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The k-spined Cubes and Its Properties

Xin-Yun QIN[1]and Dao-Yun XU[2]

[1]Department of Mathmantics and Statistics, Guizhou University,Huaxi(550025), Guiyang, P.R.China

[2]Department of Computer Science, Guizhou University,Huaxi(550025), Guiyang, P.R.China

[1]xyqin163@yeah.net, [2]dyxu@gzu.edu.cn.

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Abstract. The diameter of hypercube and its properties is an important issue for improving thecommunication efficiency of interconnection network. A new variant SQ_n^k of the n-dimensionalhypercube is introduced by a spinningfunction $\emptyset_k$. In this variant, called k-spined cubes, neighbors of any node can be quickly obtained by using matrix method. The k-spinedcubes SQ_n^k hasdiameter $(2+\left\lceil\frac{n-k}{k+1}\right\rceil)k$ for $k \geq 4$ and $n \geq 6$.The method can be freely used to constructing minimal diameter hypercube network by choosing a proper value for k.

Introduction

With the development of technology, especially the advent of VLSI circuit it madepossible to build a large parallel and distributed system involving thousands or even tens of thousandsof processors. One crucial step on designing a large-scale parallel and distributed system is todetermine the topology of the interconnection network[1,2].Because the network topological structure properties directly impact on a variety of hardware and software of the parallel and distributed systems design. Thus, an interconnection network plays important roles in a large-scale parallelcomputer system.

Thecommunication efficiency of interconnection network is a significant parameter in a large-scale parallel computer system. The efficiency of communication can be improved by minimizing thediameter. As a result, with a given fixed number of interconnection resources (i.e., nodes and edgesof an interconnection network), being able to construct an interconnection network with a diameteras small as possible is a very significant factor in the design of an interconnection network [3]. Asfar, to achieve smaller diameter with the same numbers of nodes and links, a number of hypercubevariants were proposed [4-10].

An interconnection network can be modeled as an undirected graph $Q_n = \left(V(Q_n), E(Q_n)\right)$.The*distance* between two nodes x and y of graphQ_n,denoted d(x, y) is the length of a shortest pathbetween x and y. The diameter of G is defined as the maximal value of the distances between allpairs of nodes in G, denoted *diam*(G), (i.e., *diam*(G) = max{dist(x, y) | x, $y \in$ V(G)}). Some otherdefinitions and notations not given in this paper are referred to [11, 12] and the reader is referred toref. [13] for fundamental graph-theoretic terminology.

In this paper, we present a new k-spined cubes network SQ_n^k by modifying the n-dimensionalcube $Q_n$definitions, similar but not identical definitions, to explore its properties. We explore thediameter and the minimal diameter of SQ_n^k by choosing the value *k, n*.

The remainder of this paper is organized as follows: Section 1 provides the preliminaries. Section 2 define dimension of neighbors and adjacent-decide matricesinSQ_n^k. Section 3 gives the distanceand several properties of SQ_n^k. In section 4, we give the expression for the diameter of SQ_n^k, andindicate the method of choosing *k* for the minimal diameter inSQ_n^k.

Preliminaries

In this section, we will de_ne some basic notations for BC graphs.

A string $x = x_1x_2 \dots x_n$ in $\{0, 1\}^n$ is viewed as a vector $x = x_1x_2 \dots x_n$on {0, 1}. We use$\oplus$to denote (modulo 2) sum of two bits aand b (a, b∈{0, 1}). For two strings$x = x_1x_2 \dots x_n$and$y = y_1y_2 \dots y_n$of length n in $\{0, 1\}^n$, we define$x \oplus y$(modulo 2) sum of x and y where $(x\oplus y)_i = x_i \oplus y_i$for $i = 1, \dots, n$. It is equivalent to $x \oplus y = [x_1 \oplus y_1, \dots, x_n \oplus y_n]$. For a vector $x = [x_1x_2 \dots x_n]$and two positive integersi, j $(1 \le i \le j \le n)$, denote$[i{:}j] = [x_i, x_{i+1}, \dots, x_j]$.

In this paper, a string $x_1x_2 \dots x_n$and a vector $[x_1x_2 \dots x_n]$ are viewed as the same.

In the graphQ_n, for any two distinct nodes$x$ and y, the distance between x and y is Hammingdistance $dist_H(x, y)$, and computed by

$$dist_H(x, y) = \sum_{i=1}^{n} (x_i \bigoplus y_i)$$

Proposition 1 [14] *For any two vertices* $x, y \in Z_2^n$ *in the graph* Q_n, *the distance*$d(x, y) = |\{i \in [n]: x_i \ne y_i\}|$, *and then the diameter* $diam(Q_n) = n$.

For a fixed string ω with length s, we can get a bijection $\phi_\omega: Z_2^n \to Z_2^n$defined by

$$\psi_\omega(x) = \begin{cases} x & n - s \le 0 \\ [(\omega_{(2s-n)+1} \oplus x_1), \dots, (\omega_s \oplus x_{(n-s)}), x_{(n-s)+1}, \dots, x_n] & 1 \le (n-s) \le s \\ [(\omega_1 \oplus x_1), \dots, (\omega_s \oplus x_s), x_{s+1}, \dots, x_n] & (n-s) \ge s+1 \end{cases}$$

Clearly,$\psi_\omega(\phi_\omega(x)) = x$.

In the definition ofψ_ω, the vector w is called spined factor. The following definition of thefunction comes from [10]. For any positive integer *k*, we define a bijection $\phi_k$from Z_2^n to itself asfollows:

$$\phi_k(x) = \begin{cases} x & n \le k \\ x[k+1{:}n] \oplus x[1{:}(n-k)]^{-} x[(n-k+1){:}n] & 1 \le (n-s) \le n \\ x[(n-k+1){:}n] \oplus x[1{:}k]^{-} x[(k+1){:}n] & n \ge 2k+1 \end{cases}$$

where $x^{-}y$ is a vector by contacting x with y,e.g.,$[1,0,1]^{-}[0,0,1] = [1,0,1,0,0,1]$.

In fact, $\phi_k(x) = \psi_{x[(k+1):n]}(x)$ for a vector x of length n and $1 \le k < n$.

It is seen in the definition of $\phi_k$that we take the suffix$[x_{n-k+1}, \dots, x_n]$with length *k* of thevector $x = [x_1, \dots, x_n]$itself as spined factor.

Please note under the action of ϕ_k that$x[n-k{:}n] = \phi_k(x)[n-k{:}n]$, i.e., the lost *k* bits areunchanged.

For example, $\phi_6([1,0,0,1,0,1]) = [\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}]$, $\phi_3([1,0,0,1,0,1]) = [0,0,1, \mathbf{1}, \mathbf{0}, \mathbf{1}]$, $\phi_4([1,0,0,1,0,1]) = [1,1, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}]$.

Based on the function ϕ_k, we can introduce a variant of n-dimensional hypercube,the basic ideacomes from [10].

Definition 1 (*The k-Spined cubes*) For a fixed integer *k*, the *k*-Spined cubes$SQ_n^k$is defined recursivelyas follows:

(1) SQ_1^k *is the graph* K_2*, where two vertices labeled 0 and 1;*

(2) For $n \geq 2, SQ_n^k$ *is obtained from two copies of* $SQ_{n-1}^k, 0SQ_{n-1}^k$ *and* $1SQ_{-1}^k$*, by adding edges connecting* $0x$*, and* $1\phi_k$*for* $x \in V(SQ_{n-1}^k = Z_2^{n-1})$.

It is easy to check $SQ_n^2 = SQ_n$ defined in [3], and $SQ_n^k = Q_n$ for $k \geq n$.

For example, let *x* = 10110101 and *k* = 3, we have a path in SQ_8^3. 10110101 ⟶ 01100101 ⟶ 00001101 ⟶ 00100101 ⟶ 00111101 ⟶ 00110101

Please note that both Q_n and SQ_n are n-regular graphs. The following Figure 1 gives the graph SQ_1^3.

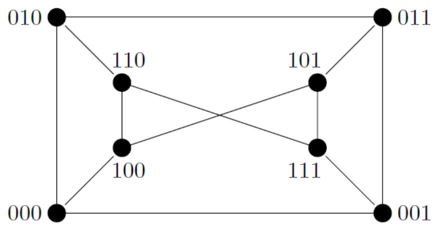

Figure 1: The graph SQ_3^1

Theorem 1*[3] For the* $n \geq 14$*, the diameter* $diam(SQ_n^2) = \lceil n/3 \rceil + 3$.

Neighbors and adjacent-decide matrices in SQ_n^k

For fixed $1 \leq k < n$we define dimension of neighbors in the k-spined cubesSQ_n^k.

(1) For two distinct vertices x and y, if$x[1:(n-k)] = y[1:(n-k)]$*, and x[(n k + 1): n] andy[(n− k + 1): n] are adjacent in* SQ_k*, then call that x and y are adjacent in 0-hierarchy adjacentor 0-dimensional adjacent.*

(2) For $1 \leq m \leq (n-k)$*, we call that x and y are m-adjacent or m-dimensional adjacent, if thefollowing three conditions are held*

(2.1) $x[1: n-(m+k+1)] = y[1: n-(m+k+1)]$,

(2.2) $x_{n-(k+m)} = 1 \oplus y_{n-(k+m)}$*, and*

(2.3) $x[n-(k+m)+1 : n] = \phi_k(y[n-(k+m)+1 : n])$.

Thus, the vector $x = [x_1, x_2, \dots, x_{n-k}, x_{n-k+1}, \dots, x_n]$has the following nneighbors:

$[(1 \oplus x_1)]^- \phi_k(x[2 : n])$ (*n –k-dimensional*),
$[x_1, (1 \oplus x_2)]^- \phi_k(x[3 : n])$(*n –k –1-dimensional*),
$[x_1, x_2, (1 \oplus x_3)]^- \phi_k(x[4 : n])$(*n –k –2-dimensional*),
……,
$[x_1, x_2, \dots, x_{n-k-1}, (1 \oplus x_{n-k})]^- \phi_k(x[(n-k+1) : n])$
$= [x_1, x_2, \dots, x_{n-k-1}, (1 \oplus x_{n-k}), x_{n-k+1}, \dots, x_n]$(*1-dimensional*),
$[x_1, x_2, \dots, x_{n-k}, (1 \oplus x_{n-k+1}), x_{n-k+2}, \dots, x_n]$(*0-dimensional*),
$[x_1, x_2, \dots, x_{n-k+1}, (1 \oplus x_{n-k+2}), x_{n-k+3}, \dots, x_n]$(*0-dimensional*),
……,
$[x_1, x_2, \dots, x_{n-1}, (1 \oplus x_n)]$(*0-dimensional*).

We see the following Figure 2.

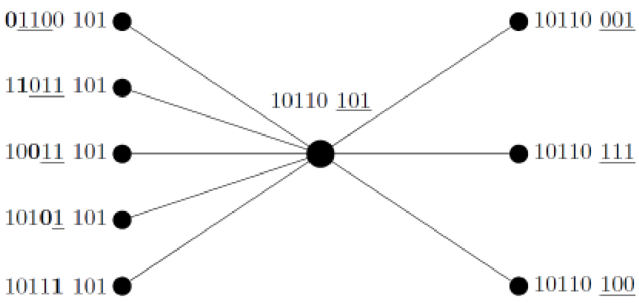

Figure 2: Neighbors of the node 10110101

Denote $\mathbf{0_m}$ as the zero-vector of length m. For a vector $\omega = [\omega_1, \omega_2, \ldots, \omega_k]$(*1* $\leq k < n$), wedefine the following ($n - k$) vectors:

$$S_\omega^{(1,k)} = [1]^\frown[\omega_1, \omega_2, \ldots, \omega_k]^\frown \mathbf{0}_{\boldsymbol{n-(2k+1)}}{}^\frown \mathbf{0}_{\boldsymbol{k}}$$
$$S_\omega^{(2,k)} = [0,1]^\frown[\omega_1, \omega_2, \ldots, \omega_k]^\frown \mathbf{0}_{\boldsymbol{n-(2k+2)}}{}^\frown \mathbf{0}_{\boldsymbol{k}}$$
$$\ldots$$
$$S_\omega^{(n-2k,k)} = \mathbf{0}_{\boldsymbol{n-2k-1}}{}^\frown[1]^\frown[\omega_1, \omega_2, \ldots, \omega_k]^\frown \mathbf{0}_{\boldsymbol{k}}$$
$$S_\omega^{(n-2k+1,k)} = \mathbf{0}_{\boldsymbol{n-2k}}{}^\frown[1]^\frown[\omega_1, \ldots, \omega_k]^\frown \mathbf{0}_{\boldsymbol{k}}$$
$$\ldots$$
$$S_\omega^{(n-k-2,k)} = \mathbf{0}_{(\boldsymbol{n-k-3})}{}^\frown[1]^\frown[\omega_{k-1}, \omega_k]^\frown \mathbf{0}_{\boldsymbol{k}}$$
$$S_\omega^{(n-k-1,k)} = \mathbf{0}_{(\boldsymbol{n-k-2})}{}^\frown[1]^\frown[\omega_k]^\frown \mathbf{0}_{\boldsymbol{k}}$$

andintroduce a matrix Spin_ω, called spined matrix with factor ω, is shown in Figure 3:

$$Spin_w = \begin{bmatrix} S_w^{(1,k)} \\ \vdots \\ S_w^{(n-k-1,k)} \\ \mathbf{e_{(k+2)}} \\ \vdots \\ \mathbf{e_n} \end{bmatrix}$$

Figure 3

$$Spin_w = \begin{bmatrix} 1 & w_1 & w_2 & w_3 & 0 & 0 & 0 & 0 \\ 0 & 1 & w_1 & w_2 & w_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & w_2 & w_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & w_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 4

Where $\boldsymbol{e_i}$ *is i-th unit vector of length n for i = k + 2,…, n.*

For example, let n = 8, and w = [w_1*,* w_2*,* w_3*], the Figure 4 gives the model.*

For a vector x = [x_1*,… ,*x_n*] and* $1 \leq k < n$*, the matrix* $Spin_k$ *(=* $Spin_w$*) is called the spinedmatrix of x with respect to k, where the spined factor w is the suffix with length k of x. The spinedmatrix is corresponded to the adjacent-decide matrix in [3].*

By the spined matrix $Spin_k$*,we can get all adjacent nods of x in* SQ_n^k*, which is defined by a matrix* B_k^x *, where the i-th row* $B_k^x(i,:)$*is defined by*

$$x \oplus Spin_k(i,:) = [x_1 \oplus Spin_k(i,1), \ldots, x_n \oplus Spin_k(i,n)]$$

For the vector *x* = [1, 0, 1, 1, 0, 1, 0, 1] in the *Figure 2*, and *k* = 3, we take*w*= [1, 0, 1], which isthe suffix with length *k* of *x*, and then show in Figure 5.

$$B_3^x = \left[\begin{array}{ccccc|ccc} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

Figure 5

It is easy to check the following lemma.

Lemma 1 *Let x be n-dimension vector on* {0,1} *and, and let Spined$_k$be the spinedmatrix of x with respect to k. Then, are* (n-k)*-dimensional,…,*1*-dimensional neighbors of x respectively, and are*0*-dimensional neighbors of x.*

Distance inSQ_n^k

In this section, we consider the distance between two nodes in the graph SQ_n^k, where $1 \leq k < n$.

We now introduce some concepts and notations. A string *y* can be spined from *x* by *w* at *i*-thbit ($1 \leq i < n$), if $y = x_1 \dots x_{i-1}(1 \oplus x_i)\phi_\omega(x[i+1:n])$, denoted by $x \vdash_{\omega,i} y$. The string *y c*an bespined from x by w, if y can be spined from *x* by *w* at i-th bit for some ($1 \leq i < n$), denoted by$x \vdash_\omega y$. The string *y* can be *k*-spined from *x,* if *y* can be spined from *x* by some string *w* of length*k*, denoted by$x \vdash_k y$.

For example, let *x*= [1, 0, 1, 0, 0, 1, 1, 1], *w* = [1, 0, 1]. We have$x \vdash_{\omega,3} [1,0,0,1,0,0,1,1]$.

A string *y* can be spin-generated from *x* by *w*, if there exists a sequence of strings *x*, $X_1, \dots, X_{m-1}$, y, denoted by $x \vdash_\omega^m y$, such that

$$x \vdash_\omega X_1, X_1 \vdash_\omega X_2, \dots, X_{m-2} \vdash_\omega X_{m-1}, X_{m-1} \vdash_\omega y.$$

Note that the sequence is associated with a sequence $i_1,\dots,i_m$ of integers, such that $x \vdash_{\omega,\ i_1} X_1, X_1 \vdash_{\omega,\ i_2} X_2, \dots, X_{m-2} \vdash_{\omega,\ i_{m-1}} X_{m-1}, X_{m-1} \vdash_{\omega,\ i_m} y$. Here, *m*is called as generated length.

For example, let *x* = [0, 1, 1, 1, 0, 1, 1, 1], *w* = [1, 0, 1] and *y* = [0, 0, 0, 0, 0, 0, 0, 0]. Then,

$$x \vdash_{\omega,\ 2} [0,0,0,1,1,1,1,1] \vdash_{\omega,\ 4} [0,0,0,0,0,1,0,1] \vdash_{\omega,\ 6} [0,0,0,0,0,0,0,0].$$

If we take proper different spined factor string for fixed length *k* = 3, then we get shortersequence:

$$[0,1,1,1,0,1,1,1] \vdash_{110,\ 2} [0,0,0,0,0,1,1,1] \vdash_{011,\ 6} [0,0,0,0,0,0,0,0].$$

Generally, for a string *x* of length *n*, we take at most $l = \left\lceil \frac{n}{k+1} \right\rceil$ strings $\omega_1, \omega_2, \dots, \omega_l$ with length *k*,and then get a zero vector [0, … , 0] of length *n* in *l* steps.

Lemma 2*Let x and y be two nodes in* SQ_n^k *and*$1 \leq k < n$*. The distance d*(*x*, *y*) *is at most*$\left(2 + \left\lceil \frac{n-k}{k+1} \right\rceil\right) k$.

Proof: Assume that *x* = [$x_1, \dots, x_n$] and *y* = [$y_1, \dots, y_n$]. Let$z = x \oplus y$, then we have$y = z \oplus x$ and $x = z \oplus y$.

Denote $x' = [x_1,\ x_2, \dots,\ x_{n-k},\ z_{(n-k)+1}, \dots, z_n]$
$= [y_1 \oplus z_1, y_2 \oplus z_2, \dots, y_{n-k} \oplus z_{n-k}, z_{(n-k)+1}, \dots, z_n]$,
$y' = [y_1, y_2, \dots, y_{n-k}, z_{(n-k)+1}, \dots, z_n]$,
$\omega_x = [x_{(n-k)+1}, \dots, x_n]$, $\omega = [z_{(n-k)+1}, \dots, z_n]$, $\omega_y = [y_{(n-k)+1}, \dots, y_n]$.
Consider the path $x \rightsquigarrow x' \rightsquigarrow y' \rightsquigarrow y$. Then,

$$d(x,x') = d_H(\omega_x, \omega) \leq k \qquad d(y,y') = d_H(\omega_y, \omega) \leq k$$
$$d(x', y') = d(z', \mathbf{0^{n-k}}) \leq \lceil \tfrac{n-k}{k+1} \rceil \cdot k,$$

where $z' = [z_1, \dots, z_{n-k}]$ and $\mathbf{0^{n-k}} = [0,0,\dots,0]$. Thus, $d(x,y) \leq (2 + \lceil \frac{n-k}{k+1} \rceil)k$. ■

We now introduce three distance function $d_H(x,y), d_\omega(x,y)$, and $d_k(x,y)$, where $|x| = |y| = n, |\omega| = k$ and $1 \leq k < n$.

(1) Hamming Distance $d_H(\cdot,\ \cdot)$ $d_H(x,y) = |\{i \in [n]: x_i \neq y_i\}|$.

(2) Spinning Distance $d_\omega(\cdot,\ \cdot)$

$$d_\omega(x,y) = \min\{m: x \vdash_\omega^m y\}, \text{ where } 1 \leq |\omega| < n.$$

(3) k-Spinning Distance $d_k(\cdot,\ \cdot)$

$$d_k(x,y) = \min\{d_\omega(x,y): \omega \in z_2^k\}, \text{ where } 1 \leq k < n.$$

Clearly, $d_H(x,y) = d_{0^k}(x,y)$ for any $1 \leq k < n$.

For $x = [x_1,\ x_2, \dots,\ x_{n-k},\ x_{(n-k)+1}, \dots, y_n]$ and $y = [y_1, y_2, \dots, y_{n-k}, y_{(n-k)+1}, \dots, y_n]$, the vector $z = x \oplus y$ is represented as $[z_1, z_2, \dots, z_{n-k}, 0, \dots, 0] \oplus [0,0,\dots,0, z_{(n-k)+1}, \dots, z_n]$.

Consider the path $x \rightsquigarrow x' \rightsquigarrow y' \rightsquigarrow y$, then, the distance between *x* and *y*
$d(x,y) \leq d_H(x,x') + d_\omega(x',y') + d_H(y',y)$, where $\omega = [z_{(n-k)+1}, \dots,\ z_n]$.
In fact, we have.

Theorem 2 *In the k-spined cubes* SQ_n^k $(1 \leq k < n)$, *for any two distinct nodes x and y, the distance* $d(x,y) \leq d_H(x,x') + d_\omega(x',y') + d_H(y',y)$,
where $x' = [x_1,\ x_2,\ \dots,\ x_{n-k},\ x_{(n-k)+1} \oplus y_{(n-k)+1}, \dots,\ x_n \oplus y_n]$, $y' = [y_1, y_2, \dots, y_{n-k}, x_{(n-k)+1} \oplus y_{(n-k)+1}, \dots, x_n \oplus y_n]$, $\omega = x[(n-k+1):n] \oplus y[(n-k+1):n]$.

Proof: For $1 \leq k \leq 4$ and $k \leq n < 5$, this is obvious. Assume $4 \leq k < n$. To prove Theorem 2, it suffices to prove the following inequality.

$$d(x,y) \leq d_H(x,x') + d_\omega(x',y') + d_H(y',y),$$

By Lemma 2, there is a path in SQ_n^k from *x* to *y* of distance at most $(2 + \lceil \frac{n-k}{k+1} \rceil)$k. Thus, we only need to prove that there exist two nodes *x* and *y* in SQ_n^k, such that the length of any path *d*(*x* ,*y*) between *x* and *y* satisfies $d(x\ ,y) \geq (2 + \lceil \frac{n-k}{k+1} \rceil)$k.

Let $d(x,y):\ x, X_1,\ X_2, \dots, X_{m-1}, y$, $|\omega| = k$, $\forall\, x = [x_1,\ x_2,\ \dots,\ x_n]$, $y = [y_1, y_2, \dots, y_n] \in SQ_n^k (4 \leq k < n)$, $\mathbf{0^n}$ is a zero vector of length *n*. Then any path $d(x\ ,y): x,\ X_1,\ X_2, \dots, X_m, y$ between *x* and *y* can be divide into the following three cases.

Case 1 The path *d(x; y)* only contains nodes of *0-dimensional*. By defining of the *0-dimensional* adjacent points in the SQ_n^k. If the X_i and X_{i+1} in *0-dimensional*, then they have atleast 1 bit the same. So *x* and *y* have at least n-1 different points for any integer *i* with $0 \leq i < k$, this means that $d(x,y) = d(z, 0^n) \geq \lceil \frac{n-k}{k+1} \rceil k$.

Because $\lceil \frac{n-1}{k+1} \rceil k - (2 + \lceil \frac{n-k}{k+1} \rceil)k = (\lceil \frac{n-k}{k+1} \rceil + (k-1))k - 2k - \lceil \frac{n-k}{k+1} \rceil k$

(The Property of supremum)

$= (k - 4)k$

Thus, when $k \geq 4,\ n \geq k + 2$, we obtain $d(x,y) \geq \lceil \frac{n-1}{k+1} \rceil k \geq (2 + \lceil \frac{n-k}{k+1} \rceil)k$.

Case 2 The path *d(x, y)* contains at least one node of *n-k-dimensional*. There must betwo edges in the path, one of which is used for routing from node of *0-dimensional* to node of*n-k-dimensional* and other of which is used for routing back to node of *0-dimensional*. Thus,we have

$$d(x,y) = d(z,0^n) \geq (\left\lceil\frac{n-k}{k+1}\right\rceil + \left\lceil\frac{k}{k+1}\right\rceil + 2)n \geq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k.$$

Case 3 The path *d(x, y)* only contains nodes of *n-k-dimensional*. By defining of the*n-k-dimensional* adjacent points in the SQ_n^k, if the X_i and X_{i+1}in *n-k-dimensional*, theyhave the same *k* bit for any integer *i* for$0 \leq i < k$. This means that $d(x,y) = d(z,0^n) \geq \left(2 + \lceil\frac{n-k}{k+1}\rceil\right)k$.

In summary, we have

$$d(x,y) \geq \left(2 + \lceil\frac{n-k}{k+1}\rceil\right)k \geq d_H(x,x') + d_\omega(x',y') + d_H(y',y).$$

By the distance between *x* and *y*, we easily get

$$d(x,y) \leq d_H(x,x') + d_\omega(x',y') + d_H(y',y)$$

Thus, we have

$$d(x,y) = d_H(x,x') + d_\omega(x',y') + d_H(y',y).\blacksquare$$

Theorem 2 is oblivious as the path selected for the interconnection network is only dependent onsource and destination nodes of the addresses, and has nothing to do with the route between them.In other words, route selections are made by a distributed computing in the course of routing. If so,the each intermediate node for selecting the next node need only linear time, so as to improve theefficiency of the Internet communication.

We also get a direct result.

Theorem 3*In the k-spined cubes* SQ_n^k $(1 \leq k < n)$, *for any two distinct nodes x and y, let* z=x ⊕y=z1, ..., zn-k, zn-k+1, ..., zn,*the distance* $d(x,y) = 2 * d_H\left(z[(n-k+1):n], \mathbf{0^k}\right) + d_k(z[1:n-k], \mathbf{0^{n-k}})$. *where* $\mathbf{0^m}$ *is a zero vector of length m.*

Proof: By Lemma 2, we obtain

$$\mathrm{d}(x,x') = d_H(\omega_x,\omega) = d_H(x[(n-k+1):n], z[(n-k+1):n])$$
$$\mathrm{d}(y',y) = d_H(\omega_y,\omega) = d_H(y[(n-k+1):n], z[(n-k+1):n])$$
$$d_\omega(x',y') = d(z', \mathbf{0^{n-k}}) = d_k(z[1:n-k], \mathbf{0^{n-k}})$$

Beause

$$d(x,y) = d_H(x,x') + d_\omega(x',y') + d_H(y',y) \text{in Theorem 2.}$$

So, we easily get

$$d(x,y) = 2 * d_H\left(z[(n-k+1):n], \mathbf{0^k}\right) + d_k(z[1:n-k], \mathbf{0^{n-k}}).$$

(where $\mathbf{0^k}$ is a zero vector of length *k*).■

Diameter and the minimal Diameter of SQ_n^k

From the argument of Definition 1 and Theorem 2, we immediately obtain

Theorem 4*Let* $x = [x_1, x_2, \ldots, x_n]$ *and* $y = [y_1, y_2, \ldots, y_n]$ *be two nodes of* SQ_n^k.

(1) If$k \geq n$, *then* $SQ_n^k = Q_n$.

(2) If $k < n$, *then* $d(x,y) = d_H(x,x') + d_\omega(x',y') + d_H(y',y)$.

Theorem 5 (1)$diam(SQ_n^k) = n$ for $k \geq n$;

(2) $diam(SQ_n^k) = \left(2 + \lceil\frac{n-k}{k+1}\rceil\right)k$ for $k \geq 4 \; and \; n \geq 6$.

Proof: (1) By Proposition 1, we have $d(x;\ y)\ = |i \in [n]: x_i \neq y_i|$ for any two vertices $x,\ y \in z_2^n$ inthe graph Q_n , and then the diameter $diam(Q_n)\ =\ n$ .So $diam(SQ_n^k)\ =\ diam(Q_n)\ =\ n$ for$k\ \geq\ n$.
(2) Let $x\ =\ [x_1,\ x_2, \ldots, x_n]$ and $y = [y_1,\ y_2, \ldots, y_n]$ be two distinct nodes of SQ_n^k for k < n. Weexamine three possibilities.

Case 1 The path *d*(*x, y*) only contains nodes of *0-dimensional*. By Theorem 2, we have$d(x,y) \geq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$ for k ≥ 4 and n $\geq\ k\ +\ 2$.

Case 2 The path *d*(*x, y*)contains at least one node of *n-k-dimensional*. It follows from Theorem 2 that$d(x,y) \geq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$.

Case 3 The path *d*(*x, y*) only contains nodes of *n-k-dimensional*. Similarly *Case 1*, we can derive$d(x,y) \geq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$.

From the above discussions, we conclude $diam(SQ_n^k) \geq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$for k $\geq$ 4 and n $\geq\ k\ +\ 2$.

On the other hand, consider the two nodes *x* and *y* in $SQ_n^k$and $1 \leq k\ <\ n$.

According to Lemma 2, we have$d(x,y) \leq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$.

So $diam(SQ_n^k) \geq \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$for k ≥ 4 and n $\geq\ k\ +\ 2$.

In summary, for k ≥ 4 and n $\geq\ 6$, $diam(SQ_n^k) = \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$ ■

Lemma 3*When k=1, 2,* $diam(SQ_n^k) = \lceil n/3 \rceil + 3$ *is the minimal diameter in* SQ_n^k *for* n ≥ 14.

Proof: By Theorem 1 and Theorem 5, we obtain

$$diam(SQ_n^k) = \lceil n/3 \rceil + 3 \text{for k = 1, 2, n} \geq 14;$$

$$diam(SQ_n^k) = \left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k \text{fork} \geq 4 \text{ and n} \geq 6.$$

Because

$$\left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k - (\lceil n/3 \rceil + 3) = (2k-3) + \left(\left\lceil\frac{n-k}{k+1}\right\rceil k - \lceil n/3 \rceil\right)$$
$$> 0 \qquad (k \geq 4, n \geq 14).$$

So, when *k*=1,2,$diam(SQ_n^k) = \lceil n/3 \rceil + 3$is the minimal diameter in $SQ_n^k$for $n \geq 14$.

The following Corollary give additional properties ofSQ_n^k.

Corollary 1*For any integer n,* $diam(SQ_n^3) \leq n - 3$.

Proof: According to mathematical induction, we can quickly verify this Corollary. ■

Conclusions and Future Works

In this paper, we have defined a new variant of *n-dimensional* cube based on the function ϕ_k, viz. the*k*-spinedcubesSQ_n^k.When$k\ \geq\ n$,$SQ_n^k = Q_n$, which the properties of $Q_n$have been developed. So,we have laid emphasis on the study of the *k*-spined cubes properties for $k\ <\ n$. We define dimensionof neighbors in $SQ_n^k$for fixed $1 \leq k\ <\ n$by two new different ways. Next, we have considered theconsidered the distance between two nodes in SQ_n^k and explored the relationship betweend$_H(x,y), d_\omega(x,y)$, and $d_k(x,y)$. On the basis of this discovery, we have shown that the diameter of SQ_n^k is$\left(2 + \left\lceil\frac{n-k}{k+1}\right\rceil\right)k$ for any integer with k ≥ 4, n $\geq\ 6$. Finally, we get minimal diameter in SQ_n^k bychoosing *k* = 1, 2.

There still have a lot of problems about SQ_n^k. Its topological propertieshave not yet been investigated.The performance of SQ_n^k should be compared with that of hypercube or other hypercube variantsin communication situation in the future.

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References

[1] Li Jing. K-yuan n-cube of the road and circle. ShanxiUniversity, 2011.

[2] Junming Xu. Topological Structure and Analysis of Interconnection Networks. Springer Publishing Company, Incorporated, 2010 7(2-3):969-970.

[3] Wujun Zhou, Jianxi Fan, XiaohuaJia, and Shukui Zhang: submitted to Information ProcessingLetters, 111(2011):561-567.

[4] S. Abraham, K. Padmanabhan: submitted to J.ParallelDistrib. Comput. 13 (1) (1991) 104-110.

[5] Efe K: submitted to IEEE Transactions on Computers, 1991, 40(11):1312-1316.

[6] P. Cull, S.M. Larson: submitted to IEEE Transactions on Computers, 54 (2) (1995) 647-659.

[7] D. Wang and L. Zhao: submitted to Journal of Computer Science and Technology, (1999), 14(2):181-187.

[8] X. Yang, D.J. Evans, G.M. Megson: submitted to International Journal of Computer Mathematics 82 (4) (2005) 401-413.

[9] Yang Li, Heng Qi, Zhiyang Li and Kequi Li, \The Exchanged Folded Hypercube,"IEEE Inter.Conf. on High Perf. Comp. and Communication.pp 45-50, 2013.

[10] Xuding Zhu: submitted to Mathematics (2015).

[11] J.A. Bondy and and U.S.R. Murty, Graph Theory with Applications (Macmillan Press, London,1976).

[12]D.B. West, Introduction to Graph Theory (Prentice Hall, 2001).

[13]Harary, F., 1969, Graph Theory (Reading: Addison-Wesley).

[14]XiaoYuan Dong: submitted to China Science and technology information, 2009 (20) 275-276.