The Comparison of Reliability of the Hypercube and Crossed Cube

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Abstract. The crossed cube is a variation of the hypercube. In this paper, we supply the topological property of the two network structures, and compare the all-terminal reliability and node-connected reliability of them. It is obtained that the crossed cube is superior to the hypercube in the lower bound of two kinds of reliability.

Introduction

The hypercube network Q_n has become one of the most popular interconnection networks for its simple structure and nice properties. The crossed cube CQ_n is an attractive alternative for the ordinary hypercube. It is a more generalized version of the twisted hypercube which is obtained by interchanging a pair of edges of the ordinary hypercube. Both the crossed cube and the ordinary hypercube have the same number of vertices and the same node degree. Recent research studies have revealed that the crossed cube, as an interconnection network, possesses many desirable properties which are not available in ordinary hypercube [1]. The crossed cube has a half diameter of the hypercube approximately [2]. Also it contains the optimal complete binary tree as subgraph whereas the ordinary hypercube does not[3].

In this paper, we supply the topological property of the two network structures by proving lower bounds of all-terminal reliability and node-connected reliability, and draw a conclusion that the crossed cube is superior to the hypercube in the lower bound of two kinds of reliability.

Basic definitions

The n-dimensional hypercube Q_n and the n-dimensional crossed cube CQ_n are n-regular graphs with 2^n vertices and $2^{n-1}n$ edges. A vertex in CQ_n or Q_n will be represented by a binary string $v_{n-1}v_{n-2}\cdots v_1v_0$ of length n in Q_n, the string is defined to be the address of the vertex, two vertices are adjacent if and only if their binary addresses differ only in one bit position .

Definition 1.The n-dimensional crossed cube CQ_n is the labeled graph defined recursively as follows.

CQ_1 is the complete graph of two vertices labeled by 0 and 1.

For $n \geq 2$, CQ_n is obtained by taking two copies of CQ_{n-1}, denoted by CQ_{n-1}^0 with vertex-set

$V(CQ_{n-1}^0) = \{0x_{n-2}x_{n-3}\cdots x_0 : x_i = 0\, or\, 1\}$

and CQ_{n-1}^1 with vertex-set

$V(CQ_{n-1}^1) = \{0x_{n-2}x_{n-3}\cdots x_0 : x_i = 0\, or\, 1\}$

respectively, and adding an edge joining

$0x_{n-2}x_{n-3}\cdots x_0 \in V(CQ_{n-1}^0)$

and $1y_{n-2}y_{n-3}\cdots y_0 \in V(CQ_{n-1}^1)$

if and only if

(1) $x_{n-2} = y_{n-2}$ if n is even, and

(2) $(x_{2i+1}x_{2i}y_{2i+1}y_{2i}) \in \{(00,00),(10,10),(01,11),(11,01)\}$ for $0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$.

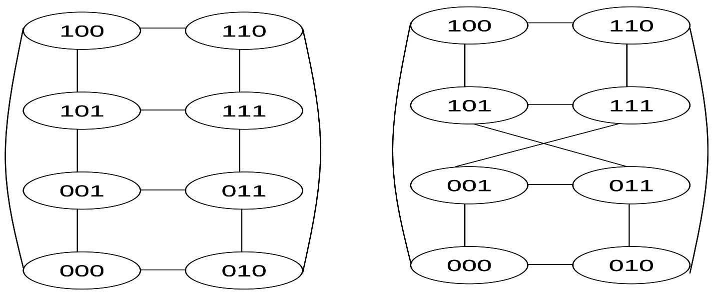
Definition 2

Suppose the vertices of graph G never fail and the edges fail independently with probability p, then the all-terminal reliability of graph G is $R(G,p) = \sum_{i=1}^{m} N_i(G)P^i q^{m-i}$, where $N_i(G)$ is the number of connected induced subgraphs of G that contain exactly r edges.

Definition 3

Suppose the edges of graph G never fail and the vertices fail independently with probability $1-p$, then the node-reliability of graph G is $R(G,p) = 1 - \sum_{i=1}^{m} B_i(G)P^i q^{m-i}$, where $B_i(G)$ is the number of disconnected subgraphs of G that contain exactly vertices.

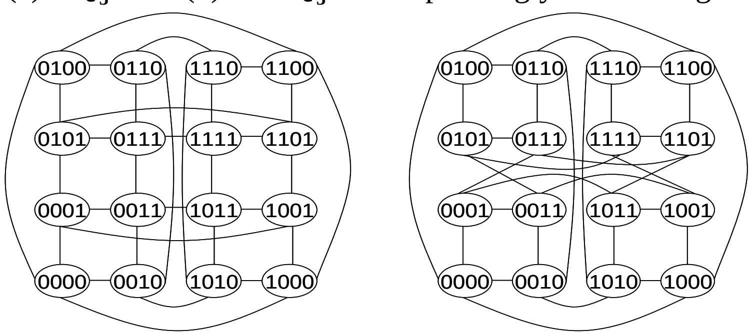
The graphs shown in Fig1.(a)and (b) are Q_3 and CQ_3 respectively.



$(a)Q_3$ $(b)CQ_3$

Fig.1 $(a)Q_3$ $(b)CQ_3$

The graphs shown in Fig2.(a)and (b) are Q_4 and CQ_4,respectively. (Take graphs in Fig.1 and Fig.2 as an example. (a) is Q_3 and (b) is CQ_3 correspondingly. It is analogous in Fig.2)



$(a)Q_4$ $(b)CQ_4$

Fig.2 $(a)Q_4$ $(b)CQ_4$

The compare for the all terminal reliability of Q_n and CQ_n

If $n < 3$, $R(Q_n) = R(CQ_n)$.

If $n = 3$, using factoring theorem[4], the all terminal reliability of graph Q_3 and CQ_3 can be obtained as follows

$$R(Q_3) = p^{12} + 12p^{11}q + 66p^{10}q^2 + 212p^9q^3 + 408p^8q^4 + 384p^7q^5$$

$$R(CQ_3) = p^{12} + 12p^{11}q + 66p^{10}q^2 + 212p^9q^3 + 409p^8q^4 + 392p^7q^5$$

Therefore $R(Q_3) \le R(CQ_3)$.

If $n \ge 4$,it is hard to calculate the all terminal reliability of graph Q_n and CQ_n, but we can obtain the lower bound of all-terminal reliability because they are recursive networks from lower dimension to higher dimension . Q_4 is obtained by taking two copies of Q_3 and adding eight edges between the two copies , so we obtain the inequality of all terminal reliability of Q_4 as follows

$$R(Q_4) \geq 8pq^7(p^{12}+12p^{11}q+66p^{10}q^2+212p^9q^3+409p^8q^4+392p^7q^5)$$

CQ_4 is also obtained by taking two copies of CQ_3 and adding eight edges in the way of definition 1, so the inequality of all terminal reliability of Q_4 can also be obtained in the same way.

$$R(CQ_4) \geq 8pq^7(p^{12}+12p^{11}q+66p^{10}q^2+212p^9q^3+408p^8q^4+384p^7q^5)$$

If $n \geq 4$, we can obtain the lower bound of $R(Q_n)$ and the lower bound of $R(Q_n)$ as follows

$$(\prod_{i=4}^{n-1} 2^{i-1} pq^{2^{i-1}-1})(p^{12}+12p^{11}q+66p^{10}q^2+212p^9q^3+409p^8q^4+392p^7q^5)$$

$$(\prod_{i=4}^{n-1} 2^{i-1} pq^{2^{i-1}-1})(p^{12}+12p^{11}q+66p^{10}q^2+212p^9q^3+409p^8q^4+384p^7q^5)$$

It is clear that the lower bound of $R(Q_n)$ is greater than he lower bound of $R(CQ_4)$.

The compare for the node-connected reliability of Q_n and CQ_n

If $n < 3$, $R(Q_n) = R(CQ_n)$.

If $n = 3$, node-connected reliability is calculated by dropping fault points .

For Q_3, $B_1(G) = 0$, $B_2(G) == 16$, $B_3(G) = 34$.

It is not easy to calculated $B_4(G)$, so we study $B_4(G)$ by the adjacent matrix of Q_3, the element 1 in the adjacent matrix of Q_3 indicates that two vertices are connected in Q_3 , and the element 0 in the adjacent matrix of Q_3 indicates that two vertices are disconnected in Q_3.

Dropping four vertices of Q_3 acts as dropping 4 rows and 4 columns simultaneously in the adjacent matrix, which gives rise to a remaining matrix of order 4.There are $c_8^4 = 70$ different remaining matrices, Among these, the matrix in which the amount of element 1 is less than 3, is disconnected , therefore $B_4(G) = 32$.

Considering that 3 fault vertices must be adjacent to the same vertices to make the remaining vertices disconnected, it is gained that $B_5(G) = 8$.

Because 3- dimensional hypercube is 3-connected, it is definite that the remained vertices become connected graph when the fault points is less than 3.Therefore

$B_6(G) = 0$, $B_7(G) = 0$, $B_8(G) = 0$

It turns out the node-connected reliability

$$R(Q_3, p) = 1 - \{16p^2(1-p)^6 + 34p^3(1-p)^5 + 32p^4(1-p)^4 + 8p^5(1-p)^3\}$$

The node-connected reliability of CQ_3 is calculated in the same way ,

$$R(CQ_3, p) = 1 - [16p^2(1-p)^6 + 34p^3(1-p)^5 + 26p^4(1-p)^4 + 8p^5(1-p)^3]$$

So we obtained $R(Q_3, p) > R(CQ_3, p)$, the reliability of graph CQ_3 is superior to Q_3.

If $n \geq 4$, the vertices increase by 2^n times .It is difficult to calculate the reliability of graph Q_n and CQ_n ,but they are recursive networks from lower dimensional to higher dimensional, we can obtain the lower bound of node-reliability

$$R(H_n, p) \geq 1 - (1-p)^{(2^n-2^3)}[16p^2(1-p)^6 + 34p^3(1-p)^5 + 32p^4(1-p)^4 + 8p^5(1-p)]$$

$$R(CQ_n, p) \geq 1 - (1-p)^{(2^n-2^3)}[16p^2(1-p)^6 + 34p^3(1-p)^5 + 26p^4(1-p)^4 + 8p^5(1-p)]$$

It is clear that the lower bound of $R(Q_n)$ is greater than he lower bound of $R(CQ_4)$.

Conclusions

We supply the topological property of the two network structures by proving lower bounds of all-terminal reliability and node-connected reliability, and draw a conclusion that the crossed cube is superior to the hypercube in the lower bound of two kinds of reliability.

Acknowledgments

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