Persistence of Global Well-Posedness for Fractional Dissipation Boussinesq System

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Abstract—The goal of this paper is to address the global existence and the uniqueness of solution to the 2D fractional dissipation Boussinesq system. And further prove the persistence in the space $H^{1+\varepsilon}(\mathbb{R}^2)\times H^{1+\varepsilon}(\mathbb{R}^2)$, $s \in (0,1)$.

Keywords—existence; uniqueness; fractional dissipation; Boussinesq system

I. INTRODUCTION

In this paper, we study the Cauchy problem for the 2D Boussinesq equations with fractional dissipation. The model reads as follow

$$
\begin{align}
\dot{u}_1 + \nu \Delta^{\alpha} u + u \cdot \nabla u + \nabla P &= \partial_x e_2, \\
\partial_t \theta + \kappa \Delta^{\beta} \theta + u \cdot \nabla \theta &= 0, \\
u(x,0) &= u_0(x), \theta(x,0) = \theta_0(x).
\end{align}
$$

where $u = (u_1, u_2)$ is the velocity vector field, $\theta$ and $P$ denote the scalar temperature and pressure of the fluid, respectively. The positive constants $\nu$ and $\kappa$ denote the viscosity and thermal diffusivity. $e_2 = (0,1)$ is the unit vector in the vertical direction. For the sake of simplicity, we denote $\Lambda = \sqrt{-\Delta}$, the square root of the negative Laplacian.

The Boussinesq system is a nonlinear partial differential equations models the thermal convection and geophysical flows, which plays an important role in the atmospheric sciences and oceanographic turbulence (see, e.g., [1, 2]). Our main focus of the research on the 2D Boussinesq system has been on the global regularity issue when only fractional dissipation is present. So far, there has been a lot of literature about this model.

II. MAIN RESULT

The following is the main result of this paper which asserts the global existence and the uniqueness of solution to the 2D fractional dissipation Boussinesq system (1).

Theorem 1 Let $\nu > 0, \kappa > 0$, $\alpha, \beta \in \left(\frac{1}{2}, 1\right)$ and $\alpha + \beta > \frac{3}{2}$. Assume that $(u_0, \theta_0) \in H^{1+s} \times H^{1+s}, s \in (0,1)$. Then there exists a unique global solution $(u, \theta)$ of the Boussinesq system (1), such that, for any $T > 0$,

$$
\begin{align}
&u \in C([0,T]; H^{1+s}) \cap L^2(0,T; H^{1+s+\alpha}), \\
&\theta \in C([0,T]; H^{1+s}) \cap L^2(0,T; H^{1+s+\beta}).
\end{align}
$$

III. PROOF OF THEOREM

The proof is divided into two main parts: the global existence and uniqueness.

A. The Proof of Global Existence

Taking $L^2$ inner product of the third equation of (1) with $\theta$, we can get

$$
\frac{d}{dt} \|\theta\|^2 + 2\kappa \|\Lambda^{\beta} \theta\|^2 = 0.
$$

Similarly, from the first equation of (1) we know that

$$
\frac{d}{dt} \|u\|^2 + 2\nu \|\Lambda^{\alpha} u\|^2 = \int \partial_x e_2 \cdot u dx \leq \frac{1}{2} \|u\|^2 + \frac{1}{2} \|\theta\|^2.
$$

Summing up (2) and (3), we have

$$
\frac{d}{dt} \left(\|u\|^2 + \|\theta\|^2\right) + 2\nu \|\Lambda^{\alpha} u\|^2 + 2\kappa \|\Lambda^{\beta} \theta\|^2 \leq \frac{1}{2} \|u\|^2 + \frac{1}{2} \|\theta\|^2.
$$

Applying the Gronwall inequality to (4), we can get, for all $t \in [0, T]$
\[ \|u\|^2 + 2\nu\int_0^T \left\| \nabla \theta \right\|^2 \, dt + 2\kappa \int_0^T \left\| \lambda^{\alpha} \theta \right\|^2 \, dt \leq C, \quad (5) \]

where \( C = C(\|u_0\|_{L^2} \| \theta_0 \|_{L^2}) \) is a positive constant.

Next, taking the curl of the Boussinesq system (1), we obtain that

\[ \omega_t + \nu \lambda^\alpha \omega + u \cdot \nabla \omega = \theta_{xx}, \quad (6) \]

where \( \omega = \partial_x u_2 - \partial_y u_1 \). Multiplying (6) by \( \omega \) and integrating the result equation by parts, we have

\[ \frac{d}{dt} \left\| \omega \right\|^2 + 2\nu \left\| \lambda^\alpha \omega \right\|^2 = \int \theta_{xx} \cdot \omega \, dx \leq \frac{1}{2\nu} \left\| \nabla \omega \right\|^2 + \frac{\nu}{2} \left\| \lambda^\alpha \omega \right\|^2. \quad (7) \]

By the condition, and use the embbedding inequality, it follows that

\[ \frac{d}{dt} \left\| \nabla \theta \right\|^2 + 2\kappa \left\| \lambda^{\alpha + \beta} \theta \right\|^2 = 0. \quad (8) \]

Taking \( L^2 \) inner product of the third equation of (1) with \( \lambda^{\alpha + \beta} \theta \), we obtain

\[ \frac{d}{dt} \left\| \lambda^{\alpha + \beta} \theta \right\|^2 + 2\kappa \int_0^T \left\| \lambda^{\alpha + \beta} \theta \right\|^2 \, dt \leq C, \quad (10) \]

where \( C = C(\|u_0\|_{L^2} \| \theta_0 \|_{L^2}) \) is a positive constant.

Taking \( L^2 \) inner product of the third equation of (1) with \( \lambda^{\alpha + \beta} \theta \), we obtain

\[ \frac{d}{dt} \left\| \lambda^{\alpha + \beta} \theta \right\|^2 + 2\kappa \int_0^T \left\| \lambda^{\alpha + \beta} \theta \right\|^2 \, dt \leq -2 \int (u \cdot \nabla \theta) \cdot \lambda^{2\alpha + \beta} \theta \, dx. \quad (11) \]

Since \( u \) is divergence free, then \( u \cdot \nabla \theta = \nabla \cdot (u \theta) \), using the Kate-Ponce inequality from [3] (see also, e.g., [4, 5]), we have

\[ \int (u \cdot \nabla \theta) \cdot \lambda^{2\alpha + \beta} \theta \, dx \leq C \left( \| \lambda^{2\alpha + \beta} u \|_{L^2} \| \theta \|_{L^2} + \| \lambda^{2\alpha + \beta} \theta \|_{L^2} \right) \| \lambda^{2\alpha + \beta} \theta \| \quad (12) \]

Applying the fractional embedding theorems and the Young inequality, we arrive at

\[ \int (u \cdot \nabla \theta) \cdot \lambda^{2\alpha + \beta} \theta \, dx \leq \frac{\nu}{4} \left( \| \lambda^{2\alpha + \beta} u \|_{L^2} + \nu \| \lambda^{2\alpha + \beta} \theta \|_{L^2} \right)^2 + F \left( \| u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \right), \quad (13) \]

where

\[ F = C \left( \| \lambda^{2\alpha + \beta} \|_{L^2} \| \nabla \theta \|_{L^2} + \nu \| \lambda^{2\alpha + \beta} \theta \|_{L^2} \right)^2 + \| \nabla \theta \|_{L^2} \| \lambda^{2\alpha + \beta} \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \right)^2 \]

is an explicit polynomial. Inserting (13) into (11), we can get

\[ \frac{d}{dt} \left\| \lambda^{\alpha + \beta} \theta \right\|^2 + \frac{\nu}{2} \left\| \lambda^{\alpha + \beta} \theta \right\|^2 \leq -\int \lambda^{2\alpha + \beta} u \cdot \lambda^{2\alpha + \beta} \theta \, dx \quad (14) \]

Applying the operator \( \lambda^{\alpha + \beta} \) to the first equation of (1), and taking \( L^2 \) inner product with \( \lambda^{\alpha + \beta} u \), then, integrating by parts, we obtain

\[ \frac{d}{dt} \left\| \lambda^{\alpha + \beta} u \right\|^2 + 2\nu \left\| \lambda^{\alpha + \beta} u \right\|^2 \leq -2 \int \lambda^{\alpha + \beta} \theta \cdot \lambda^{2\alpha + \beta} u \, dx \quad (15) \]

Using the Kate-Ponce inequality again, and applying the fractional embedding theorems and the Young inequality, we can get

\[ \frac{d}{dt} \left\| \lambda^{\alpha + \beta} u \right\|^2 + 2\nu \left\| \lambda^{\alpha + \beta} u \right\|^2 \leq -\int \lambda^{\alpha + \beta} \theta \cdot \lambda^{2\alpha + \beta} u \, dx \quad (15) \]

where

\[ \int \lambda^{\alpha + \beta} \theta \cdot \lambda^{2\alpha + \beta} u \, dx \leq C \left( \| \lambda^{2\alpha + \beta} u \|_{L^2} \| \theta \|_{L^2} + \| \lambda^{2\alpha + \beta} \theta \|_{L^2} \right) \| \lambda^{2\alpha + \beta} \theta \| \quad (16) \]

\[ \int \lambda^{\alpha + \beta} \theta \cdot \lambda^{2\alpha + \beta} u \, dx \leq \frac{\nu}{4} \left( \| \lambda^{2\alpha + \beta} u \|_{L^2} + \nu \| \lambda^{2\alpha + \beta} \theta \|_{L^2} \right)^2 + F \left( \| \lambda^{2\alpha + \beta} u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \right), \quad (13) \]

where

\[ F = C \left( \| \lambda^{2\alpha + \beta} \|_{L^2} \| \nabla \theta \|_{L^2} + \nu \| \lambda^{2\alpha + \beta} \theta \|_{L^2} \right)^2 + \| \nabla \theta \|_{L^2} \| \lambda^{2\alpha + \beta} \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \right)^2 \]

is an explicit polynomial. Inserting (13) into (11), we can get

\[ \frac{d}{dt} \left\| \lambda^{\alpha + \beta} \theta \right\|^2 + \frac{\nu}{2} \left\| \lambda^{\alpha + \beta} \theta \right\|^2 \leq -\int \lambda^{\alpha + \beta} \theta \cdot \lambda^{2\alpha + \beta} u \, dx \quad (14) \]

Applying the operator \( \lambda^{\alpha + \beta} \) to the first equation of (1), and taking \( L^2 \) inner product with \( \lambda^{\alpha + \beta} u \), then, integrating by parts, we obtain
\[
G = C \left( \|M^{2}u\|^{2} \right)
\]
\[
+ \left\|\frac{\partial M^{2}u}{\partial x}\right\|^{2}
\]
is an explicit polynomial. Using the Hölder inequality and the Cauchy inequality, we have
\[
\int \Lambda^{1+\delta}(\partial_{\alpha}+\partial_{\beta})\Lambda^{1+\delta} udx \leq \frac{1}{2} \left\|\Lambda^{1+\delta}(\partial_{\alpha}+\partial_{\beta})\right\|^{2} + \frac{1}{2} \left\|\Lambda^{1+\delta} u\right\|^{2}.
\]  
(17)

Inserting (15) and (16) into (14), we obtain that
\[
\frac{d}{dt} \left( \left\|\Lambda^{1+\delta} u\right\|^{2} + \frac{3\nu}{2} \left\|\Lambda^{1+\delta} \theta\right\|^{2} \right) = C \left( \left\|\Lambda^{1+\delta} \theta\right\|^{2} + \left\|\Lambda^{1+\delta} u\right\|^{2} \right) + F + G.
\]  
(18)

Summing (14) and (18) together, we get
\[
\frac{d}{dt} \left( \left\|\Lambda^{1+\delta} u\right\|^{2} + \left\|\Lambda^{1+\delta} \theta\right\|^{2} \right) = C \left( \left\|\Lambda^{1+\delta} u\right\|^{2} + \left\|\Lambda^{1+\delta} \theta\right\|^{2} \right) + F + G.
\]  
(19)

Applying the Gronwall inequality, and by (5) and (10), we find that, for all \( t \in [0, T] \)
\[
\left\|\Lambda^{1+\delta} u\right\|^{2} + \left\|\Lambda^{1+\delta} \theta\right\|^{2} = C
\]
where
\[
C = C \left( \left\|\Lambda^{1+\delta} u_{0}\right\| + \left\|\Lambda^{1+\delta} \theta_{0}\right\| \right)^{2}(\kappa, s, T)
\]
is a positive constant.

**B. The Proof of Uniqueness**

For any fixed \( T > 0 \), suppose there are two solution \((u_{1}, \theta_{1}, \rho_{1})\) and \((u_{2}, \theta_{2}, \rho_{2})\) to the Boussinesq system (1). Setting \( \ddot{u} = u_{1} - u_{2}, \ddot{\theta} = \theta_{1} - \theta_{2}, \) and \( \ddot{P} = \ddot{u} \cdot \ddot{u} + \nabla \ddot{u} \cdot \nabla \ddot{u} \), then \((\ddot{u}, \ddot{\theta}, \ddot{P})\) satisfies
\[
\begin{align*}
\dddot{u} &+ \nu \lambda^{2} \ddot{u} + u_{1} \cdot \nabla \ddot{u} + \ddot{u} \cdot \nabla u_{2} + \nabla \ddot{P} = \ddot{\theta}_{2}, \\
\dddot{\theta} &+ \nu \lambda^{2} \ddot{\theta} + u_{1} \cdot \nabla \ddot{\theta} + \ddot{u} \cdot \nabla \ddot{\theta} = 0, \\
\dddot{u}(x,0) = 0, \quad \dddot{\theta}(x,0) = 0.
\end{align*}
\]  
(20)

Taking the \( L^{2} \) inner product of the first equation of (20) with \( \ddot{u} \) and the third equation with \( \ddot{\theta} \), respectively, we have
\[
\frac{d}{dt} \left( \left\|\ddot{u}\right\|^{2} + \left\|\ddot{\theta}\right\|^{2} \right) + 2\nu \left\|\nabla \ddot{u}\right\|^{2} + 2\kappa \left\|\nabla \ddot{\theta}\right\|^{2} \leq \int_{\Omega} \ddot{\theta} \cdot \ddot{u}_{2} \cdot \ddot{u}_{1} dx + \int_{\Omega} \ddot{u} \cdot \nabla \ddot{u}_{2} \cdot \ddot{u} dx.
\]  
(21)

Using the Hölder inequality and the Cauchy inequality, we can get as follows
\[
\int_{\Omega} \ddot{\theta} \cdot \ddot{u}_{2} \cdot \ddot{u}_{1} dx \leq \frac{1}{2} \left\|\ddot{u}_{1}\right\|^{2} + \frac{1}{2} \left\|\ddot{u}_{2}\right\|^{2},
\]  
(22)

\[
-\int_{\Omega} \ddot{u} \cdot \nabla \ddot{u}_{2} \cdot \ddot{u} dx \leq C \left\|\nabla \ddot{u}_{2}\right\| \left\|\ddot{u}\right\| \left\|\ddot{u}\right\|.
\]  
(23)

Inserting (22)-(24) into (21), we arrive at
\[
\frac{d}{dt} \left( \left\|\ddot{u}\right\|^{2} + \left\|\ddot{\theta}\right\|^{2} \right) + \nu \left\|\nabla \ddot{u}\right\|^{2} + \kappa \left\|\nabla \ddot{\theta}\right\|^{2} \leq C \left( \left\|\nabla \ddot{u}_{2}\right\|^{2} + \left\|\ddot{u}_{2}\right\|^{2} \right).
\]  
(25)

Using the Gronwall inequality and the estimates for \( \ddot{u}_{2} \) and \( \ddot{u}_{2} \), (25) implies that, for any \( t \geq 0 \),
\[
e^{-Ct} \left( \left\|\ddot{u}_{2}\right\|^{2} + \left\|\ddot{\theta}_{2}\right\|^{2} \right) \leq \left( \left\|\ddot{u}(0)\right\|^{2} + \left\|\ddot{\theta}(0)\right\|^{2} \right),
\]  
(26)

\text{i.e., } \ddot{u} = 0, \ddot{\theta} = 0, \theta_{1} = \theta_{2}, u_{1} = u_{2}. \text{ So the solution of the Boussinesq system (1) is unique.}

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