Large Time Stability Control for a Class of Quasilinear Parabolic and Hyperbolic Equations

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Abstract. In this paper, we concerned with the stability control of solutions for a class of quasi-linear wave equations with weak attenuation. We use a so-called energy perturbation method to establish weak controllability of solutions in terms of energy norm for a class of nonlinear functions. We show the weak controllability with the help of differential inequalities by estimating the relationship between energy inequalities and attenuating property of weak solutions.

Introduction

We consider the control problem of weak decay to the following so-called quasi-linear wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$

$$u_{tt} - \lambda \Delta u_t + u \cdot \nabla u = f(x, t) \quad (1.1)$$

$$\operatorname{div} u = 0 \quad (1.2)$$

$$u|_{\partial\Omega} = 0 \quad (1.3)$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1 \quad (1.4)$$

where u_0, u_1, f are given functions, Δu is a Laplacian with respect to the variable $x \in \Omega$, $u = u(t, x)$ is an unknown function, $\lambda > 0$ is a fixed positive number, $f \in L^2(\Omega)$ are given external forces, and satisfying the following conditions:

$$f \in L_2, \quad u \in L_q(R^+, L_p) \quad \text{and} \quad \frac{2}{p} + \frac{n}{q} \leq 1 \quad (1.5)$$

For some positive $a, p \in [1/2, 2), C > 0$, and $q > 0$, we have a weak solution, which fulfills additionally

$$d + a|x|^p \leq f'(x, t) \leq d(1 + |x|^q), \quad \forall s \in \mathbb{R}^3 \quad (1.6)$$

And

$$u_{tt} - \operatorname{div} \lambda \nabla u^2 - \Delta u_t = f(x, t) \quad (1.7)$$

We have several techniques to prove the existence of weak decay solutions with respect to the phase space $X^\ell \times X^\tau$, $X^\varepsilon := D(\Omega^{q/p})$, and have additional nice properties with energy inequality for almost all times or solutions with weak decay properties for $t \rightarrow \infty$, this has been studied recently by several people, e.g. Y.M. Qin, Ebihara, Xin Liu [1-6] etc.

As a model of the quasi-linear equation, for $N = 1/2, 1$ and $f = 0, 1$ equation (1.1)-(1.4) admits a global weak decay solution as large initial data, which was proved by Y.M. Qin, Xin Liu, X.G. Yang, Lan Huang etc [2-3]. TG Wang, Ming Zhang, M.J. Wang simplified the above arguments and give the proof of control with exponential decay. From the perspective and background of physics, this represents an implementation of an axial movement of the viscoelastic material, this causes the form of above equations, in the one-dimensional case, their model of longitudinal vibration of a uniform rod with nonlinear stress function f . In two, three-dimensional case, they describe the viscoelastic solid of anti-plane shear action. While $n = 1$ and $f = 0$, S.T. Li prove the existence of

weak periodic decay strong solution on the periodicity condition, X.K. Su and JL Zhang[3-6] proved the controllability of a smooth solution in the method of Cauchy problems in the case of smooth and small data.

Ulteriorly, while $n > 1/2$ and $f \neq 0$, M.J. Wang and X.G. Yang gave the proof of global controllability with a smooth solution in the case of small initial data. Make use of combining L^p -theorem of Sobolev space and semigroup theorem of operators, Nakao [4] and T. Kato [5] devised certain decay rate of the energy of global solutions with large data under a specific condition which is certainly satisfied if the mean curvature of the boundary $\partial\Omega$ is non-positive. For $n > 1$ and $f = 1$, nonlinear elliptic equation with periodicity conditions was studied [10],

$$d'(t) + k_1 \|w_1(t) - w_2(t)\|_p^\nu d(t) \leq k_2 d(t)^{p+q-1} + \kappa d(t)^\mu \|w_1(t)\|_2^{\nu+1} \quad (1.8)$$

$$\|w(t) - w_0(t)\|_p \leq k(\lambda + x)^{\frac{d}{2} - \frac{n}{2}(\frac{\kappa}{2} - \frac{1}{p})} \quad (1.9)$$

$$w(x, t + \omega) = w(x, t) \quad (1.10)$$

$$w|_{\partial} = 0 \quad (1.11)$$

In the case of $\|w(t)\| \leq k_0(1+t)^{-\frac{\lambda}{2}}$, and $\|w(t)\|_p \leq \tau(1+t)^{(\frac{\tau}{2} + \frac{n}{2}(\frac{1}{2} - \frac{1}{p}))}$. Here, Ω is a bounded domain in R^n with a smooth, $\partial\Omega$ is said to be C^2 class boundary, which satisfies the following uniform hyperbolic assumption:

For some constants $\rho_0, M > 0$, $\tau \in H^4[0, +\infty)$ satisfies:

$$x'(t) + \tau_0(1+t)^{(\ell + \frac{d}{2})} x(t)^{(2 + \frac{\gamma}{p})} \leq \sigma y(t)^{\frac{\nu}{2}} + \tau_1(1+t)^\delta y(t)^\gamma \quad (1.12)$$

With $w(t) = u(t) - u_0(t)$, $y(t) = \|w(t)\|_p^m$ and

$$\|w_t\| - \Delta w + \|u(r)\|_q = \frac{n}{2} + 1 - pn + 2p\delta$$

$$\tau(v^2) \geq \sigma_0 > 0 \text{ and } \tau(v^2) + 2\sigma_0(v)v^2 \geq \delta_0 > 0 \quad (1.13)$$

$$\tau(v^2) + 2\sigma_0(v^2)v + \tau''(v^2) \leq M < \infty \quad (1.14)$$

Thus, we can discuss and conclude some results with lemmas in the next text[4-6].

Preliminary Knowledge

Lemma 1. Let $f(x, t) \in H^1([0, \omega], L^2(\cdot))$, and then with the above assumption, the problem admits a ω -periodic solution $u(t)$ and the following estimates hold

$$\sup_{t \in [0, \omega]} \{ \|u(t)\|^2 + \|\nabla u_t(t)\|^2 \} + \int_0^\omega \|u_t(s)\|^2 ds \leq CM_0^2 \quad (1.15)$$

$$\sup_{t \in [0, \omega]} \{ \|u_t(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \} + \int_0^\omega \|u_{tt}(s)\|^2 ds \leq C(M_0^2 + M_1^2) \quad (1.16)$$

Where

$$M_i^2 = \int_0^\omega \left\| \frac{\partial^i}{\partial t^i} f(s) \right\|^2 ds, \quad i = 0, 1 \quad (1.17)$$

Lemma 2. Let $u_0, u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f(x, t) \in H^1(I, L^2(\cdot))$, then under assumption H^1 , the problem admits a unique solution $u(t)$ and furthermore, if $\|f(t)\|^2 + \|f_t(t)\|^2 = O(e^{-\alpha t})$ for some constant $\alpha > 0$ as $t \rightarrow \infty$, the estimate

$$\|u_n\|^2 + \|\Delta u\|^2 + \|\nabla u_t(t)\|^2 + \|\Delta u_t\|^2 + \int_t^\infty \|\nabla u_n\|^2 ds \leq C e^{-\beta t} \quad (1.18)$$

Holds [3] for some constant $\beta > 0$, where $C > 0$ is a constant depending on $\|u_0\|_{H^2} + \|u_1\|_{H^2}$.

It turns out that this has an influence on the control of weak solution with structural damping. We can continue the following argument.

Main Results

We focus on the control of weak decay stability in (1.1)-(1.4), in order to describe the maneuverability; let us define global weak solution with decay for equations (1.1)-(1.4):

If the initial data satisfies $u_0, u_1 \in H^2 \cap H_0^4$, construct a weak solution space with

$$\Theta = \left\{ \Gamma : \times w_{1/2}^2 \rightarrow R \mid \Gamma \in L^\infty(w_{1/2}^2, H^2(\cdot) \cap H_0^4(\cdot)), \Gamma' \in L^\infty(w_{1/2}^2, H_0^4(\cdot)) \right\} \quad (2.1)$$

The function ω is said to be a weak solution of Problem (1.1)-(1.4), if it satisfies the following conditions:

(a) $\omega \in \Theta$

$$\begin{aligned} (b) \quad E(t) &\equiv \frac{1}{2} \left\{ \|\omega_t(t)\|^2 + \int_\Omega \int_0^t |\nabla \omega|^2 \tau(\xi) d\xi dy \right\} \\ &= \int_0^t (f, \omega(t)) ds dy - \int_0^t \|\nabla \omega_t(t)\|^2 ds + \frac{1}{2} \left\{ \|\nabla \omega_1(t)\|^2 + \int_\Omega \int_0^t |\nabla \omega_0|^2 \tau(\xi) d\xi dy \right\} \end{aligned} \quad (2.2)$$

$$(c) \quad (\omega_t, \mu)_0^t + \int_0^t \left\{ (\omega_t, \mu_t) - (\operatorname{div}(\tau \nabla \omega), \mu) - (\Delta \omega_t, \mu) - (f, \mu) \right\} = 0 \quad (2.3)$$

It is well known that the existence of such a weak solution with decay for all times is assured. Once this is known, one can identify this solution with the global weak one and continue this process to get that

$$(\partial_t^2 u, \phi) = \rho(\partial_t v, \Delta_x \phi) + (v, \Delta_x \phi) - (\varphi'(\nabla_x \omega), \nabla_x \phi) - (f(u), \phi) + (g, \phi) \quad (2.4)$$

$$\text{And} \quad t^2 \|\partial_t v\|_H^2 \leq \lambda t^2 \|\partial_t v\|_{L^\infty}^{1/2} \|\partial_t v\|_{L^2}^{3/2} \leq \delta t^3 \|\partial_t v\|_{L^2}^3 + C^3 \rho^2 \delta^{-2} \|\partial_t \omega\|_{L^\infty}^2 \quad (2.5)$$

Obviously, we will construct the controllability of the local weak solution with decay for the semi-group generated by weak energy solutions.

Theorem 1. Let ω be a weak solution with the decay of (1.1)-(1.4). Denote by $\omega_0(t) = e^{-t\Delta}$ the solution of the wave equation and suppose $\|\omega_0(t)\|_2 \leq \lambda_0(1+t)^{-\tau/2}$, $\omega_0 \in H^2(\cdot) \cap H_0^4(\cdot)$, $\omega_1 \in H_0^4(\cdot)$, $f(x, t) \in H_0^2(\Theta, L^2(\cdot))$ and $f'(x, t) \in W_0^2(\Theta, L^2(\cdot))$, then problem(1.1)-(1.4) hold a unique local weak solution $\omega(t)$ with the following estimate:

$$y'(t) + \lambda_0(1+t)^{\frac{\kappa}{p}} y(t)^{(1+\frac{\varepsilon}{2}\kappa)} \leq \lambda_1 y(t)^{\frac{\gamma}{2}\mu} + \lambda_2(1+t)^{-q} y(t)^\sigma \cdot \nabla_x \omega(t) \quad (2.6)$$

$$\omega \in L^\infty([0, T], H^2(\cdot) \cap H_0^4(\cdot)), \omega' \in L^\infty(\Theta, H_0^4(\cdot)), \omega'' \in W_0^2(\Theta, L^2(\cdot)) \quad (2.7)$$

Conclusions

In this section, we will draw our conclusions by means of some methods in the preliminary knowledge.

In fact, for $n=3$ $\omega \in L_1 \cap L_p$, and $\|v(t) - v_0(t)\| \leq \sigma(1+t)^{\frac{\gamma}{3}(1-\frac{1}{q})}$ note that the initial data is dense, hence $\|v_0(t)\|$ attenuates exponentially fast, Indeed, using $\omega|\omega|^{\frac{2p}{n(p-2)}}$ and $\partial_t \omega$ multiply the equation (1.7) and integrate by parts over $x \in \Omega$. One gets

$$\frac{\partial}{\partial t} \left(\|\partial_t(\omega)\| + \lambda(\theta(\nabla u), 1) + \sigma \|\nabla \omega\|_{L^\infty}^2 + \frac{1}{2} (G(v), 1) + (\varphi, \varepsilon) \right) + \eta \|\nabla \partial_t h\|_{H_0^2}^2 = 0 \quad (3.1)$$

$$\delta'(t) + \lambda_1 (1+t)^{\kappa \frac{d}{2}} y(t)^{\frac{\lambda+\kappa}{p}} \leq \lambda_2 \delta(t)^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2} \right)} + \varepsilon (1+t)^{-(p-r)} \delta(t)^{\frac{1}{q}} \quad (3.2)$$

Where ε is a small positive number which will be fixed, Then we arrive at

$$\begin{aligned} \frac{\partial}{\partial t} \left(\delta \frac{\rho}{2} \|\partial_t(\omega)\|^2 + \lambda(\partial_t(\omega), \omega) \right) - \sigma \|\partial_t \omega\|_{L^\infty}^2 + \theta \|\nabla u\|^2 \\ + \tau (\varphi'(\nabla \omega), \nabla \omega) + \frac{1}{2} \nu (G(v), v) = \beta(g, \omega) \end{aligned} \quad (3.3)$$

And notice that

$$E(\xi_u(t)) = \frac{p-2}{p} \|\partial_t \omega\|_{L^2}^p + (\varphi(\nabla \omega), \kappa) - \frac{pn+4}{p+4-n} \|\nabla \omega\|_{L^2}^2 + \lambda(\partial_t \omega, \omega) \quad (3.4)$$

Let λ, α be small enough, one gets

$$\begin{aligned} \int g(|v|^2 + |\omega_0|^4) |u|^{\frac{pn}{n-2}} dx &\leq \sigma \left(\|v\|_{L^2}^{\frac{p}{2}} + \|\omega_0\|_p^4 \right) \|\omega\|_{H_0^2}^{\frac{p}{2}-1} \\ &\leq C \left(1 + \|g\|_{L^2}^n \right) + \|\partial_t v\|_{L^2}^{n-2} + \|\nabla_x u\|_{L^p}^{n+1} + \|v\|_{L^q}^{\frac{n+1}{p}} \\ &\leq C \|v\|_{L^p}^{n+1-p} \left(\beta(u)^{\frac{p}{n}} + \|v_0\|_{L^p}^{p+1} \right) \beta(u)^{2+\frac{n}{q}} \end{aligned} \quad (3.5)$$

Where the constant $C_\varepsilon, \lambda_\varepsilon$ depend only on the ε .

Hence by the standard Galerkin method, interpolation inequality, Young's and Sobolev's inequality, this section can be estimated by

$$(\varphi'(\xi_2) - \varphi'(\xi_1))(\xi_2 - \xi_1) \leq \theta(|\xi_2| - |\xi_1|)^p |\xi_2 - \xi_1|^2, \quad \forall \xi_1, \xi_2 \in R^3 \quad (3.6)$$

Noting that the energy solution of problem (1.1) is unique and the Lipschitz continuity in a weak space, There holds (2.6) and (2.7).

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