard function*

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h'_F(x)}{h_F(x)}-h_F(x). \tag{2.6}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following proposition establishes a non-trivial characterization of the Generalized Beta-distribution family in terms of the hazard function, which is not of the trivial form given in (2.6).

Proposition 2.3. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The pdf of X is (1.2) if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$h_F'(x) - \frac{p_\tau'(x)}{p_\tau(x)} h_F(x) = \frac{a(p_\tau(x))^2 (P_\tau(x))^{ac-2} (1-(P_\tau(x))^a)^{b-2}}{B(c,b)(1-I(P_\tau(x)^a;c,b))} \times$$
$$\times \left\{ ac - 1 - (P_\tau(x))^a (ac-1+ab-a) + \frac{a(P_\tau(x))^{ac}(1-(P_\tau(x))^a)^b}{B(c,b)(1-I(P_\tau(x)^a;c,b))} \right\}. \quad (2.7)$$

Proof. If X has pdf (1.2), then clearly (2.7) holds. Now, if (2.7) holds, then

$$\frac{d}{dx}\left\{ (p_\tau(x))^{-1} h_F(x) \right\} = \frac{d}{dx}\left\{ \frac{a(P_\tau(x))^{ac-1}(1-(P_\tau(x))^a)^{b-1}}{B(c,b)(1-I(P_\tau(x)^a;c,b))} \right\}. \quad (2.8)$$

or, equivalently,

$$h_F(x) = \frac{ap_\tau(x)(P_\tau(x))^{ac-1}(1-(P_\tau(x))^a)^{b-1}}{B(c,b)(1-I(P_\tau(x)^a;c,b))},$$

which is the hazard function of the Generalized Beta-generated family; see [1].

□

Now we consider the special case of the Kumaraswamy skew-normal, which is obtained by replacing the distribution function of the skew-normal $\Phi_\lambda(x)$ in (1.2) and setting $c = 1$, its cdf and pdf are given, respectively, by

$$G_{\Phi_\lambda(x)}(x;\lambda,a,b) = 1 - [1-(\Phi_\lambda(x))^a]^b, \quad x \in \mathbb{R} \quad (2.9)$$

and

$$g_{\Phi_\lambda(x)}(x;\lambda,a,b) = ab\phi_\lambda(x)(\Phi_\lambda(x))^{a-1}(1-\Phi_\lambda(x)^a)^{b-1}, \quad x \in \mathbb{R}. \quad (2.10)$$

Due to the closed form of the distribution function of the Kumaraswamy generated family, here we establish a characterization of the Kumaraswamy skew-normal distribution in terms of hazard rate which has a simple form.

Proposition 2.4. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is (2.10) if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$h_F'(x) - \frac{\phi_\lambda'(x)}{\phi_\lambda(x)} h_F(x) = ab(\phi_\lambda(x))^2 (\Phi_\lambda(x))^{a-2} (1-(\Phi_\lambda(x))^a)^{-2} \{a-1+(\Phi_\lambda(x))^a\}. \quad (2.11)$$

2.3. *Characterizations based on a single function of the random variable*

In this subsection we state characterization results in terms of a function of the random variable X with cdf given by (1.1). The following proposition can be easily proved by using results in [12], which applies in the general case of a random variable X with cdf F.

Proposition 2.5. *Let $X : \Omega \to (d,e)$ be a continuous random variable with cdf given by* (1.1).

(1) *Let $\psi(x)$ be a differentiable function on (d,e) such that $\lim_{x\to d^+}\psi(x)=1$. Then for $\delta\neq 1$ and for $x\in(d,e)$*

$$E[\psi(X) \mid X\geq x]=\delta\psi(x)\Longleftrightarrow \psi(x)=\left(1-G_{P_\tau(x)}(x;a,b,c)\right)^{\frac{1}{\delta}-1}.$$

(2) *Let $\varsigma(x)$ be a differentiable function on (d,e) with $\lim_{x\to d^+}\varsigma(x)=1$. Then for $\varepsilon>1$ and for $x\in(d,e)$*

$$E\left[(\varsigma(X))^{\delta} \mid X\geq x\right]=\varepsilon(\varsigma(x))^{\delta}\Longleftrightarrow \varsigma(x)=\left(1-G_{P_\tau(x)}(x;a,b,c)\right)^{\frac{1-\varepsilon}{\varepsilon\delta}}.$$

(3) *Let $\kappa(x)$ be a differentiable function on (d,e) with $\lim_{x\to d^+}\kappa(x)=0$ and $\lim_{x\to e^-}\kappa(x)=1$. Then for $0<\varepsilon<1$ and for $x\in(d,e)$*

$$E[\kappa(X) \mid X\geq x]=\varepsilon+(1-\varepsilon)\kappa(x)\Longleftrightarrow \kappa(x)=1-\left(1-G_{P_\tau(x)}(x;a,b,c)\right)^{\frac{\varepsilon}{1-\varepsilon}}.$$

Note that characterizations of the Kumaraswamy skew-normal and of the Beta skew-normal distributions, can be obtained by taking $c=1$ and $a=1$, respectively, with $P_\tau(x)=\Phi_\lambda(x)$ in the above formulae.

2.4. *Characterization based on truncated moments of certain functions of the $n-$th order statistics.*

The following section deals with a characterization result for the Generalized Beta-generated class of distributions based on truncated moments of the $n-$th order statistic. We refer the reader to [11] and [13] and references therein for an account of characterizations of other well-known continuous distributions along this direction. In particular, it should be pointed that [13] gives a characterization of the exponentiated distributions which is a special case of the Generalized Beta-generated class of distribution when $a=1$ and $b=1$.
Let $X_1,\ldots,X_n$ be an i.i.d. sample from the distribution function F and let $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. Our result is based on the following proposition.

Proposition 2.6. *Let $X:\Omega\longrightarrow(d,e)$ be a continuous random variable with distribution function F and let $k(x)$ be a differentiable function such that $\lim_{x\longrightarrow d}k(x)(F(x)^n)=0$. Let $q(x,n)$ be a real-valued function which is differentiable with respect to x and $\int_d^e\frac{k'(x)}{q(x,n)}dx=\infty$. Then*

$$E[k(X_{(n)})|X_{(n)}<t]=k(t)-q(t,n),\quad d<t<e$$

implies that

$$F(x)=\left(\frac{q(e,n)}{q(x,n)}\right)^{\frac{1}{n}}\exp\left\{-\int_x^e\frac{k'(t)}{nq(t,n)}dt\right\},\quad x\geq d.$$

Let $X_1,\ldots,X_n$ be an i.i.d. sample from the distribution function G_{P_τ} in eq. (1.1) and let $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. By setting $k(x)=I(P_\tau^a(x);c,b)$ and $q(x,n)=\frac{1}{n+1}I(P_\tau^a(x);c,b)$ in the previous proposition we obtain a characterization of the Generalized Beta-generated distribution. Note that when $b=1$ and $c=1$ we obtain a characterization for the class of exponentiated distributions [13]. Note that characterizations of the Kumaraswamy skew-normal and of the Beta skew-normal distributions, can be obtained by taking $c=1$ and $a=1$, respectively,

with $P_\tau(x) = \Phi_\lambda(x)$ in the above proposition.
The previous proposition furnishes a characterization of the skew-normal random variable with parameter λ, with $k(x) = \Phi_\lambda(x)$ and $q(x,n) = \frac{1}{n+1}\Phi_\lambda(x)$. In particular, the condition

$$E[\Phi_\lambda(X_{(n)})|X_{(n)} < t] = \frac{n}{n+1}\Phi_\lambda(t).$$

implies that $F(x) = \Phi_\lambda(x)$.
Note that the distribution of the largest order statistic, $X_{(n)}$, from a skew-normal distribution with parameter λ has Beta skew-normal distribution with parameters λ, n and 1; see [16].

3. Conclusions

In this article, we have studied characterizations results for the generalized Beta-generated family of distributions, which extends the Beta-generated and the Kumaraswamy generated family of distributions. The results presented in this paper can be used to obtain characterizations of other distributions belonging to the generalized Beta-generated class of distributions, such as the Generalized Beta-normal and the Generalized Beta-Weibull distributions; see [1] for other special *GBG* distributions. A possible line of research could be the investigation of the proposed characterizations in the bivariate setting of Beta-generated family (see for instance [17]), and in the bivariate setup of Kumaraswamy generated family (see for instance [14]).

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