A class of trees having strongly super total graceful labellings

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Total graceful labelling is a new graph labelling of graph theory. We define strongly the total graceful labelling and determine the existence of a class of trees having perfect matchings and strongly total graceful labellings. Our methods can be easily transferred into efficient algorithms.

Keywords: $S(P_n)$ Qerfect Matching; Strongly Graceful Labelling; Super Total Graceful Graphs.

1. Introduction and concepts

1.1. *Producing hardcopy using MS-Word*

Rosa[1], in 1976, introduced the concept of β-labeling of a simple, finite, connected and undir ected graph. The graceful labeling was introduced in attacking famous Ringel's conjecture: A complete graph K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges, and it is becoming famous since the graceful tree conjecture due to Rosa. As known graph labelings have been applied to many areas such as coding theory, radar, radio astronomy, and circuit design, some algorithms with labelling techniques, design of highly accurate optical gauging systems for use on automatic drilling machines, date security, mobile telecommunication systems, bio-informatics and X-ray crystallographs, complex networks. Graph colorings/labellings are forming a useful branch of graph theory [2].

Security of networks was known as an important topic in researching real networks. Complex networks need more and more crystallographs for keeping their running formally each day. Such topic is attracting attention of many researchers [3, 4, 5, 6, 7, 8]. In this paper, we find a new method which is called the suspension-split method to prove our results. We have shown some necessary or sufficient conditions for a graph G to be strongly super total graceful, as well as the particular trees S(Pn) are strongly super total graceful.

All graphs mentioned in this article are simple, undirected and finite. A (p, q)-graph has p vertices and q edges. The short hand notation [m, n] denotes an integer set {m, m+1, … , n} with integers n>m≥0.

2. Definition

([9, 10, 11]) If a (p, q) -graph G admits a mapping $f: V(G) \rightarrow [0, q]$ such that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and the edge labelling set $\{f(uv)=|f(u)-f(v)|: uv \in E(G)\}=[1, q]$. Then f is called a *graceful labelling*, so say, G is *graceful*. Furthermore, if G has a perfect matching M and f is a graceful labelling such that $f(x)+f(y)=q$ for every edge $xy \in M$, so we call f a *strongly graceful labelling*. We write the vertex label set $f(V(G))=\{f(u): u \in V(G)\}$ and the edge label set $f(E(G))=\{f(uv): uv \in E(G)\}$ hereafter. Suppose that a bipartite graph G admits a graceful labelling f such that $\max\{f(x): x \in X\}<\min\{f(y): y \in Y\}$, where (X, Y) is the bipartition of $V(G)$, then we call f a *set-ordered graceful labelling* (SoG-labelling), and this case is denoted as $f(X)<f(Y)$ directly.

If a (p, q)-graph G has a bijection f from $V(G) \cup E(G)$ to the set $[1, p+q]$ such that $f(uv)=|f(u)-f(v)|$ for every edge $uv \in E(G)$, then f is called a *total graceful labelling* of G. A total graceful labelling is called *super* if $f(E(G))=[1, p]$([12]).

A super total graceful labelling defined in Definition 2.2 can be abbreviated as a STG-labelling hereafter. We can define a new labelliing in this way: Suppose T be a tree with a perfect matching M. If T admits a STG-labelling f such that

$$f(u)+f(v)=3|V(T)|-1 \tag{1}$$

for each $uv \in M$, then f is called a *strongly STG-labelling*. Now, we introduce the graphs that will be discussed in this paper. A vertex of degree one is called a *hang vertex*, or a *leaf* in trees. T is a *caterpillar tree*, and if we delete all its hang vertices, then we get a path. A tree H is called *lobster tree*, if we delete all hang vertices of H, the resulting graph is just a caterpillar tree. An I-tree $T(a_i)$ is a spider tree having a 2-leg $P_3=a_iu_iv_i$ and m_i 4-legs $P_5^j=a_ia_{i,j}b_{i,j}c_{i,j}d_{i,j}$ with $j \in [1, m_i]$, where the vertex a_i is called the "body" of $T(a_i)$. A II-tree $T(a_i)$ is also a spider tree having its body a_i and m_i 4-legs $P_5^j=a_ia_{i,j}b_{i,j}c_{i,j}d_{i,j}$ with $j \in [1, m_i]$ (see Fig.1).

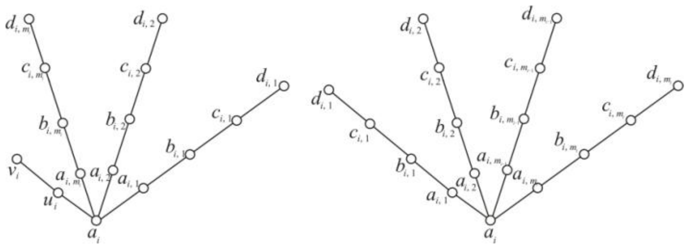

Fig. 1 The left graph is an I-tree $T(a_i)$, the right graph is a II-tree $T(a_i)$.

In this paper, we discuss the super lobster trees $S(P_n)$ with perfect matchings. Connecting some I-trees and some II-trees by edges will produce the desired trees $S(P_n)$ with $n=2m$ in the way: join the body a_i of $T(a_i)$ with the body a_{i+1} of $T(a_i+1)$ for $i\in[1, n-1]$ by an edge a_ia_{i+1}, here $T(a_1)$ is a II-tree. When $i\geq 2$, if $i-2\equiv 0(\mathrm{mod}4)$, so $T(a_i)$ and $T(a_{i+1})$ both are I-trees, otherwise $T(a_i)$ is a II-tree. Notice that $S(P_n)$ has a *main path* $P_n=a_1a_2\ldots a_n$. Because of I-trees and II-trees have perfect matchings, which indicate that each tree $S(P_n)$ has a perfect matching for when $n=2m$ with $m=1, 2, 3, \ldots$.

$$M=M^* \cup M1 \cup \Big(\bigcup_{i\equiv 0(\mathrm{mod}\,4)} (Mi \cup Mi+1)\Big) \cup \Big(\bigcup_{i-2\equiv 0(\mathrm{mod}\,4)} \big(M_i^{(2)} \cup M_{i+1}^{(2)}\quad (2)$$

In formula (2), $n\geq 2$ is even, the path P_n has a perfect matching $M^*=\{a_{2j-1}a_{2j} : j\in[1, n/2]\}$. M_1 and M_i are perfect matchings on II-trees, where $M_1=\{a_{1,j}b_{1,j}, c_{1,j}d_{1,j}: j\in[1, m_i]\}$ and $M_i=\{a_{i,j}b_{i,j}, c_{i,j}d_{i,j}: j\in[1, m_i]\}$. $M_i^{(2)}$ has a perfect matching on I-tree and $M_i^{(2)}=\{u_iv_i\}\cup\{a_{i,j}b_{i,j}, c_{i,j}d_{i,j}: j\in[1, m_i]\}$. A super lobster tree (P_6) is shown in Fig.2, in which $T(a_1)$ is a II-tree having no a 2-leg. In order to ensure the super lobster tree $S(P_n)$ has a perfect matching for even n in the following argument.

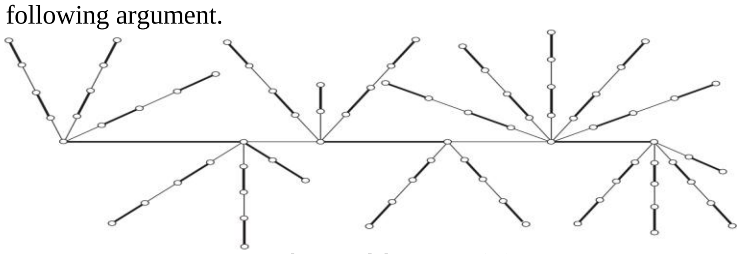

Fig. 2 The super lobster tree $S(P_6)$.

- **Theorems and Proofs**

- Theorem1. *The super lobster tree* $S(P_{2m})$ (m=1, 2, 3,…) *is a strongly* STG-*tree*.

Proof. When m=1, we say $S(P_{2m})$ is a strong STG-tree. We will show this claim in the following steps.

P_1: First of all, we construct a star T_1 with vertex set $V(T_1)=\{a_2, x_1, x_2, \ldots, x_{m_1+m_2}\}$ and edge set $E(T_1)=\{a_2, x_i \mid i=1, 2, \ldots, m_1+m_2\}$ (see Fig.3). Next, we define a labelling $f_1(x)$ to T_1 by setting $f_1(a_2)=0$, $f_1(x_j)=j$ with $j\in [1, m_1+m_2]$.

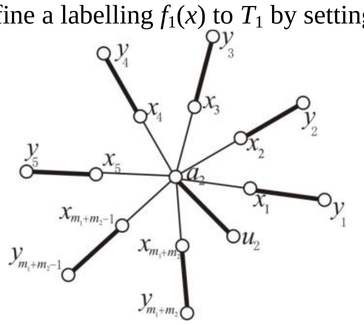

Fig. 3 A star T_1.

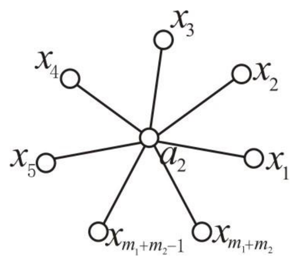

Fig. 4 A tree T_2.

P_2: Define another labeling $f_2(x)$ to T_1 as: $f_2(x)=2f_1(x)$ for each vertex $x\in V(T_1)$. matching for even n in the following argument.

P_3: Adding a hang vertex to each vertex of T_1, then we get a new tree T_2 (see Fig.4). In T_2, a hang vertex u_2 is added to a_2, each hang vertex y_j is added to x_j with $j\in [1, m_1+m_2]$. We label u_2 and y_j as:

$$\begin{cases} f_2(u_2)=2(m_1+m_2)+1-f_2(a_2)=2(m_1+m_2)+1, \\ f_2(y_j)=2(m_1+m_2)+1-f_2(x_j)=2(m_1+m_2)+1-2j. \end{cases}$$

We will show that f_2 is a strongly graceful labelling of T_2 in the following argument. It is not hard to see that the tree T_2 has a perfect matching $M_2=\{a_2u_2, x_jy_j : j\in [1, m_1+m_2]\}$, and $f_2(E(T_2)\setminus M_2)=\{2, 4,\ldots, 2(m_1+m_2)\}$, $f_2(M_2)=\{1, 3, \ldots, 2(m_1+m_2)+1\}$, which mean that $f_2(E(T_2))=[1,2(m_1+m_2)+1]$. Let $\lambda_1=2(m_1+m_2)+1$. For the edges of the perfect matching M_2, we always have $f_2(a_2)+f_2(u_2)=\lambda_1$ and $f_2(x_j)+f_2(y_j)=\lambda_1$ with $j\in [1, m_1+m_2]$. Thereby, we conclude that T_2 is a strongly graceful tree.

P4: Define a new labelling f3(ω)=2f2(ω)+1 with ω∈V(T2) for the tree T2 such that

$$\begin{cases} f_3(a_2)=2f_2(a_2)+1=1, \\ f_3(x_j)=2f_2(x_j)+1=4j+1, \\ f_3(y_j)=2f_2(y_j)+1=4(m_1+m_2-j)+3, \\ f_3(u_2)=2f_2(u_2)+1=4(m_1+m_2)+3. \end{cases} \quad j\in[1, m_1+m_2].$$

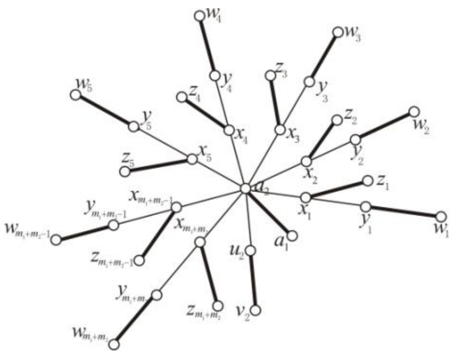

Fig. 5 A tree T_3.

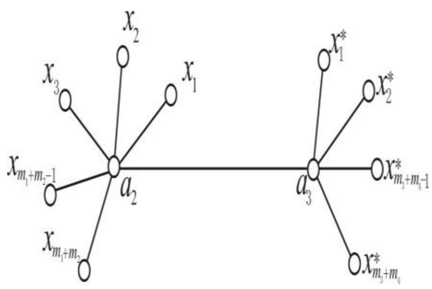

Fig. 6 A bi-star T_1^*.

Construct another tree T_3 by adding a hang vertex to each vertex of T_2 in which it adds a hang vertex a_1 to a_2, and adds the hang vertex v_2 to u_2, as well as adds the hang vertex z_j to x_j, adds the hang vertex ω_j to y_j for $j\in [1, m_1+m_2]$. Fig.5 shows a diagram of T_3. Next, we label the newly added vertices a_1, v_2, z_j, ω_j in the following ways:

$$\begin{cases} f_3(z_j)=4(m_1+m_2)+3-f_3(x_j)=4(m_1+m_2-j)+2, \\ f_3(w_j)=4(m_1+m_2)+3-f_3(y_j)=4j, \\ f_3(v_2)=4(m_1+m_2)+3-f_3(u_2)=0, \\ f_3(a_1)=4(m_1+m_2)+3-f_3(a_2)=4(m_1+m_2)+2. \end{cases} \quad j\in[1,m_1+m_2].$$

Next, we prove that f_3 is a strongly graceful labelling of T_3. It is not difficult to see that T_3 has a perfect matching $M_3=\{a_1a_2, x_jz_j, y_j\omega_j, u_2v_2: j\in[1, m_1+m_2]\}$, and

$$\begin{cases} f_3(a_1a_2)=|f_3(a_1)-f_3(a_2)|=4(m_1+m_2)+1, \\ f_3(x_jz_j)=|f_3(x_j)-f_3(z_j)|=|4(m_1+m_2-2j)+1|, \\ f_3(y_j\omega_j)=|f_3(y_j)-f_3(\omega_j)|=|4(m_1+m_2-2j)+3|, \\ f_3(u_2v_2)=|f_3(u_2)-f_3(v_2)|=4(m_1+m_2)+3. \end{cases} \quad j\in[1,m_1+m_2].$$

and

$$\begin{cases} f_3(a_2x_j)=|f_3(a_2)-f_3(x_j)|=4j, \\ f_3(x_jy_j)=|f_3(x_j)-f_3(y_j)|=|4(m_1+m_2-2j)+2|, j\in[1,m_1+m_2]. \\ f_3(a_2u_2)=|f_3(a_2)-f_3(u_2)|=4(m_1+m_2)+2. \end{cases}$$

So, $f_3(E(T_3))=[1, 4(m_1+m_2)+3]$. Let $\lambda_2=4(m_1+m_2)+3$. For the perfect matching edges in M_3, we can calculate that $f_3(a_1)+f_3(a_2)=\lambda_2$, $f_3(u_2)+f_3(v_2)=\lambda_2$, $f_3(x_j)+f_3(z_j)=\lambda_2$ and $f_3(y_j)+f_3(\omega_j)=\lambda_2$ with $j\in [1, m_1+m_2]$. According to the definition (1), T_3 is a strongly graceful tree.

P_5: Split T_3 into $S(P_2)$ gradually. It has two vertices a_1 and a_2 on the main path, and the vertex a_1 connect m_1 4-legs $a_1z_jx_jy_j\omega_j$ with $j\in$ [1, m_1]; the vertex a_2 connect m_2+1 legs, where m_2 4-legs $a_2x_jz_j\omega_jy_j$ with $j\in$ [m_1+1, m_1+m_2] and one 2-legs of $a_2u_2v_2$. Since $|f_3(a_2x_j)|=4j$ and $|f_3(a_1z_j)|=4j$ for $j\in$ [1, m_1], so, the tree $T(a_1)$ also has the property of strongly graceful tree, and

$$\begin{cases} |f_3(x_jy_j)|=|4(m_1+m_2-2j)+2|, \\ |f_3(z_j\omega_j)|=|4(m_1+m_2-2j)+2|. \end{cases} j\in[m_1+1, m_1+m_2].$$

Hence, the tree $T(a_2)$ holds the property of strongly graceful tree. Thereby, $S(P_2)$ is a strongly graceful tree when m=1.

P_6: Define a labelling h of the tree $S(P_2)$ as: $h(V(S(P_2)))=f_3(V(S(P_2)))+|V(S(P_2))|$, $h(E(S(P_2)))=f_3(E(S(P_2)))$, where $|V(S(P_2))|=4(m_1+m_2+1)$. Thus $h(V(S(P_2)))\rightarrow[|V(S(P_2))|, 2|V(S(P_2))|-1]$, $h(E(S(P_2)))=[1, |V(S(P_2))|-1]$.

So, $h(V(S(P_2)))\cup h(E(S(P_2)))=[1, 2|V(S(P_2))|-1]$. Each perfect matching edge $uv\in S(P_2)$ holds $h(u)+h(v)=3|V(S(P_2))|-1$ true. We claim that the tree $S(P_2)$ is a strongly STG-tree. We prove that $S(P_4)$ is the strongly STG-tree, when m=2.

Q_1: Construct a binary star T_1^* having vertex set $V(T_1^*)=\{a_2, a_3, x_1, x_2, \ldots, x_{m_1+m_2}, x_1^*, x_2^*, \ldots, x_{m_3+m_4}^*\}$ and edge set $E(T_1^*)=\{a_2x_j, a_2a_3, a_3x_k^* : j\in [1, m_1+m_2], k\in [1, m_3+m_4]\}$ (see Fig.6). Define a labelling g_1 of T_1^* by letting $g_1(a_2)=0$, and for $j\in$ [1, m_1+m_2] and $k\in$[1, m_3+m_4], setting $g_1(x_k^*)=k$, $g_1(a_3)=m_3+m_4+1$ and $g_1(x_j)=j+m_3+m_4+1$.

Q_2: Define another labelling $g_2(x)$ of T_1^* by $g_2(x)=2g_1(x)$ fir each vertex $x\in V(T_1^*)$.

Q_3: Adding a hang vertex to each vertex of T_1^*, we get T_2^* (see Fig.7). In T_2^*,a hang vertex u_2 is added to a_2, a hang vertex y_j is added to x_j, a hang vertex u_3 is added to a_3, a hang vertex y_k^* is added to x_k^*. For $j\in$ [1, m_1+m_2] and $k\in$[1, m_3+m_4], we label the new vertices as $g_2(u_2)=2(m_1+m_2+m_3+m_4)+3-g_2(a_2)=2(m_1+m_2+m_3+m_4)+3$, $g_2(y_j)=2(m_1+m_2+m_3+m_4)+3-g_2(x_j)=2(m_1+m_2-j)+1$, $g_2(y_k^*)=2(m_1+m_2+m_3+m_4)+3-g_2(x_k^*)=2(m_1+m_2+m_3+m_4-k)+3$, $g_2(u_3)=2(m_1+m_2+m_3+m_4)+3-g_2(a_3)=2(m_1+m_2)+1$.

We can show that g_2 is a strongly graceful labelling of T_2^* in the following. Clearly, the tree T_2^* has a perfect matching $M_2^*=\{a_2u_2, x_jy_j, a_3u_3, x_k^*y_k^* : j\in [1, m_1+m_2], k\in [1, m_3+m_4]\}$. For $j\in$ [1, m_1+m_2] and $k\in$ [1, m_3+m_4], by the definition of the labelling g_1, we know that the tree T_1^* meets $g_1(a_2x_j)$=[m_3+m_4+2,

m_1+m_2+m_3+m_4+1], $g_1(a_2a_3)$=m_3+ m_4+1 and $g_1(a_3 x_k^*)$=[1, m_3+m_4]. Therefore, we obtain $g_2(a_2x_j)$={2(m_3+m_4)+2), 2(m_3+m_4+3), ..., 2(m_1+m_2+m_3+m_4)+2}, $g_2(a_2a_3)$=2(m_3+m_4+1), $g_2(a_3 x_k^*)$={2, 4, ..., 2(m_3+m_4)}. And $g_2(a_2u_2)$=|$g_2(a_2)$–$g_2(u_2)$|=2(m_1+m_2+m_3+m_4)+3, $g_2(x_jy_j)$=|$g_2(x_j)$–$g_2(y_j)$|=|2(m_3+ m_4–m_1–m_2+2j) +1|, $g_2(a_3u_3)$=|$g_2(a_3)$–$g_2(u_3)$|=|2(m_3+m_4–m_1– m_2)+1|, $g_2(x_k^*y_k^*)$=|$g_2(x_k^*)$–$g_2(y_k^*)$|= |2(m_1+m_2 +m_3+m_4–2k)+3| with $j\in$ [1, m_1+m_2] and $k\in$ [1, m_3+m_4]. Finally, we have shown that $g_2(E(T_2^*))$=[1, 2(m_1+m_2+m_3+m_4)+3]. Let λ_3=2(m_1+ m_2+ m_3+ m_4)+3, for the perfect matching edges in M_2^*, we have

$$g_2(x_j)+g_2(y_j)=\lambda_3, g_2(x_k^*)+g_2(y_k^*)=\lambda_3,\ g_2(a_2)+g_2(u_2)=\lambda_3, g_2(a_3)+g_2(u_3)=\lambda_3.$$

According to the definition of strongly graceful tree, T_2^* is a strongly graceful tree.

Q_4: Define a new labelling $g_3(\omega')$ of T_2^* as: $g_3(\omega')=2g_2(\omega')+1$ for $\omega'\in V(T_2^*)$, for $j\in$ [1, m_1+m_2] and $k\in$ [1, m_3+m_4], we have $g_3(a_2)=2g_2(a_2)+1=1$,

$$g_3(x_j)=2g_2(x_j)+1=4(m_3+m_4+j)+5,$$

$$g_3(y_j)=2g_2(y_j)+1=4(m_1+m_2\text{-}j)+3$$

$$g_3(u_2)=2g_2(u_2)+1=4(m_1+m_2+m_3+m_4)+7,$$

$$g_3(x_k^*)=2g_2(x_k^*)+1=4k+1\ \ g_3(a_3)=2g_2(a_3)+1=4(m_3+m_4)+5,$$

$$g_3(u_3)=2g_2(u_3)+1=4(m_1+m_2)+3$$

$$g_3(y_k^*)=2g_2(y_k^*)+1=4(m_1+m_2+m_3+m_4\text{-}k)+7.$$

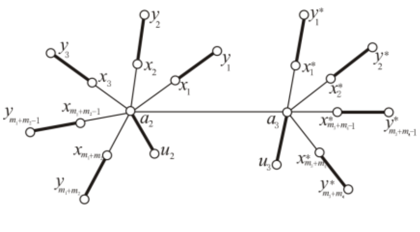

Fig. 7 The tree T_2^*.

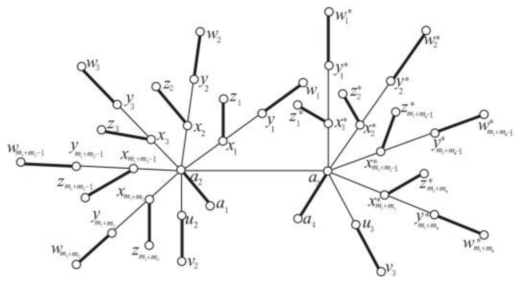

Fig. 8 The tree T_3^*.

We construct the tree T_3^* by adding a hang vertex to each vertex of T_2^*, add a hang vertex a_1 to a_2, a hang vertex v_2 to u_2, a hang vertex z_j to x_j and a hang vertex w_j to y_j with $j\in$ [1, m_1+m_2]. Similarly, we add a hang vertex a_4 to a_2, add a hang vertex v_3 to u_3, add a hang vertex z_k^* to x_k^*, and a hang vertex ω_k^* to y_k^*

for $k\in[1, m_3+m_4]$. Fig.8 gives a diagram of T_3^*. For $j\in[1, m_1+m_2]$ and $k\in[1, m_3+m_4]$, we label the new vertices by

$$\begin{cases} g_3(z_j)=4(m_1+m_2+m_3+m_4)+7-g_3(x_j)=4(m_1+m_2-j)+2, \\ g_3(\omega_j)=4(m_1+m_2+m_3+m_4)+7-g_3(y_j)=4(m_3+m_4+j)+4, \\ g_3(v_2)=4(m_1+m_2+m_3+m_4)+7-g_3(u_2)=0, \\ g_3(a_1)=4(m_1+m_2+m_3+m_4)+7-g_3(a_2)=4(m_1+m_2+m_3+m_4)+6, \\ g_3(z_k^*)=4(m_1+m_2+m_3+m_4)+7-g_3(x_k^*)=4(m_1+m_2+m_3+m_4-k)+6, \\ g_3(\omega_k^*)=4(m_1+m_2+m_3+m_4)+7-g_3(y_k^*)=4k, \\ g_3(v_3)=4(m_1+m_2+m_3+m_4)+7-g_3(u_3)=4(m_3+m_4)+4, \\ g_3(a_4)=4(m_1+m_2+m_3+m_4)+7-g_3(a_3)=4(m_1+m_2)+2. \end{cases}$$

Notice that T_3^* has a perfect matching $M_3^*=\{a_1a_2, x_jz_j, y_jw_j, u_2v_2, a_3a_4, x_k^*z_k^*, y_k^*w_k^*, u_3v_3: j\in[1, m_1+m_2], k\in[1, m_3+m_4]\}$. Let $\lambda_4=4(m_1+m_2+m_3+m_4)+7$. For the perfect matching edges in M_3^*, we can calculate $g_3(a_1)+g_3(a_2)=\lambda_4$, $g_3(x_j)+g_3(z_j)=\lambda_4$, $g_3(y_j)+g_3(w_j)=\lambda_4$, $g_3(u_2)+g_3(v_2)=\lambda_4$, $g_3(a_3)+g_3(a_4)=\lambda_4$, $g_3(x_k^*)+g_3(z_k^*)=\lambda_4$, $g_3(y_k^*)+g_3(w_k^*)=\lambda_4$, $g_3(u_3)+g_3(v_3)=\lambda_4$. Since $g_3(E(T_3^*))=[1, 4(m_1+m_2+m_3+m_4)+7]$, so g_3 is a strongly graceful labelling of T_3^*.

Q_5: Split T_3^* into $S(P_4)$ gradually. The main path has four vertices a_1, a_2, a_3, a_4, the vertex a_1 has m_1 4-legs: $a_1z_jx_jy_jw_j$ with $j\in[1, m_1]$; the vertex a_2 connects m_2+1 legs including m_2 4-legs $a_2x_jz_jqw_jy_j$ with $j\in[m_1+1, m_1+m_2]$ and one 2-legs $a_2u_2v_2$; the vertex a_3 connects m_3+1 legs including m_3 4-legs $a_3x_k^*z_k^*w_k^*y_k^*$ for $k\in[1, m_3]$, and one 2-legs $a_3u_3v_3$; the vertex a_4 connects m_4 4-legs $a_4z_k^*x_k^*y_k^*w_k^*$ with $k\in[m_3+1, m_3+m_4]$. Since

$$\begin{cases} |g_3(x_j)-g_3(a_2)|=4(m_3+m_4+j)+4, \\ |g_3(a_1)-g_3(z_j)|=4(m_3+m_4+j)+4. \end{cases} j\in[1,m_1]$$

Hence, the tree $T(a_1)$ satisfies the nature of the strongly graceful tree, and

$$\begin{cases} |g_3(x_j)-g_3(y_j)|=|4(m_3+m_4-m_1-m_2+2j)+2|, \\ |g_3(z_j)-g_3(\omega_j)|=|4(m_3+m_4-m_1-m_2+2j)+2|. \end{cases} j\in[m_1+1,m_1+m_2]$$

Furthermore, the tree $T(a_2)$ has the nature of the strongly graceful tree. In the same way

$$\begin{cases} |g_3(y_k^*)-g_3(x_k^*)|=4(m_1+m_2+m_3+m_4-m_1-m_2-2k)+6, \\ |g_3(z_k^*)-g_3(\omega_k^*)|=4(m_1+m_2+m_3+m_4-m_1-m_2-2k)+6. \end{cases} k\in[1,m_3]$$

$$\begin{cases} | g_3(a_4) - g_3(z_k^*) | = 4(m_3 + m_4 - k) + 4, \\ | g_3(a_3) - g_3(x_k^*) | = 4(m_3 + m_4 - k) + 4. \end{cases} k \in [m_3 + 1, m_3 + m_4]$$

Therefore, tree $T(a_3)$ and $T(a_4)$ satisfy the definition of the strongly graceful tree. Since the labels of the perfect matching edges a_1a_2 and a_3a_4 on the main path of T_3^* keep no change, so $S(P_4)$ is a strongly graceful tree.

Q_6: Similarly to the step P_6, the labelling of the tree P_4 is defined as:

$$h(V(S(P_4)))=g_3(V(S(P_4)))+|V(S(P_4))|,$$
$$h(E(S(P_4)))=g_3(E(S(P_4))),$$

where $|V(S(P_4))|=4(m_1+m_2+ m_3+m_4)+8$. Obviously, we obtain $h:V(S(P_2)) \rightarrow [|V(S(P_4))|,\ 2|V(S(P_4))|-1]$, $h(E(S(P_2)))=[1,\ |V(S(P_4))|-1]$.

Therefore, $h(V(S(P_4)) \cup h(E(S(P_4)))=[1,\ 2|V(S(P_4))|-1]$. Notice that each perfect matching edge $uv \in S(P_4)$ holds $h(u)+h(v)=3|V(S(P_4))|-1$ true. The tree $S(P_4)$ is strongly STG-tree.

When T_1 is a caterpillar tree, and the main path of T_1 is $P=a_2a_3\ldots a_ia_{i+1}$ ($i \geq 2$, $i-2 \equiv 0$ (mod4)). Since T_1 has a SoG-labelling $f_1(x)$, we can construct a new labeling $f_2(x)=2f_1(x)$ for each vertex $x \in V(T_1)$, and add a new hang vertex to each vertex of T_1, then we get the tree T_2.

Obviously, the tree T_2 has a perfect matching M_2. Next, we define a new labelling of T_2 as $f_3(w)=2f_2(w)+1$ for every vertex $w \in V(T_2)$, and add a hang vertex to each vertex of tree T_2 for producing a tree T_3. Clearly, the tree T_3 has a perfect matching M_3. Next, we split tree T_3 into the tree $S(P_{2m})$ for $m \geq 1$. Finally, we label each vertex of $S(P_{2m})$ by $h(\mathrm{V}(S(P_{2m})))=f_3(\mathrm{V}(S(P_{2m})))+|V(S(P_{2m}))|$, so the edge labels of $S(P_{2m})$ are not changed. By induction, $S(P_{2m})$ is a strongly STG-tree.

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