

## Hesitant Fuzzy Filters in $BE$ -algebras

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### Abstract

In this paper, we introduce the notion of hesitant fuzzy (implicative) filters and get some results on  $BE$ -algebras and show that every hesitant fuzzy implicative filter is a hesitant fuzzy filter but not the converse. Finally, we state and prove the relationship between hesitant fuzzy (implicative) filters and  $\gamma$ -inclusive sets.

*Keywords:*  $BE$ -algebra, Hesitant fuzzy (implicative) filter, Hesitant level subset,  $\gamma$ -inclusive.

### 1. Introduction

H. S. Kim and Y. H. Kim introduced the notion of a  $BE$ -algebra as a generalization of a dual  $BCK$ -algebra [3]. A. Borumand Saeid et al. defined some types of filters in  $BE$ -algebras and showed the relationship between them [2]. B. L. Meng give a procedure which generated a filter by a subset in a transitive  $BE$ -algebra [5]. Recently, A. Walendziak introduced the notion of a normal filter in  $BE$ -algebras and showed that there is a bijection between congruence relations and filters in commutative  $BE$ -algebras [13].

Fuzzy sets were introduced in 1965 by Zadeh [15] and then fuzzification ideas have been applied to other algebraic structures such as groups and  $BL$ -algebras. Fuzzy sets and its extensions have provided successful results dealing with uncertainty

in different problems. Worldwide, there has been a rapid growth in interest in applications of fuzzy sets and some generalization of this is discussed by authors such as intuitionistic fuzzy sets, interval-valued fuzzy sets, type- $n$  fuzzy sets and fuzzy multi-sets. Also, another generalization of this theory was proposed by Torra and Narukawa [11] and Torra [10]. The relationships among hesitant fuzzy sets and other generalizations of fuzzy sets such as intuitionistic fuzzy sets, type-2 fuzzy sets and fuzzy multi sets were discussed. They showed that the envelope of a hesitant fuzzy set is an intuitionistic fuzzy set. Also, they proved that the operations they proposed are consistent with the ones of intuitionistic fuzzy sets when applied to the envelopes of hesitant fuzzy sets. Then some researchers who have defined divers concepts, extensions, aggregation operators and measures to handle with hesitant information.

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It may be mentioned that hesitant fuzzy sets can reflect the humans hesitancy more objectively than the other classical extensions of fuzzy sets and suit the modeling of quantitative settings. We can try to manage those situation, where a set of values are possible in the definition process of the membership of an element with this theory.

In this paper, we introduce the notion of hesitant fuzzy (implicative) filters and get some useful properties. In fact, we show that in self distributive BE-algebras two concepts of hesitant fuzzy implicative filter and hesitant fuzzy filter are equivalent. Also, the notion of  $\gamma$ -inclusive set which denoted by  $i_A(h_A; \gamma)$  is defined.

## 2. Preliminaries

In this section, we cite the fundamental definitions that will be used in the sequel:

**Definition 1.** (Kim and Kim [3]) By a BE-algebra we shall mean an algebra  $(X; *, 1)$  of type  $(2, 0)$  satisfying the following axioms:

- (BE1)  $x * x = 1$ ,
- (BE2)  $x * 1 = 1$ ,
- (BE3)  $1 * x = x$ ,
- (BE4)  $x * (y * z) = y * (x * z)$ , for all  $x, y, z \in X$ .

From now on  $X$  is a BE-algebra, unless otherwise is stated. We introduce a relation " $\leq$ " on  $X$  by  $x \leq y$  if and only if  $x * y = 1$ . A BE-algebra  $X$  is said to be a self distributive if  $x * (y * z) = (x * y) * (x * z)$ , for all  $x, y, z \in X$ . A BE-algebra  $X$  is said to be commutative if satisfies:

$$(x * y) * y = (y * x) * x, \text{ for all } x, y \in X.$$

**Proposition 1.** (Walendziak [13]) If  $X$  is a commutative BE-algebra, then for all  $x, y \in X$ ,

$$x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

We note that " $\leq$ " is reflexive by (BE1). If  $X$  is self distributive then relation " $\leq$ " is a transitive ordered set on  $X$ , because if  $x \leq y$  and  $y \leq z$ , then

$$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1.$$

Hence  $x \leq z$ . If  $X$  is commutative then by Proposition 1, relation " $\leq$ " is antisymmetric. Hence if  $X$

is a commutative self distributive BE-algebra, then relation " $\leq$ " is a partial ordered set on  $X$ .

**Proposition 2.** (Kim and Kim [3]) In a BE-algebra  $X$ , the following holds:

- (i)  $x * (y * x) = 1$ ,
- (ii)  $y * ((y * x) * x) = 1$ , for all  $x, y \in X$ .

A subset  $F$  of  $X$  is called a filter of  $X$  if it satisfies: (F1)  $1 \in F$ , (F2)  $x \in F$  and  $x * y \in F$  imply  $y \in F$ . Define

$$A(x, y) = \{z \in X : x * (y * z) = 1\},$$

which is called an upper set of  $x$  and  $y$ . It is easy to see that  $1, x, y \in A(x, y)$ , for any  $x, y \in X$ . Every upper set  $A(x, y)$  need not be a filter of  $X$  in general.

**Definition 2.** (Borumand and Rezaei [2]) A non-empty subset  $F$  of  $X$  is called an implicative filter if satisfies the following conditions:

- (IF1)  $1 \in F$ ,
- (IF2)  $x * (y * z) \in F$  and  $x * y \in F$  imply that  $x * z \in F$ , for all  $x, y, z \in X$ .

If we replace  $x$  of the condition (IF2) by the element 1, then it can be easily observed that every implicative filter is a filter. However, every filter is not an implicative filter as shown in the following example.

**Example 1.** Let  $X = \{1, a, b\}$  be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	a	1

Then  $F = \{1, a\}$  is a filter of  $X$ , but it is not an implicative filter, since  $1 * (a * b) = 1 * a = a \in F$  and  $1 * a = a \in F$  but  $1 * b = b \notin F$ .

**Definition 3.** Let  $(X_1; *, 1)$  and  $(X_2; \circ, 1')$  be two BE-algebras. Then a mapping  $f : X_1 \rightarrow X_2$  is called a homomorphism if  $f(x_1 * x_2) = f(x_1) \circ f(x_2)$ , for all  $x_1, x_2 \in X_1$ . It is clear that if  $f : X_1 \rightarrow X_2$  is a homomorphism, then  $f(1) = 1'$ .

**Definition 4.** (Rezaei and Borumand [6]) A fuzzy set  $\mu$  of  $X$  is called a fuzzy filter if satisfies the following conditions:

- (FF1)  $\mu(1) \geq \mu(x)$ ,  
(FF2)  $\mu(y) \geq \min\{\mu(x*y), \mu(x)\}$ , for all  $x, y \in X$ .

**Definition 5.** (Rao [9]) A fuzzy set  $\mu$  of  $X$  is called a fuzzy implicative filter of  $X$  if satisfies the following conditions:

- (FIF1)  $\mu(1) \geq \mu(x)$ ,  
(FIF2)  $\mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(x*y)\}$ , for all  $x, y, z \in X$ .

If we replace  $x$  of the condition (FIF2) by the element 1, then it can be easily observed that every fuzzy implicative filter is a fuzzy filter. However, every fuzzy filter is not a fuzzy implicative filter as shown in the following example.

**Example 2.** (Rao [9]) Let  $X = \{1, a, b, c, d\}$  be a BE-algebra with the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can be easily verified that  $(X; *, 1)$  is a BE-algebra. Define a fuzzy set  $\mu$  on  $X$  as follows:

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = 1, a \\ 0.2 & \text{otherwise} \end{cases}$$

Then clearly  $\mu$  is a fuzzy filter of  $X$ , but it is not a fuzzy implicative filter of  $X$ , since

$$\mu(b*c) \not\geq \min\{\mu(b*(d*c)), \mu(b*d)\}.$$

**Definition 6.** (Torra [10]) Let  $X$  be a reference set. Then a hesitant fuzzy set  $HFS$   $A$  in  $X$  is represented mathematical as:

$$A = \{ \langle x, h_A(x) \rangle : h_A(x) \in \rho([0, 1]), x \in X \},$$

where  $\rho([0, 1])$  is the power set of  $[0, 1]$ .

So, we can define a set of fuzzy sets an  $HFS$  by union of their membership functions.

**Definition 7.** (Torra [10]) Let  $A = \{\mu_1, \mu_2, \dots, \mu_n\}$  be a set of  $n$  membership functions. The  $HFS$  that is associated with  $A$ ,  $h_A$ , is defined as

$$\begin{aligned} h_A : X &\rightarrow \rho([0, 1]) \\ h_A(x) &= \bigcup_{\mu \in A} \{\mu(x)\}. \end{aligned}$$

It is remarkable that this definition is quite suitable to decision making, when experts have to assess a set of alternatives. In such a case,  $A$  represents the assessments of the experts for each alternative and  $h_A$  the assessments of the set of experts. However, note that it only allows to recover those  $HFS$ s whose memberships are given by sets of cardinality less than or equal to  $n$ .

For convenience, Xia and Xu in [14] named the set  $h = h_A(x)$  as a hesitant fuzzy element  $HFE$ . The family of all hesitant fuzzy elements defined on  $X$  by  $HFE(X)$ .

**Definition 8.** (Verma and Dev Sharma [12]) Let  $h, h_1, h_2 \in HFE(X)$  and  $\lambda \in [0, 1]$ . Then the operations complement, union and intersection are defined as follows:

- (i)  $h^c = \{1 - \gamma : \gamma \in h\}$ ,
- (ii)  $h_1 \sqcup h_2 = \{\max(\gamma_1, \gamma_2) : \gamma_1 \in h_1, \gamma_2 \in h_2\}$ ,
- (iii)  $h_1 \sqcap h_2 = \{\min(\gamma_1, \gamma_2) : \gamma_1 \in h_1, \gamma_2 \in h_2\}$ ,
- (iv)  $h_1 \oplus h_2 = \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2 : \gamma_1 \in h_1, \gamma_2 \in h_2\}$ ,
- (v)  $h_1 \otimes h_2 = \{\gamma_1 \gamma_2 : \gamma_1 \in h_1, \gamma_2 \in h_2\}$ ,
- (vi)  $\lambda h = \{1 - (1 - \gamma)^\lambda : \gamma \in h\}$ .

### 3. Hesitant Fuzzy filters

In what follows, we introduce binary operation " $\sqsubseteq$ " as follow:

$$\begin{aligned} \sqsubseteq : \rho([0, 1]) \times \rho([0, 1]) &\rightarrow \rho([0, 1]) \\ (A, B) &\mapsto A \sqsubseteq B \end{aligned}$$

and

$$A \sqsubseteq B \text{ if and only if } x \in A \text{ imply } x \in B.$$

It is obvious that  $\sqsubseteq$  is a partial order set on  $\rho([0, 1])$ .

**Definition 9.** Hesitant fuzzy set  $A$  of  $X$  is called a *hesitant fuzzy filter* if satisfies the following conditions:

- (HFF1)  $h_A(x) \sqsubseteq h_A(1)$ ,
- (HFF2)  $h_A(x) \sqcap h_A(x*y) \sqsubseteq h_A(y)$ , for all  $x, y \in X$ .

Denote the set of all hesitant fuzzy filters of  $X$  by  $HFF(X)$ .

**Note.** If  $|h_A(x)| = 1$ , for all  $x \in X$ , then  $h_A$  is a fuzzy filter. In fact in this case we put “ $\leq := \sqsubseteq$ ” and “ $\min := \sqcap$ ”.

**Example 3.** Let  $X = \{1, a, b\}$  be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

Let  $t_1 = \{0.2, 0.3\}$ ,  $t_2 = \{0.4, 0.5\}$  and  $t_3 = \{0.6, 0.8\}$ . Define  $A$  as  $h_A(1) = t_3$ ,  $h_A(a) = t_2$  and  $h_A(b) = t_1$ . Then  $A$  is a hesitant fuzzy filter.

**Proposition 3.** Let  $A \in HFF(X)$  and  $x, y, z, a_i \in X$  for  $i = 1, \dots, n$ . Then

- (i) if  $x \leq y$ , then  $h_A(x) \sqsubseteq h_A(y)$ ,
- (ii)  $h_A(x) \sqsubseteq h_A(y*x)$ ,
- (iii)  $h_A(x) \sqcap h_A(y) \sqsubseteq h_A(x*y)$ ,
- (iv)  $h_A(x) \sqsubseteq h_A((x*y)*y)$ ,
- (v)  $h_A(x) \sqcap h_A(y) \sqsubseteq h_A((x*(y*z))*z)$ ,
- (vi) if  $h_A(y) \sqcap h_A((x*y)*z) \sqsubseteq h_A(z*x)$ , then  $h_A$  is antitonic (i.e. if  $x \leq y$ , then  $h_A(y) \sqsubseteq h_A(x)$ ),
- (vii) if  $z \in A(x, y)$ , then  $h_A(x) \sqcap h_A(y) \sqsubseteq h_A(z)$ ,
- (viii) if  $\prod_{i=1}^n a_i * x = 1$ , then  $\sqcap_{i=1}^n h_A(a_i) \sqsubseteq h_A(x)$ , where

$$\prod_{i=1}^n a_i * x = a_n * (a_{n-1} * (\dots (a_1 * x) \dots)).$$

**Proof.** (i). Let  $x \leq y$ . Then  $x*y = 1$  and so by using (BE2) and Definition 9 (HFF2), we have

$$h_A(x) = h_A(x) \sqcap h_A(1) = h_A(x) \sqcap h_A(y*1) \sqsubseteq h_A(y).$$

(ii). Since  $x \leq (y*x)$ , by using (i) we have  $h_A(x) \sqsubseteq h_A(y*x)$ .

(iii). By using (ii) we have

$$h_A(x) \sqcap h_A(y) \sqsubseteq h_A(y) \sqsubseteq h_A(x*y).$$

(iv). It follows from Definition 9,

$$\begin{aligned} h_A(x) &= h_A(x) \sqcap h_A(1) \\ &= h_A(x) \sqcap h_A((x*y)*(x*y)) \\ &= h_A(x) \sqcap h_A(x*((x*y)*x)) \\ &\sqsubseteq h_A((x*y)*y). \end{aligned}$$

(v). From (iv) we have

$$\begin{aligned} h_A(x) \sqcap h_A(y) &\sqsubseteq h_A((x*(y*x))*(y*x)) \sqcap h_A(y) \\ &\sqsubseteq h_A((x*(y*z))*z). \end{aligned}$$

(vi). Let  $x \leq y$ , that is,  $x*y = 1$ .

$$\begin{aligned} h_A(y) &= h_A(y) \sqcap h_A(1*1) \\ &= h_A(y) \sqcap h_A((x*y)*1) \\ &\sqsubseteq h_A(1*x) \\ &= h_A(x). \end{aligned}$$

(vii). Let  $z \in A(x, y)$ . Then  $x*(y*z) = 1$ . Hence

$$\begin{aligned} h_A(x) \sqcap h_A(y) &= h_A(x) \sqcap h_A(y) \sqcap h_A(1) \\ &= h_A(x) \sqcap h_A(y) \sqcap h_A(x*(y*z)) \\ &\sqsubseteq h_A(y) \sqcap h_A(y*z) \\ &\sqsubseteq h_A(z). \end{aligned}$$

(viii). The proof is by induction on  $n$ . By (vii) it is true for  $n = 1, 2$ . Assume that it satisfies for  $n = k$ , that is,

$$\prod_{i=1}^k a_i * x = 1 \Rightarrow \sqcap_{i=1}^k h_A(a_i) \sqsubseteq h_A(x),$$

for all  $a_1, \dots, a_k, x \in X$ .

Suppose that  $\prod_{i=1}^{k+1} a_i * x = 1$ , for all  $a_1, \dots, a_k, a_{k+1}, x \in X$ . Then

$$\sqcap_{i=2}^{k+1} h_A(a_i) \sqsubseteq h_A(a_1 * x).$$

Since  $A$  is a hesitant fuzzy filter of  $X$ , we have

$$\begin{aligned} \sqcap_{i=1}^{k+1} h_A(a_i) &= (\sqcap_{i=2}^{k+1} h_A(a_i)) \sqcap h_A(a_1) \\ &\sqsubseteq h_A(a_1 * x) \sqcap h_A(a_1) \\ &\sqsubseteq h_A(x). \end{aligned}$$

□

In the following example shows that if  $h_A \in HFF(X)$ , then  $h_A^c \notin HFF(X)$ , in general.

**Example 4.** In Example 3, we have  $h_A^c(1) = \{0.4, 0.2\}$ ,  $h_A^c(a) = \{0.6, 0.5\}$  and  $h_A^c(b) = \{0.8, 0.7\}$  and so  $h_A^c \notin HFF(X)$  because  $h_A^c(1) \sqsubset h_A^c(a)$ .

**Theorem 4.** Let  $h, h_1, h_2 \in HFF(X)$  and  $\lambda \in [0, 1]$ . Then

- (i)  $h_1 \sqcup h_2 \in HFF(X)$ ,
- (ii)  $h_1 \sqcap h_2 \in HFF(X)$ ,
- (iii)  $h_1 \oplus h_2 \in HFF(X)$ ,
- (iv)  $h_1 \otimes h_2 \in HFF(X)$ ,
- (v)  $\lambda h \in HFF(X)$ .

**Proof.** (i). Assume that  $h_1, h_2 \in HFF(X)$  and  $x \in X$ . Then

$$\begin{aligned} (h_1 \sqcup h_2)(x) &= \{\max(\gamma_1, \gamma_2) : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \\ &\sqsubseteq \{\max(\gamma_1, \gamma_2) : \gamma_1 \in h_1(1), \gamma_2 \in h_2(1)\} \\ &= (h_1 \sqcup h_2)(1). \end{aligned}$$

Now, we have

$$\begin{aligned} (h_1 \sqcup h_2)(x * y) \sqcap (h_1 \sqcup h_2)(x) &= \{\max(\gamma_1, \gamma_2) : \gamma_1 \in h_1(x * y), \gamma_2 \in h_2(x * y)\} \\ &\sqcap \{\max(\eta_1, \eta_2) : \eta_1 \in h_1(x), \eta_2 \in h_2(x)\} \\ &= \{\max(\beta_1, \beta_2) : \beta_1 \in h_1(x * y) \sqcap h_1(x), \beta_2 \in h_2(x * y) \sqcap h_2(x)\} \\ &\sqsubseteq \{\max(\beta_1, \beta_2) : \beta_1 \in h_1(y), \beta_2 \in h_2(y)\} \\ &= (h_1 \sqcup h_2)(y). \end{aligned}$$

Therefore,  $h_1 \sqcup h_2 \in HFF(X)$ .

(ii). Assume that  $h_1, h_2 \in HFF(X)$  and  $x \in X$ . Then

$$\begin{aligned} (h_1 \sqcap h_2)(x) &= \{\min(\gamma_1, \gamma_2) : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \\ &\sqsubseteq \{\min(\gamma_1, \gamma_2) : \gamma_1 \in h_1(1), \gamma_2 \in h_2(1)\} \\ &= (h_1 \sqcap h_2)(1). \end{aligned}$$

Now, we have

$$\begin{aligned} (h_1 \sqcap h_2)(x * y) \sqcap (h_1 \sqcap h_2)(x) &= \{\min(\gamma_1, \gamma_2) : \gamma_1 \in h_1(x * y), \gamma_2 \in h_2(x * y)\} \\ &\sqcap \{\min(\eta_1, \eta_2) : \eta_1 \in h_1(x), \eta_2 \in h_2(x)\} \\ &= \{\min(\beta_1, \beta_2) : \beta_1 \in h_1(x * y) \sqcap h_1(x), \beta_2 \in h_2(x * y) \sqcap h_2(x)\} \\ &\sqsubseteq \{\min(\beta_1, \beta_2) : \beta_1 \in h_1(y), \beta_2 \in h_2(y)\} \\ &= (h_1 \sqcap h_2)(y). \end{aligned}$$

$$= (h_1 \sqcap h_2)(y).$$

Therefore,  $h_1 \sqcap h_2 \in HFF(X)$ .

(iii). Assume that  $h_1, h_2 \in HFF(X)$  and  $x \in X$ . Then  $(h_1 \oplus h_2)(x)$

$$\begin{aligned} &= \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \\ &\sqsubseteq \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2 : \gamma_1 \in h_1(1), \gamma_2 \in h_2(1)\} \\ &= (h_1 \oplus h_2)(1). \end{aligned}$$

Now, we have

$$\begin{aligned} (h_1 \oplus h_2)(x * y) \sqcap (h_1 \oplus h_2)(x) &= \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2 : \gamma_1 \in h_1(x * y), \gamma_2 \in h_2(x * y)\} \\ &\sqcap \{\eta_1 + \eta_2 - \eta_1 \eta_2 : \eta_1 \in h_1(x), \eta_2 \in h_2(x)\} \\ &= \{\beta_1 + \beta_2 - \beta_1 \beta_2 : \beta_1 \in h_1(x * y) \sqcap h_1(x), \beta_2 \in h_2(x * y) \sqcap h_2(x)\} \\ &\sqsubseteq \{\beta_1 + \beta_2 - \beta_1 \beta_2 : \beta_1 \in h_1(y), \beta_2 \in h_2(y)\} \\ &= (h_1 \oplus h_2)(y). \end{aligned}$$

Therefore,  $h_1 \oplus h_2 \in HFF(X)$ .

(iv). Assume that  $h_1, h_2 \in HFF(X)$  and  $x \in X$ . Then

$$\begin{aligned} (h_1 \otimes h_2)(x) &= \{\gamma_1 \gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \\ &\sqsubseteq \{\gamma_1 \gamma_2 : \gamma_1 \in h_1(1), \gamma_2 \in h_2(1)\} \\ &= (h_1 \otimes h_2)(1). \end{aligned}$$

Now, we have

$$\begin{aligned} (h_1 \otimes h_2)(x * y) \sqcap (h_1 \otimes h_2)(x) &= \{\gamma_1 \gamma_2 : \gamma_1 \in h_1(x * y), \gamma_2 \in h_2(x * y)\} \\ &\sqcap \{\eta_1 \eta_2 : \eta_1 \in h_1(x), \eta_2 \in h_2(x)\} \\ &= \{\beta_1 \beta_2 : \beta_1 \in h_1(x * y) \sqcap h_1(x), \beta_2 \in h_2(x * y) \sqcap h_2(x)\} \\ &\sqsubseteq \{\beta_1 \beta_2 : \beta_1 \in h_1(y), \beta_2 \in h_2(y)\} \\ &= (h_1 \otimes h_2)(y). \end{aligned}$$

Therefore,  $h_1 \otimes h_2 \in HFF(X)$ .

(v). Assume that  $h \in HFF(X)$ ,  $x \in X$  and  $\lambda \in [0, 1]$ .

$$\begin{aligned} \lambda h(x) &= \{1 - (1 - \gamma)^\lambda : \gamma \in h(x)\} \\ &\sqsubseteq \{1 - (1 - \gamma)^\lambda : \gamma \in h(1)\} \\ &= \lambda h(1). \end{aligned}$$

Now, we have

$$\begin{aligned} \lambda h(x * y) \sqcap \lambda h(x) &= \{1 - (1 - \gamma)^\lambda : \gamma \in h(x * y)\} \\ &\sqcap \{1 - (1 - \eta)^\lambda : \eta \in h(x)\} \\ &= \{1 - (1 - \beta)^\lambda : \beta \in h(x * y) \sqcap h(x)\} \\ &\sqsubseteq \{1 - (1 - \beta)^\lambda : \beta \in h(y)\} \\ &= \lambda h(y). \end{aligned}$$

Therefore,  $\lambda h \in HFF(X)$ .  $\square$

**Lemma 5.** If  $\{h_i\}_{i \in \Lambda} \in HFF(X)$ , then  $\bigcap_{i \in \Lambda} h_i$ , is too.

**Proof.** Straightforward.  $\square$

Since the set  $HFF(X)$  is closed under arbitrary intersections, we have the following theorem.

**Theorem 6.**  $(HFF(X); \sqsubseteq)$  is a complete lattice, but it is not a Boolean algebra.

**Proof.** By Theorem 4 and Lemma 5, the proof is obvious. Example 4 shows that it is not a Boolean algebra.  $\square$

**Theorem 7.** Let  $A \in HFF(X)$ . Then the set

$$X_{h_A} = \{x \in X : h_A(x) = h_A(1)\},$$

is a filter of  $X$ .

**Proof.** Obviously,  $1 \in X_{h_A}$ . Let  $x, x * y \in X_{h_A}$ . Then  $h_A(x) = h_A(x * y) = h_A(1)$ . Now, by Definition 9, we have

$$h_A(1) = h_A(x) \sqcap h_A(x * y) \sqsubseteq h_A(y) \sqsubseteq h_A(1).$$

Hence  $h_A(y) = h_A(1)$ . Therefore,  $y \in X_{h_A}$ .  $\square$

Let  $\gamma \in \rho([0, 1])$ . For a hesitant fuzzy filter  $A$  of  $X$ ,  $\gamma$ -inclusive set which denoted by  $i_A(h_A; \gamma)$  is defined as follows:

$$i_A(h_A; \gamma) := \{x \in A : \gamma \sqsubseteq h_A(x)\}.$$

It is obvious that if  $\beta \sqsubseteq \gamma$ , then  $i_A(h_A; \gamma) \sqsubseteq i_A(h_A; \beta)$ , for all  $\gamma, \beta \in \rho([0, 1])$ .

**Example 5.** In Example 3,  $\gamma := \{0.1, 0.4\}$ , we have  $i_A(h_A; \gamma) = \{1, a\}$ .

**Theorem 8.** Let  $A \in HFS(X)$ . The following are equivalent:

- (i)  $A \in HFF(X)$ ,
- (ii)  $(\forall \gamma \in \rho([0, 1])) i_A(h_A; \gamma) \neq \emptyset$  imply  $i_A(h_A; \gamma)$  is a filter of  $X$ .

**Proof.** (i)  $\Rightarrow$  (ii). Let  $x, y \in X$  be such that  $x, x * y \in i_A(h_A; \gamma)$ , for any  $\gamma \in \rho([0, 1])$ . Then  $\gamma \sqsubseteq h_A(x)$  and  $\gamma \sqsubseteq h_A(x * y)$ . Hence

$$\gamma \sqsubseteq h_A(x) \sqcap h_A(x * y) \sqsubseteq h_A(y).$$

Since  $h_A$  is a hesitant fuzzy filter, we have  $y \in i_A(h_A; \gamma)$ .

(ii)  $\Rightarrow$  (i). Let  $i_A(h_A; \gamma)$  be a filter of  $X$ , for any  $\gamma \in \rho([0, 1])$  with  $i_A(h_A; \gamma) \neq \emptyset$ . Put  $h_A(x) = \gamma$ , for any  $x \in X$ . Then  $x \in i_A(h_A; \gamma)$ . Since  $i_A(h_A; \gamma)$  is a filter of  $X$ , we have  $1 \in i_A(h_A; \gamma)$  and so  $h_A(x) = \gamma \sqsubseteq h_A(1)$ .

Now, for any  $x, y \in X$ , let  $h_A(x * y) = \gamma_{x*y}$  and  $h_A(x) = \gamma_x$ . Put  $\gamma = \gamma_{x*y} \sqcap \gamma_x$ . Then  $x, x * y \in i_A(h_A; \gamma)$ , so  $y \in i_A(h_A; \gamma)$ . Hence  $\gamma \sqsubseteq h_A(y)$  and so

$$h_A(x * y) \sqcap h_A(x) = \gamma_{x*y} \sqcap \gamma_x = \gamma \sqsubseteq h_A(y).$$

Therefore,  $A \in HFF(X)$ .  $\square$

**Theorem 9.** Let  $A \in HFF(X)$ . Then for all  $a, b \in X$  and  $\gamma \in \rho([0, 1])$

$$(a, b \in i_A(h_A; \gamma) \Rightarrow A(a, b) \sqsubseteq i_A(h_A; \gamma)).$$

**Proof.** Assume that  $A \in HFF(X)$ . Let  $a, b \in X$  be such that  $a, b \in i_A(h_A; \gamma)$ . Then  $\gamma \sqsubseteq h_A(a)$  and  $\gamma \sqsubseteq h_A(b)$ . Let  $c \in A(a, b)$ . Hence  $a * (b * c) = 1$ . Now, by Proposition 3 (v), we have

$$\begin{aligned} \gamma &\sqsubseteq h_A(a) \sqcap h_A(b) \\ &\sqsubseteq h_A((a * (b * c)) * c) \\ &= h_A(1 * c) \\ &= h_A(c). \end{aligned}$$

Then  $c \in i_A(h_A; \gamma)$ . Therefore,  $A(a, b) \subseteq i_A(h_A; \gamma)$ .  $\square$

**Corollary 10.** Let  $A \in HFF(X)$ . Then for all  $\gamma \in \rho([0, 1])$

$$(i_A(h_A; \gamma) \neq \emptyset \Rightarrow i_A(h_A; \gamma) = \bigsqcup_{a, b \in i_A(h_A; \gamma)} A(a, b)).$$

**Proof.** It is sufficient prove that

$$i_A(h_A; \gamma) \sqsubseteq \bigsqcup_{a, b \in i_A(h_A; \gamma)} A(a, b).$$

For this, assume that  $x \in i_A(h_A; \gamma)$ . Since  $x * (1 * x) = 1$ , we have  $x \in A(x, 1)$ . Hence

$$\begin{aligned} i_A(h_A; \gamma) &\subseteq A(x, 1) \\ &\subseteq \bigsqcup_{x \in i_A(h_A; \gamma)} A(x, 1) \\ &\subseteq \bigsqcup_{x, y \in i_A(h_A; \gamma)} A(x, y). \end{aligned}$$

□

**Theorem 11.** Let  $A \in HFS(X)$ . Define a Hesitant fuzzy set  $h_{A^*}$  of  $X$  as follows

$$h_{A^*} : X \rightarrow \rho([0, 1]), \quad x \mapsto \begin{cases} h_A(x) & \text{if } x \in i_A(h_A; \gamma) \\ \eta & \text{otherwise} \end{cases}$$

where  $\gamma, \eta \in \rho([0, 1])$  satisfying  $\eta \sqsubseteq \sqcap_{x \notin i_A(h_A; \gamma)} h_A(x)$ . If  $A \in HFF(X)$ , then  $A^* \in HFF(X)$ .

**Proof.** Let  $A \in HFF(X)$  and  $x, y \in X$ . If  $x * y, x \in i_A(h_A; \gamma)$ , then  $y \in i_A(h_A; \gamma)$  by Theorem 8 (ii). Hence

$$h_{A^*}(x) \sqcap h_{A^*}(x * y) = h_A(x) \sqcap h_A(x * y) \subseteq h_A(y) = h_{A^*}(y).$$

If  $x * y \notin i_A(h_A; \gamma)$  or  $x \notin i_A(h_A; \gamma)$ , then  $h_{A^*}(x * y) = \eta$  or  $h_{A^*}(x) = \eta$ . Thus

$$h_{A^*}(x) \sqcap h_{A^*}(x * y) = \eta \subseteq h_{A^*}(y).$$

Therefore,  $A^* \in HFF(X)$ . □

#### 4. Hesitant Fuzzy implicative filters

**Definition 10.** Hesitant fuzzy set  $A$  of  $X$  is called a *hesitant fuzzy implicative filter* if satisfies the following conditions:

- (HFIF1)  $h_A(x) \subseteq h_A(1)$ ,
- (HFIF2)  $h_A(x * (y * z)) \sqcap h_A(x * y) \subseteq h_A(x * z)$ , for all  $x, y, z \in X$ .

Denote the set of all hesitant fuzzy implicative filters on  $X$  by  $HFIF(X)$ . It can be seen that every hesitant fuzzy implicative filter is a hesitant fuzzy filter.

**Note.** If  $|h_A(x)| = 1$ , for all  $x \in X$ , then  $h_A$  is an implicative fuzzy filter. In fact in this case we put “ $\leq := \sqsubseteq$ ” and “ $\min := \sqcap$ ”.

**Example 6.** Let  $X = \{1, a, b, c\}$  with the following table:

*	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	b	1

Then  $(X; *, 1)$  is a *BE*-algebra. Let  $t_1 = \{0.6, 0.9\}$ ,  $t_2 = \{0.2, 0.3\}$  and  $t_3 = \{0.5\}$ . Define  $A$  as

$$h_A(1) = t_1, \quad h_A(a) = t_3 \text{ and } h_A(b) = h_A(c) = t_2.$$

Then  $A$  is a hesitant fuzzy implicative filter.

**Theorem 12.** Let  $X$  be a self distributive *BE*-algebra. Then every hesitant fuzzy filter is a hesitant fuzzy implicative filter.

**Proof.** Let  $A \in HFF(X)$ . Obvious that  $h_A(x) \subseteq h_A(1)$ , for all  $x \in X$ . By using self distributivity and (HFF2), we have  $h_A(x * (y * z)) \sqcap h_A(x * y) = h_A((x * y) * (x * z)) \sqcap h_A(x * y) \subseteq h_A(x * z)$ . Therefore,  $A \in HFIF(X)$ . □

In the following example shows that the condition self distributivity of Theorem 12, is necessary.

**Example 7.** (Rao [9]) Let  $X = \{1, a, b, c, d\}$  with the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then  $(X; *, 1)$  is a *BE*-algebra but it is not self distributive because,

$$a * (b * d) = a * a = 1 \neq (a * b) * (a * d) = a.$$

Let  $t_1 = \{0.8, 0.7\}$  and  $t_2 = \{0.4\}$ . Define  $A$  as  $h_A(1) = h_A(a) = t_1$  and  $h_A(b) = h_A(c) = h_A(d) = t_2$ . Then  $A$  is a hesitant fuzzy filter, but it is not a hesi-

tant implicative filter because,

$$\begin{aligned} h_A(b * (d * c)) \sqcap h_A(b * d) &= h_A(1) \sqcap h_A(a) \\ &= t_1 \\ &\sqsubseteq h_A(b * c) \\ &= h_A(b) \\ &= t_2. \end{aligned}$$

**Theorem 13.** Let  $F$  be a (implicative) filter of  $X$ . Then there exists a hesitant (implicative) fuzzy filter  $h_A$  of  $X$  such that  $i_A(h_A; \gamma) = F$ , for some  $\gamma \in \rho([0, 1])$ .

**Proof.** Define hesitant fuzzy set  $h_A$  as follows

$$h_A(x) = \begin{cases} B & \text{if } x \in F \\ \emptyset & \text{otherwise} \end{cases}$$

where  $\gamma \in \rho([0, 1])$  is a fixed subset. Since  $1 \in F$ , we have  $h_A(x) \sqsubseteq h_A(1) = \gamma$ , for all  $x \in X$ . Now, we consider the following cases.

Case 1. If  $x * y, x \in F$ , then  $y \in F$ . Hence

$$h_A(x * y) \sqcap h_A(x) = \gamma = h_A(y).$$

Case 2. If  $x * y \in F$  and  $x \notin F$ , Then  $h_A(x * y) = F$  and  $h_A(x) = \emptyset$ . Hence

$$h_A(x * y) \sqcap h_A(x) = \gamma \sqcap \emptyset = \emptyset \sqsubseteq h_A(y).$$

Case 3. If  $x * y \notin F$  and  $x \in F$ , Then  $h_A(x * y) = \emptyset$  and  $h_A(x) = F$ . Hence

$$h_A(x * y) \sqcap h_A(x) = \emptyset \sqcap \gamma = \emptyset \sqsubseteq h_A(y).$$

Case 4. If  $x * y \notin F$  and  $x \notin F$ , Then  $h_A(x * y) = \emptyset$  and  $h_A(x) = \emptyset$ . Hence

$$h_A(x * y) \sqcap h_A(x) = \emptyset \sqcap \emptyset = \emptyset \sqsubseteq h_A(y).$$

Clearly  $i_A(h_A; \gamma) = F$ .  $\square$

**Theorem 14.** Let  $X$  be a self distributive BE-algebra and  $A \in HFF(X)$ . Then the following conditions are equivalent:

- (i)  $A \in HFIF(X)$ ,
- (ii)  $h_A(y * (y * x)) \sqsubseteq h_A(y * x)$ ,

- (iii)  $h_A((z * (y * (y * x))) \sqcap h_A(z) \sqsubseteq h_A(y * x)$ , for all  $x, y, z \in X$ .

**Proof.** (i)  $\Rightarrow$  (ii). Let  $A \in HFIF(X)$ . By using (HFIF1) and (BE1) we have

$$\begin{aligned} h_A(y * (y * x)) &= h_A(y * (y * x)) \sqcap h_A(1) \\ &= h_A(y * (y * x)) \sqcap h_A(y * y) \\ &\sqsubseteq h_A(y * x). \end{aligned}$$

(ii)  $\Rightarrow$  (iii). Let  $A$  be a hesitant fuzzy filter of  $X$  satisfying the condition (ii). By using (HFIF2) and (ii) we have

$$h_A(z * (y * (y * x))) \sqcap h_A(z) \sqsubseteq h_A(y * (y * x)) \sqsubseteq h_A(y * x).$$

(iii)  $\Rightarrow$  (i). Since

$$x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z)).$$

Hence  $h_A(x * (y * z)) \sqsubseteq h_A((x * y) * (x * (x * z)))$ , by Proposition 3 (i). Thus

$$\begin{aligned} h_A(x * (y * z)) \sqcap h_A(x * y) &\sqsubseteq h_A(((x * y) * (x * (x * z))) \sqcap h_A(x * y)) \\ &\sqsubseteq h_A(x * z). \end{aligned}$$

Therefore,  $A \in HFIF(X)$ .  $\square$

Let  $f : X \rightarrow Y$  be a homomorphism of BE-algebras and  $A \in HFS(Y)$ . Define a mapping  $h_{A^f} : X \rightarrow \rho([0, 1])$  such that  $h_{A^f}(x) = h_A(f(x))$ , for all  $x \in X$ .

Then  $h_{A^f}(x)$  is well-defined and  $A^f \in HFS(X)$ , in which  $A^f = \{x \in X : f(x) \in A\}$ .

**Theorem 15.** Let  $f : X \rightarrow Y$  be an onto homomorphism of BE-algebras and  $A \in HFS(Y)$ . Then  $A \in HFF(Y)$  (resp.  $A \in HFIF(Y)$ ) if and only if  $A^f \in HFF(X)$  (resp.  $A^f \in HFIF(X)$ ).

**Proof.** Assume that  $A \in HFF(Y)$ . For any  $x, y \in X$ , we have

$$h_{A^f}(x) = h_A(f(x)) \sqsubseteq h_A(1_Y) = h_A(f(1_X)) = h_{A^f}(1_X).$$

Hence (HFF1) is valid. Now, let  $x, y \in X$

$$\begin{aligned} h_{A^f}(x * y) \sqcap h_{A^f}(x) &= h_A(f(x * y)) \sqcap h_A(f(x)) \\ &= h_A(f(x) * f(y)) \sqcap h_A(f(x)) \\ &\sqsubseteq h_A(f(y)) \\ &= h_{A^f}(y) \end{aligned}$$



Therefore,  $A^f \in HFF(X)$ .

Conversely, Assume that  $A^f \in HFF(X)$ . Let  $y \in Y$ . Since  $f$  is onto, there exists  $x \in X$  such that  $f(x) = y$ . Then

$$\begin{aligned} h_A(y) &= h_A(f(x)) \\ &= h_{A^f}(x) \\ &\sqsubseteq h_{A^f}(1_X) \\ &= h_A(f(1_X)) \\ &= h_A(1_Y). \end{aligned}$$

Now, let  $x, y \in Y$ . Then there exists  $a, b \in X$  such that  $f(a) = x$  and  $f(b) = y$ . Hence we have

$$\begin{aligned} h_A(x * y) \sqcap h_A(x) &= h_A(f(a) * f(b)) \sqcap h_A(f(a)) \\ &= h_A(f(a * b)) \sqcap h_A(f(a)) \\ &= h_{A^f}(a * b) \sqcap h_{A^f}(a) \\ &\sqsubseteq h_{A^f}(b) \\ &= h_A(f(b)) \\ &= h_A(y). \end{aligned}$$

Therefore,  $A \in HFF(Y)$ .  $\square$

Let  $A \in HFS(X)$ . Denote

$$\top := 1 - \sup\{\sup\{h_A(x) : x \in X\}\}.$$

Then for any  $\beta \in [0, \top]$ , define

$h_{A_\beta}(x) := h_A(x) + \beta = \{a + \beta : a \in h_A(x)\}$ , for all  $x \in X$ .

Obviously,  $h_{A_\beta}$  is a mapping from  $X$  to  $[0, 1]$ , that is,  $A_\beta \in HFS(X)$ .  $h_{A_\beta}(x)$  is well define. Assume that  $\beta \in [0, \top]$  and  $x_1 = x_2$ . Then  $h_A(x_1) = h_A(x_2)$  and so  $h_{A_\beta}(x_1) = h_A(x_1) + \beta = h_A(x_2) + \beta = h_{A_\beta}(x_2)$ .

Hence  $h_{A_\beta}$  is well define. Let  $\beta_1, \beta_2 \in [0, \top]$  be such that  $\beta_1 \leq \beta_2$ . Then  $h_{A_{\beta_1}} \sqsubseteq h_{A_{\beta_2}}$ .

**Example 8.** In Example 6,

$\sup\{\sup\{h_A(x) : x \in X\}\} = \sup\{0.9, 0.3, 0.5\} = 0.9$  and so  $\top = 1 - 0.9 = 0.1$ .

**Theorem 16.** Let  $\beta \in [0, \top]$ . If  $A \in HFF(X)$  (resp.  $HFIF(X)$ ), then

$h_{A_\beta} \in HFF(X)$  (resp.  $HFIF(X)$ ), too.

**Proof.** Let  $x, y \in X$ . Then

$$\begin{aligned} h_{A_\beta}(x * y) \sqcap h_{A_\beta}(x) &= (h_A(x * y) + \beta) \sqcap (h_A(x) + \beta) \\ &= (h_A(x * y) \sqcap h_A(x)) + \beta \\ &\sqsubseteq (h_A(y) + \beta) \\ &= h_{A_\beta}(y). \end{aligned}$$

Also,  $h_{A_\beta}(x) = h_A(x) + \beta \sqsubseteq h_A(1) + \beta = h_{A_\beta}(1)$ . Therefore,  $h_{A_\beta} \in HFF(X)$ .  $\square$

**Theorem 17.** If there exists  $\beta \in [0, \top]$  such that  $h_{A_\beta} \in HFF(X)$  (resp.  $HFIF(X)$ ), then  $h_A \in HFF(X)$  (resp.  $HFIF(X)$ ), too.

**Proof.** Assume that  $h_{A_\beta} \in HFF(X)$ , for some

$\beta \in [0, \top]$ . Let  $x, y \in X$ . Since

$h_{A_\beta}(x * y) \sqcap h_{A_\beta}(x) \sqsubseteq h_{A_\beta}(y)$ , we can see that

$$\begin{aligned} h_{A_\beta}(x * y) \sqcap h_{A_\beta}(x) &= (h_A(x * y) + \beta) \sqcap (h_A(x) + \beta) \\ &= (h_A(x * y) \sqcap h_A(x)) + \beta \\ &\sqsubseteq (h_A(y) + \beta) \end{aligned}$$

Now, by canceling  $\beta$  we have

$$h_A(x * y) \sqcap h_A(x) \sqsubseteq h_A(y).$$

Also, by a similar way,  $h_A(x) \sqsubseteq h_A(1)$ . Therefore,  $h_A \in HFF(X)$ .  $\square$

## 5. Conclusion

Uncertainty usually appears in many real world problems. Fuzzy sets and its extensions have provided successful results dealing with uncertainty in different problems. We have paid attention to one of them, *HFS*, that manages hesitant situations that often appear when the membership degree of an element to a set must be established. Additionally, it is known that many operators for hesitant fuzzy sets and their extensions have been introduced to deal with such a type of information in different applications where decision making has been the most remarkable one.

In this paper, we applied the theory of hesitant fuzzy sets to *BE*-algebras and introduced the notions of hesitant fuzzy (implicative) filters and  $\gamma$ -inclusive sets in *BE*-algebras and many related properties are introduced.

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