

# A Relation Between Binomial Coefficients and Fibonacci Numbers to the Higher Power

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**Abstract.** In this paper, we calculate high power of Fibonacci numbers by elementary mathematical methods and prove an interesting identity between the binomial coefficients and the high power of Fibonacci numbers.

## Introduction

Let us begin by recalling the binomial theorem  $(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$ , for arbitrary integers  $n$  and  $i$  with  $n \geq i \geq 0$ . The coefficient  $\binom{n}{i}$  are known as binomial coefficient and is given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}. \quad (1)$$

It is clear from the above formula (1) that  $\binom{n}{i}$  must be a positive integer. The binomial theorem, for arbitrary complex number  $x$  is given by  $(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$ .

Let  $F_l$  (for arbitrary non-negative  $l$ ) be the  $l$ -th Fibonacci number given by

$$F_l = \frac{\alpha^l - \beta^l}{\alpha - \beta},$$

and  $L_l$  be the  $l$ -th Lucas number defined by

$$L_l = \alpha^l + \beta^l.$$

For a long time, properties of binomial coefficients and Fibonacci numbers had been very interesting topics in 'combinatorics' and in 'number', and the identities relating binomial coefficients and Fibonacci numbers attracted many experts. We refer the reader to [1],[2],[3],[4],[5],[6] and [7] for more exhaustive details.

For arbitrary fixed nonnegative integer  $k$  and arbitrary fixed positive integer  $n$ , we denote the convolution of sequence  $\left\{ \binom{n}{i} \right\}_{i=0}^n$  and  $\{F_{k+i}^m\}_{i=0}^n$  by  $f(k, m, n)$  and define that as follows:

$$f(k, m, n) = \binom{n}{0} F_k^m + \binom{n}{1} F_{k+1}^m + \dots + \binom{n}{n} F_{k+n}^m = \sum_{i=0}^n \binom{n}{i} F_{k+i}^m \quad (2)$$

The question of calculating  $f(k, m, n)$  is being considered in [8],[9],[11],[12] and in [13]. For example when  $m=1, 2, 3$ , it had been proved that:

$$f(k, 1, n) = F_{k+2n}$$

$$f(k, 2, n) = \begin{cases} 5^{\frac{n}{2}} L_{2k+n}, 2 \mid n \\ 5^{\frac{n-1}{2}} F_{2k+n}, 2 \nmid n \end{cases}$$

$$f(k, 3, n) = \begin{cases} \frac{1}{5}(2^n F_{3k+2n} - (-1)^{k+n} 3F_{k-n}) & k \geq n \\ \frac{1}{5}(2^n F_{3k+2n} + 3F_{n-k}) & k < n. \end{cases}$$

Here  $F_{k+2n}, F_{2k+n}, F_{3k+2n}, F_{k-n}$  denotes the  $(k+2n)$ -th,  $(2k+n)$ -th,  $(3k+2n)$ -th and  $(k-n)$ -th Fibonacci numbers respectively. Also,  $L_{2k+n}$  denotes the  $(2k+n)$ -th Lucas numbers. Inspired by the above work, a natural question would be whether there exist a similar conclusion to the higher powers of the Fibonacci numbers as well. We try to explore this here in this work. To this end, let  $f(k, 4, n)$  be the

convolution of sequence  $\left\{ \binom{n}{i} \right\}_{i=0}^{i=n}$  and  $\{F_{k+i}^4\}_{i=0}^n$ , i.e.

$$f(k, 4, n) = \binom{n}{0} F_k^4 + \binom{n}{1} F_{k+1}^4 + \dots + \binom{n}{n} F_{k+n}^4 = \sum_{i=0}^{i=n} F_{k+i}^4. \quad (3)$$

In this paper, we consider this question and prove an identity between the binomial coefficients and the quartic of Fibonacci numbers. In other words, we prove the following result.

**Theorem** For arbitrary fixed nonnegative integer  $k$  and positive integer  $n$ , let  $f(k, 4, n)$  be the convolution of sequence  $\left\{ \binom{n}{i} \right\}_{i=0}^n$  and  $\{F_{k+i}^4\}_{i=0}^n$ , then

$$f(k, 4, n) = \frac{1}{25} [3^n L_{4k+2n} - 4(-1)^{(k+i)} \sqrt{5}^{n+1} F_{2k+n} + 6]. \quad (4)$$

## Proof of the theorem

**Proof.** It is known that,

$$F_l = \frac{\alpha^l - \beta^l}{\alpha - \beta}, \quad (5)$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, \quad (6)$$

Now using (5), (6) we get

$$\begin{aligned} F_l^4 &= \frac{1}{25} (\alpha^{4l} - 4\alpha^{3l} \beta^l + 6\alpha^{2l} \beta^{2l} - 4\alpha^l \beta^{3l} + \beta^{4l}) \\ &= \frac{1}{25} (\alpha^{4l} - 4(-1)^l \alpha^{2l} + 6(-1)^{2l} - 4\alpha^l \beta^{3l} + \beta^{4l}) \end{aligned} \quad (7)$$

According (2) and (7) we see that

$$f(k, 4, n) = \sum_{i=0}^n \binom{n}{i} F_{k+i}^4 = \frac{1}{25} \sum_{i=0}^n \binom{n}{i} X \quad (8)$$

where

$$X = \alpha^{4(k+i)} - 4(-1)^{(k+i)} \alpha^{2(k+i)} + 6(-1)^{2(k+i)} - 4(-1)^{(k+i)} \beta^{2(k+i)} + \beta^{4(k+i)}$$

We know from (2) that

$$\sum_{i=0}^n \binom{n}{i} \alpha^{4i} = (1 + \alpha^4)^n \quad (9)$$

and

$$\sum_{i=0}^n \binom{n}{i} \beta^{4i} = (1 + \beta^4)^n. \quad (10)$$

Now if we use the relations (6) and (7) we find that

$$1 - \alpha = \beta, 1 - \beta = \alpha, \alpha\beta = -1, 1 + \alpha^4 = 3\alpha^2, 1 + \beta^4 = 3\beta^2, 1 + \alpha^2 = \sqrt{5}\alpha \quad (11)$$

Substituting these into (11) we have,

$$\begin{aligned} f(k, 4, n) &= \frac{1}{25} \sum_{i=0}^n \binom{n}{i} (\alpha^{4k+4i} - 4(-1)^{(k+i)} \alpha^{2k+2i} + 6(-1)^{2k+2i} - 4(-1)^{k+i} \beta^{2k+2i} + \beta^{4k+4i}) \\ &= \frac{1}{25} [\alpha^{4k} \sum_{i=0}^n \binom{n}{i} \alpha^{4i} - 4(-1)^{(k+i)} \alpha^{2k} \sum_{i=0}^n \binom{n}{i} \alpha^{2i} + 6(-1)^{2k+2i} \sum_{i=0}^n \binom{n}{i} \alpha^{2i} \\ &\quad - 4(-1)^{k+i} \beta^{2k} \sum_{i=0}^n \binom{n}{i} \beta^{2i} + \beta^{4k} \sum_{i=0}^n \binom{n}{i} \beta^{4i}] \\ &= \frac{1}{25} [\alpha^{4k} (1 + \alpha^4)^n - 4(-1)^{(k+i)} \alpha^{2k} (1 + \alpha^2)^n + 6(-1)^{2k+2i} - 4(-1)^{k+i} \beta^{2k} (1 + \beta^2)^n + \beta^{4k} (1 + \beta^4)^n] \\ &= \frac{1}{25} [\alpha^{4k} (3\alpha^2)^n - 4(-1)^{(k+i)} \alpha^{2k} (1 + \alpha^2)^n + 6(-1)^{2k+2i} - 4(-1)^{k+i} \beta^{2k} (1 + \beta^2)^n + \beta^{4k} (3\beta^2)^n] \\ &= \frac{1}{25} [3^n \alpha^{4k+2n} - 4(-1)^{k+i} \alpha^{2k} (1 + \alpha^2)^n + 6 - 4(-1)^{k+i} \beta^{2k} (1 + \beta^2)^n + 3^n \beta^{4k+2n}] \\ &= \frac{1}{25} [3^n (\alpha^{4k+2n} + \beta^{4k+2n}) - 4(-1)^{k+i} \{5^{\frac{n}{2}} (\alpha^{2k+n} - \beta^{2k+n})\} + 6] \end{aligned}$$

Thus finally we have

$$f(k, 4, n) = \frac{1}{25} [3^n L_{4k+2n} - 4(-1)^{(k+i)} \sqrt{5}^{n+1} F_{2k+n} + 6]$$

The question of calculating  $f(k, m, n)$  for  $m = 5$  or for even higher powers could be solved in the same way but the form may be too complicated.

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