A Computational Algorithm for the Inverse of a Sevendiagonal Matrix

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Abstract—In this article, we investigate the inverse of the sevendiagonal matrix. Under certain conditions, the efficient algorithm which is suitable for implementation using computer algebra systems, is proposed. Also, the inverse of an anti-sevendiagonal matrix is obtained based on the special relation between the sevendiagonal matrix and the anti-sevendiagonal matrix. A numerical example is provided to further illustrate the validity of the corresponding algorithm.

Keywords—a sevendiagonal matrix; inverse matrix; determinant; computer algebra system

I. INTRODUCTION

A large number of banded matrices are appeared in the process of numerical analysis and numerical calculation, such as tridiagonal, pentadiagonal and sevendiagonal matrices. They play an important role in theory, and they are applied in spline approximation, parallel computing, partial differential equations(PDE) and boundary value problems(BVP). For a long time, the inverses of tridiagonal and pentadiagonal matrices have been researched[1-11]. Recently, the inverses of Toeplitz tridiagonal and pentadiagonal matrices are also investigated[12]. And the inverses of sevendiagonal and generalized sevendiagonal matrices have been studied[13-15]. Many of them gave algorithms to obtain the inverse matrices by LU factorization[3-5,9,10,13-15]. A few by dividing blocks of diagonal matrices[6,12]. And Ran Rui-sheng[7] used the Doolittle factorization to get the inverse matrix of a general tridiagonal matrix.

In this paper, based on the numerical algorithms given in [1,13], we present a fast algorithm for computing the inverse of the sevendiagonal matrix. Moreover, the inverse of anti-sevendiagonal matrix is also investigated.

The rest of this paper is organized as follows. In Section 2, we present an efficient algorithm for the inverse of an n-by-n sevendiagonal matrix, and the inverse of the anti-sevendiagonal matrix is also given. One numerical example is provided to illustrate the proposed algorithm in Section 3. Finally, some conclusions of the work are given.

II. INVERSE OF A SEVENDIAGONAL MATRIX

In this section, we mainly focus on giving some recurrence formulas for the inverse of an n-by-n sevendiagonal matrix in terms of their columns.

\[
\begin{bmatrix}
  a_1 & b_1 & c_1 & d_1 & 0 & \cdots & 0 \\
  a_2 & a_1 & b_2 & c_2 & d_2 & \cdots & 0 \\
  \beta_1 & a_1 & a_2 & b_1 & c_1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  \beta_2 & \beta_1 & a_1 & a_2 & b_1 & c_1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
  \alpha_1 & \alpha_2 & \cdots & 0 & \cdots & \cdots & \cdots & a_n \\
  \gamma_1 & 0 & \cdots & \cdots & \gamma_2 & \cdots & \cdots & \gamma_n \\
  \gamma_2 & \gamma_1 & \cdots & \cdots & \gamma_3 & \cdots & \cdots & \gamma_n \\
  \gamma_3 & \gamma_2 & \cdots & \cdots & \gamma_3 & \cdots & \cdots & \gamma_n \\
  \gamma_n & \gamma_{n-1} & \cdots & \cdots & \gamma_n & \cdots & \cdots & \gamma_n \\
\end{bmatrix}
\]

(1)

Without loss of generality, a sevendiagonal matrix with order \( n \) can be given as

\[
\begin{align*}
\text{where } a_i, b_i, c_i, d_i, \alpha_i, \beta_i, \text{ and } \gamma_i (i = 1, 2, \ldots, n) & \text{ are finite sequences of numbers such that } d_1 \neq 0 (i = 1, 2, \ldots, n-3), \\
d_{n-2} & = d_{n-1} = d_n = 1, \alpha_1 = \beta_1 = \beta_2 = 0, \gamma_1 = \gamma_2 = \gamma_3 = 0, \\
\text{ and } b_n & = c_{n-1} = c_n = 0.
\end{align*}
\]

Suppose that the sevendiagonal matrix \( \mathbf{T} \) is non-singular and the inverse is

\[
\mathbf{T}^{-1} = (C_1, C_2, \ldots, C_n),
\]

where \( C_j \) denote the \( j \)th column of \( \mathbf{T}^{-1} \), \( j = 1, 2, \ldots, n \). Here \( C_j \) also can be written as

\[
C_j = (C_1, C_2, \ldots, C_n)E_j,
\]

where \( E_j = (\delta_{j1}, \delta_{j2}, \ldots, \delta_{jn})' \), \( j = 1, 2, \ldots, n \), and \( \delta_j \) is the Kronecker symbol.

Through the relation \( \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}_n \) ( here \( \mathbf{I}_n \) is the \( n \times n \) identity matrix), we get the relations as

\[
\begin{align*}
C_{i+3} & = (E_i - c_{i+3} C_{i+2} - b_{i+3} C_{i+1} - a_i C_i) / d_{i+3} \\
C_{i+4} & = (E_i - c_{i+4} C_{i+3} - b_{i+4} C_{i+2} - a_i C_{i+1} - a_i C_i) / d_{i+4} \\
C_{i+5} & = (E_i - c_{i+5} C_{i+4} - b_{i+5} C_{i+3} - a_i C_{i+2} - a_i C_{i+1} - a_i C_i - \beta_i C_i) / d_{i+5} \\
C_{i+k} & = (E_i - c_{i+k} C_{i+k-2} - b_{i+k} C_{i+k-1} - a_i C_{i+k-3} - \beta_i C_i) / d_{i+k} & \text{for } 4 \leq k \leq n-3
\end{align*}
\]

(2)
where $E_j = (\delta_j, \delta_{j}, \cdots, \delta_{j}), j = 1, 2, \cdots, n$, and $\delta_j$ is the Kronecker symbol.

It follows from (2) that if we get the last three columns $C_{n}, C_{n+1}$ and $C_{n+2}$, then we can recursively determine all other columns $C_{n-3}, \cdots, C_1$. Now we shall give the recurrence formulas for computing $C_{n}, C_{n+1}$ and $C_{n+2}$.

Consider the three finite sequences of numbers $A_0, B_0$ and $D_i (i = 0, 1, \cdots, n + 2)$ with the initial conditions of $A_0 = A_1 = B_0 = B_2 = D_1 = D_2 = 0$ and $A_2 = B_1 = D_0 = 1$, which satisfy

$$
(T_{nn} T_{nn} T_{nn}) \begin{pmatrix} A_{n+1} \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \end{pmatrix},
$$

$$
(T_{nn} T_{nn} T_{nn}) \begin{pmatrix} B_{n+1} \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \end{pmatrix},
$$

$$
(T_{nn} T_{nn} T_{nn}) \begin{pmatrix} D_{n+1} \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \end{pmatrix},$$

where $T = \begin{pmatrix} A', B' \\ \end{pmatrix}$, $A = [A_0, A_1, \ldots, A_{n+1}]^T$, $B = [B_0, B_1, \ldots, B_{n+1}]^T$, $D = [D_0, D_1, \ldots, D_{n+1}]^T$, $A' = [A_1, A_2, A_{n+1}]^T$, $B' = [B_1, B_2, \ldots, B_{n+1}]^T$, and $D' = [D_1, D_2, D_{n+1}]^T$. Here $I_{3n}$ is the $3 \times 3$ identity matrix.

On the basis of (3), (4) and (5), we can get the Eqs

$$
\begin{align*}
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0 \\
[a_i D_0 + b_i D_1 + c_i D_2 + d_i D_3 &= 0
\end{align*}
$$

From the recurrence equations (6), (7) and (8), we can give a matrix form as

$$
\begin{align*}
TA &= -A_n E_{n-2} - A_{n+1} E_{n-1} - A_{n+2} E_n, \\
TB &= -B_n E_{n-2} - B_{n+1} E_{n-1} - B_{n+2} E_n, \\
TD &= -D_n E_{n-2} - D_{n+1} E_{n-1} - D_{n+2} E_n
\end{align*}
$$

Multiply (9),(10),(11) by $Q_{11}, Q_{22}, Q_{23}$ respectively to get the following equations

$$
\begin{align*}
TAQ_{11} &= -A_n Q_{11} E_{n-2} - A_{n+1} Q_{11} E_{n-1} - A_{n+2} Q_{11} E_n, \\
TBQ_{12} &= -B_n Q_{12} E_{n-2} - B_{n+1} Q_{12} E_{n-1} - B_{n+2} Q_{12} E_n, \\
TDQ_{13} &= -D_n Q_{13} E_{n-2} - D_{n+1} Q_{13} E_{n-1} - D_{n+2} Q_{13} E_n
\end{align*}
$$

multiply (9),(10),(11) by $Q_{21}, Q_{22}, Q_{23}$ respectively to get the following equations

$$
\begin{align*}
TAQ_{21} &= -A_n Q_{21} E_{n-2} - A_{n+1} Q_{21} E_{n-1} - A_{n+2} Q_{21} E_n, \\
TBQ_{22} &= -B_n Q_{22} E_{n-2} - B_{n+1} Q_{22} E_{n-1} - B_{n+2} Q_{22} E_n, \\
TDQ_{23} &= -D_n Q_{23} E_{n-2} - D_{n+1} Q_{23} E_{n-1} - D_{n+2} Q_{23} E_n
\end{align*}
$$

multiply (9),(10),(11) by $Q_{31}, Q_{32}, Q_{33}$ respectively to get the following equations

$$
\begin{align*}
TAQ_{31} &= -A_n Q_{31} E_{n-2} - A_{n+1} Q_{31} E_{n-1} - A_{n+2} Q_{31} E_n, \\
TBQ_{32} &= -B_n Q_{32} E_{n-2} - B_{n+1} Q_{32} E_{n-1} - B_{n+2} Q_{32} E_n, \\
TDQ_{33} &= -D_n Q_{33} E_{n-2} - D_{n+1} Q_{33} E_{n-1} - D_{n+2} Q_{33} E_n
\end{align*}
$$
\[ T D Q_{13} = -D_n Q_{13} E_{n-2} - D_{n1} Q_{13} E_{n-1} - D_{n2} Q_{13} E_n, \]  
where
\[ Q_{11} = B_{n2} D_{n1} - B_{n1} D_{n2}, Q_{12} = A_{n1} D_{n2} - A_{n2} D_{n1}, Q_{13} = A_{n2} B_{n1} - A_{n1} B_{n2}, \]
\[ Q_{21} = B_{n2} D_{n2} - B_{n1} D_{n1}, Q_{22} = A_{n1} D_{n1} - A_{n2} D_{n2}, Q_{23} = A_{n1} B_{n1} - A_{n2} B_{n2}, \]
\[ Q_{31} = B_{n1} D_{n1} - B_{n2} D_{n2}, Q_{32} = A_{n1} D_{n2} - A_{n2} D_{n1}, Q_{33} = A_{n1} B_{n2} - A_{n2} B_{n1}, \]

Let \((12)+(13)+(14),(15)+(16)+(17)\) and \((18)+(19)+(20)\), thus
\[ T X = -X_n E_{n-2}, \]
\[ T Y = -Y_{n1} E_{n-1}, \]
\[ T Z = -Z_{n2} E_n, \]
where \(X = [X_0, X_1, \ldots, X_n]\), \(Y = [Y_0, Y_1, \ldots, Y_n]\), \(Z = [Z_0, Z_1, \ldots, Z_{n-2}]\), and for \(0 \leq i \leq n + 2\)
\[ X_i = \text{det} \left( \begin{array}{ccc} A_{n2} & A_{n1} & A_i \\ B_{n2} & B_{n1} & B_i \\ D_{n2} & D_{n1} & D_i \end{array} \right), \]
\[ Y_i = \text{det} \left( \begin{array}{ccc} A_{n2} & A_n & A_i \\ B_{n2} & B_n & B_i \\ D_{n2} & D_n & D_i \end{array} \right), \]
\[ Z_i = \text{det} \left( \begin{array}{ccc} A_{n1} & A_n & A_i \\ B_{n1} & B_n & B_i \\ D_{n1} & D_n & D_i \end{array} \right). \]

We can easily get \(X_n = -Y_{n1} = Z_{n2}\) by (24), (25) and (26).

**Lemma 2.1.** If \(X_n \neq 0\), then the matrix \(T\) is singular.

Proof. We have
\[ X_0 = A_{n2} B_{n1} - A_{n1} B_{n2}, Y_0 = A_{n2} B_n - A_n B_{n2}, Z_0 = A_{n1} B_n - A_n B_{n1}, \]

If \(A_{n2} \neq 0, B_{n1} \neq 0\) and \(A_{n1} = 0\), then \(X\) is non-null and \(T X = 0\). If \(A_{n1} \neq 0, B_n \neq 0\) and \(A_n = 0\), then \(Z\) is non-null and \(T Z = 0\).

If \(A_{n2} = A_{n1} = A_n = 0\), then \(A\) is non-null and \(T A = 0\). The proof is completed.

**Theorem 2.2.** Suppose that \(X_n \neq 0\), then \(T\) is invertible and
\[ C_n = [-Z_0 / Z_{n2}, -Z_1 / Z_{n2}, \ldots, -Z_{n-1} / Z_{n2}], \]
\[ C_{n+1} = [-Y_0 / Y_{n1}, -Y_1 / Y_{n1}, \ldots, -Y_{n-1} / Y_{n1}], \]
\[ C_{n+2} = [-X_0 / X_{n1}, -X_1 / X_{n1}, \ldots, -X_{n-1} / X_{n1}], \]

Proof. According to the determinant of tridiagonal and pentadiagonal matrix (see [1]), we can get
\[ \text{det}(T) = (-1)^n \prod_{i=1}^{n} d_i X_n \neq 0 \] so \(T\) is invertible. From the (21), (22), (23) and (27), we have \(T C_n = E_n, T C_{n+1} = E_{n+1}\) and \(T C_{n+2} = E_{n+2}\). The proof is completed.

**Algorithm 2.3.** To find the \(n \times n\) inverse matrix of the sevendiagonal matrix in (1) by using the relations (2)-(27).

**INPUT:** Order of the matrix \(n\) and the components \(a_i, b_i, c_i, d_i, \alpha_i, \beta_i, \gamma_i, i = 1,2,\ldots,n\), where \(d_{n-2} = d_{n-1} = d_n = 1\), \(c_{n+1} = c_n = b_n = \alpha_1 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \gamma_3 = 0\).

**OUTPUT:** Inverse matrix \(T^{-1}\).

**Step 1:** If \(d_k = 0\) for any \(k = 1,2,\ldots,n-3\), then set \(d_j = t\) (\(t\) is just a symbolic name).

**Step 2:** Set \(A_0 = A_1 = 0, A_2 = 1\), then for \(i = 3,\ldots,n+2\), compute the components \(A_i\) by using (6).

**Step 3:** Set \(B_0 = B_1 = 0, B_2 = 1\), then for \(i = 3,\ldots,n+2\), compute the components \(B_i\) by using (7).

**Step 4:** Set \(D_0 = 1, D_1 = D_2 = 0\), then for \(i = 3,\ldots,n+2\), compute the components \(D_i\) by using (8).

**Step 5:** For \(0 \leq i \leq n+2\), compute the components \(X_i, Y_i\) and \(Z_i\) by using (24), (25) and (26).

**Step 6:** Substitute the actual value \(t = 0\), and compute
\[ \text{det}(T) = (-1)^n \prod_{i=1}^{n} d_i X_n \] If the matrix \(T\) is singular, then OUTPUT ('Singular Matrix'), Stop.
**Step 7:** Compute the components $C_{n-2}$, $C_{n-1}$, and $C_n$ by using (27).

**Step 8:** Compute the components $C_{n-3}$, $C_{n-4}$, and $C_{n-5}$ by using the first three equations in (2).

**Step 9:** For $j = n - 3$, $1, 2, 4$, compute the components $C_{j-3}$ by using the last equation in (2).

**Remark 2.4.** Here the algorithm complexity of algorithm 2.3 will be computed.

The algorithm complexity in step 1 is $n - 3$, in each step 2-4 is $n + 3$, in step 5 is $3n + 9$, in the step 6-8 are $n, 3n, 3n$ respectively, and in the last step is $n^2 - 6n$. So the algorithm complexity of algorithm 2.3 is $n^2 + 8n + 15$.

Here we will discuss the inverse of $n \times n$ anti-sevendiagonal matrix, the anti-sevendiagonal matrix of $T$ is denoted by $H$, take the form

$$H = \begin{pmatrix}
0 & \cdots & 0 & d_1 & c_1 & b_1 & a_1 \\
0 & d_2 & c_2 & b_2 & a_2 & & \\
0 & d_3 & c_3 & b_3 & a_3 & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
\end{pmatrix}$$

To do this we take into account the fact that for the $n \times n$ matrix $R = (r_1)$ given by

$$R = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & & & \vdots \\
0 & & & \\
1 & 0 & \cdots & 0 \\
\end{pmatrix} = R'$$

From [1] we can get

$$H = TR.$$  

Moreover, if the matrix $T$ is invertible and its inverse is $T^{-1}$, then we obtain

$$H^{-1} = R'T^{-1} = RT^{-1}.$$  

**Example 3.1.** Consider the $6 \times 6$ matrices $T$ and $H$ given by

$$T = \begin{pmatrix}
1 & 1 & -2 & 1 & 0 & 0 \\
1 & 1 & -1 & -1 & 1 & 0 \\
2 & 0 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & 2 & 1 & 1 \\
0 & -1 & 1 & 2 & 1 & 4 \\
0 & 0 & 1 & 1 & -1 & 1 \\
\end{pmatrix}$$

$$H = \begin{pmatrix}
0 & 0 & 1 & -2 & 1 & 1 \\
0 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & 0 & 1 & 0 \\
1 & 1 & 2 & 1 & -1 & 1 \\
4 & 1 & 2 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}$$

By applying the algorithm 2.3, we get

- $A = [0, 0, 1, 2, 3, 4]'$ and $A' = [-12, -24, -4]'$,
- $B = [0, 1, 0, -1, -2, -3]'$ and $B' = [8, 17, 2]'$,
- $D = [1, 0, 0, -1, -2, -5]'$ and $D' = [8, 24, 4]'$,
- $X = [-20, 0, -20, -20, -20, 0]'$ and $X_6 = 80, X_7 = 0, X_8 = 0$
- $Y = [8, -16, -16, -8, 24]'$ and $Y_6 = 0, Y_7 = -80, Y_8 = 0$
- $Z = [12, -96, -56, -28, 0, 4]'$ and $Z_6 = 0, Z_7 = 0, Z_8 = 80$

**III. ONE ILLUSTRATIVE EXAMPLE**

In this section we are going to give one illustrative example.
$C_3 = [-0.15, 1.2, 0.7, 0.35, 0, -0.05]^T$

$C_4 = [\begin{array}{ccccc}
-0.1 & -0.2 & -0.1 & 0 & 0.3 \\
0.25 & 0.25 & 0.25 & -0.25 & 0.05 \\
2.0 & 0.5 & 0.1 & 1.5 & 0.0 \\
0 & -0.4 & 0.5 & 0.2 & 0.0 \\
0.1 & -0.2 & -0.45 & 0.15 & -0.25 & 0.05 \\
\end{array}]$ 

So,

$$C_3 = \begin{bmatrix} 0.3 & -0.4 & -0.15 & -0.2 & -0.25 & 0.1 \end{bmatrix}^T$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0.5 & 0 & 0.5 & 0 \end{bmatrix}^T$$

$$C_4 = \begin{bmatrix} 0.15 & -0.2 & -0.45 & 0.15 & -0.25 & 0.05 \end{bmatrix}^T$$

Hence, using (31) gives

$$H = \begin{bmatrix} 1 & 0 & 2 & -5 & 6 & -1 \\
-5 & 10 & -5 & 5 & 0 & 0 \\
3 & 0 & -4 & 5 & -2 & 7 \\
-9 & 10 & -3 & 5 & -4 & 14 \\
-4 & 20 & -8 & 0 & -4 & 24 \\
3 & 0 & 6 & 5 & -2 & -3 \\
\end{bmatrix}$$

IV. CONCLUSION

In this paper, we mainly give a fast algorithm for the inverse of a sevendiagonal matrix, and the fast algorithm is appropriate for implementation using computer algebra systems. What’s more, in the light of algorithm 2.3, then we can obtain that the algorithm complexity is $n^2 + 8n + 15$. Base on the illustrative example of section 3 which confirm the validity of this algorithm. The recent algorithm presented in [1] for inverting a tridiagonal and pentadiagonal matrix is a special case of this new algorithm.

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