Multi-valued Neutrosophic Sets and Power Aggregation Operators with Their Applications in Multi-criteria Group Decision-making Problems

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Abstract

In recent years, hesitant fuzzy sets (HFSs) and neutrosophic sets (NSs) have become a subject of great interest for researchers and have been widely applied to multi-criteria group decision-making (MCGDM) problems. In this paper, multi-valued neutrosophic sets (MVNSs) are introduced, which allow the truth-membership, indeterminacy-membership and falsity-membership degree have a set of crisp values between zero and one, respectively. Then the operations of multi-valued neutrosophic numbers (MVNNs) based on Einstein operations are defined, and a comparison method for MVNNs is developed depending on the related research of HFSs and Atanassov’s intuitionistic fuzzy sets (IFSs). Furthermore, the multi-valued neutrosophic power weighted average (MVNPWA) operator and the multi-valued neutrosophic power weighted geometric (MVNPWG) operator are proposed and the desirable properties of two operators are also discussed. Finally, an approach for solving MCGDM problems is explored by applying the power aggregation operators, and an example is provided to illustrate the application of the proposed method, together with a comparison analysis.

Keywords: Multi-criteria group decision-making, multi-valued neutrosophic sets, power aggregation operators.

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1. Introduction

In many cases, it is difficult for decision-makers to precisely express a preference when solving multi-criteria decision-making (MCDM) and multi-criteria group decision-making (MCGDM) problems with inaccurate, uncertain or incomplete information. Under these circumstances, Zadeh’s fuzzy sets (FSs), where the membership degree is represented by a real number between zero and one, are regarded as an important tool for solving MCDM and MCGDM problems. Furthermore, the non-standard unit interval \([0, 1]\), the non-membership degree and hesitation degree that he or she is not sure is 0.2\(^3\) and the possibility that the statement is false is 0.6 and the degree that he or she is not sure is 0.2\(^2\). This issue is beyond the scope of the FSs and IFSs. Then Smarandache proposed neutrosophic logic and neutrosophic sets (NSs)\(^3\)\(^1\)\(^3\)\(^2\) and subsequently Rivieccio pointed out that an NS is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and it lies in \([0, 1][1], the non-standard unit interval\(^3\)\(^4\). Clearly, this is the extension of the standard interval \([0, 1]\). Furthermore, the uncertainty presented here, i.e. indeterminacy factor, is dependent on of truth and falsity values, whereas the incorporated uncertainty is dependent on the degrees of belongingness and degree of non-belongingness of IFSs\(^3\)\(^5\). Additionally, the aforementioned example of NSs can be expressed as \(x(0.5, 0.2, 0.6)\). However, without specific description, NSs are difficult to apply to real-life situations. Therefore, single-valued neutrosophic sets (SVNSs) were proposed, which are an extension of NSs\(^3\)\(^1\)\(^3\)\(^3\) and subsequently Rivieccio introduced a measure of entropy of SVNSs\(^3\)\(^5\). Furthermore, the correlation coefficients of SVNSs as well as a decision-making method using SVNSs were introduced\(^3\)\(^6\). In addition, Ye also introduced the concept of simplified neutrosophic sets (SNSs), which can be described by three real numbers in the real unit interval \([0,1]\), and proposed an MCDM method using the aggregation operators of SNSs\(^7\). Wang et al and Lupiáñez proposed the concept of interval neutrosophic sets (INSs) and provided the set-theoretic operators of INSs\(^3\)\(^8\),\(^3\)\(^9\). Broumi and Smarandache discussed the correlation coefficient of INSs\(^3\)\(^0\). Furthermore, Ye proposed the cross-entropy of SVNSs and similarity of INSs respectively\(^4\)\(^1\)\(^3\)\(^2\). However, in certain cases, the operations of SNSs provided by Ye may be unreasonable\(^3\)\(^7\). For example, the sum of any element and the maximum value should be equal to the maximum value, but this is not always the case during operations. The similarity measures and distances of SVNSs that are based on those operations may also be unrealistic. Peng et al developed novel operations, outranking relations and aggregation operators of SNSs\(^4\)\(^3\),\(^4\)\(^4\), which were based on the operations in Ye\(^3\)\(^7\) and applied them to MCGDM problems. Zhang et al introduced a MCDM method with INSs\(^4\)\(^5\). Liu and Wang investigated single-valued neutrosophic normalized weighted Bonferroni mean and applied it to MCDM problems\(^4\)\(^6\). Liu et al developed some Hamacher aggregation operators with NSs\(^4\)\(^7\). Tian
et al. developed simplified neutrosophic linguistic normalized weighted Bonferroni mean operator and applied it to MCDM problems.

However, decision-makers can also be hesitant when expressing their evaluation values for each parameter in SNSs. For example, if the possibility of a statement being true is 0.6 or 0.7, the possibility of it being false is 0.2 or 0.3 and the degree that he or she is not sure is 0.1 or 0.2, this will be beyond the capability of SNSs. If the operations and comparison method of SNSs were extended to multiple values, the shortcomings discussed earlier would still exist. Therefore, Wang and Li developed the definition of multi-valued neutrosophic sets (MVNSs), based on which, the Einstein operations and comparison method, and power aggregation operators for multi-valued neutrosophic numbers (MVNNs) are defined in this paper. Consequently, a MCGDM method is established based on the proposed operators. An illustrative example is also given to demonstrate the applicability of the proposed method.

The rest of paper is organized as follows. In Section 2 some basic concepts and operations of SNSs are briefly reviewed. Then the definition of MVNSs is introduced, and the operations, a comparison method and distance of MVNNs are defined in Section 3. Section 4 contains the operations of SNSs are also defined by Ye. An illustrative example and a comparison analysis are presented to verify the proposed approach. Finally, the conclusions are drawn in Section 6.

2. Preliminaries

In this section, the definitions and operations of NSs and SNSs are introduced, which will be utilized in the latter analysis.

Definition 1. Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. An NS $A$ in $X$ is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$ as follows:

$$A = \{ (x, T_A(x), I_A(x), F_A(x)) \mid x \in X \},$$

(1)

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $[0^0, 1^1]$, that is,

$$T_A(x) : X \rightarrow [0^0, 1^1], \quad I_A(x) : X \rightarrow [0^0, 1^1], \quad F_A(x) : X \rightarrow [0^0, 1^1].$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, therefore $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Considering the applicability of NSs, Ye reduced NSs of nonstandard intervals into SNSs of standard intervals, which can preserve the operations of NSs properly.

Definition 2. Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. An NS $A$ in $X$ is characterized by $T_A(x)$, $I_A(x)$ and $F_A(x)$, which are singleton subintervals/subsets in the real standard $[0, 1]$, that is $T_A(x) : X \rightarrow [0, 1]$, $I_A(x) : X \rightarrow [0, 1]$ and $F_A(x) : X \rightarrow [0, 1]$. Then, a simplification of $A$ is denoted by:

$$A = \{ (x, T_A(x), I_A(x), F_A(x)) \mid x \in X \},$$

(2)

which is called an SNS and is a subclass of NSs. For convenience, the SNSs is denoted by the simplified symbol $A = \{ (T_A(x), I_A(x), F_A(x)) \}$. The set of all SNSs is represented as SNSSS.

The operations of SNSs are also defined by Ye.

Definition 3. Let $A$, $A_1$ and $A_2$ be three SNSs. For any $x \in X$, the following operations can be defined:

1. $A_1 + A_2 = \{ T_{A_1}(x) + T_{A_2}(x), I_{A_1}(x) + I_{A_2}(x), F_{A_1}(x) + F_{A_2}(x) \} ;$

2. $A_1 \cdot A_2 = \{ T_{A_1}(x) \cdot T_{A_2}(x), I_{A_1}(x) \cdot I_{A_2}(x), F_{A_1}(x) \cdot F_{A_2}(x) \} ;$

3. $\lambda \cdot A = \{ \lambda \cdot T_{A}(x), I_{\lambda}(x) \cdot I_{A}(x), F_{\lambda}(x) \cdot F_{A}(x) \} ;$

4. $A^\lambda = \{ T_{A}(x), I_{A}(x), F_{A}(x) \} , \lambda > 0 ;$

There are some limitations related to Definition 3 and these are now outlined.

(i) In some situations, operations such as $A_1 + A_2$ and $A_1 \cdot A_2$ might be impractical. This is demonstrated in Example 1.
Example 1. Let $A_1 = \{x, 0.5, 0.5, 0.5\}$ and $A_2 = \{x, 1, 0, 0\}$ be two SNSs. Clearly, $A_2 = \{x, 1, 0, 0\}$ can be the larger of these SNSs. Theoretically, the sum of any number and the maximum number should be equal to the maximum one. However, according to Definition 3, $A_1 + A_2 = \{x, 1, 0.5, 0.5\} \neq A_2$, therefore the operation “+” cannot be accepted. Similar contradictions exist in other operations of Definition 3, and thus those defined above are incorrect.

(ii) The correlation coefficient of SNSs\(^{36}\), which is based on the operations of Definition 3, cannot be accepted in some specific cases.

Example 2. Let $A_1 = \{x, 0.8, 0, 0\}$ and $A_2 = \{x, 0.7, 0, 0\}$ be two SNSs, and $A = \{x, 1, 0, 0\}$ be the largest one of the SNSs. According to the correlation coefficient of SNSs\(^{36}\), $W(A_1, A) = W(A_2, A) = 1$ can be obtained, but this does not indicate which one is the best. However, it is clear that $A_1$ is superior to $A_2$.

(iii) In addition, the cross-entropy measure for SNSs\(^{41}\), which is based on the operations of Definition 3, cannot be accepted in special cases.

Example 3. Let $A_1 = \{x, 0.1, 0, 0\}$ and $A_2 = \{x, 0.9, 0, 0\}$ be two SNSs, and $A = \{x, 1, 0, 0\}$ be the largest one of the SNSs. According to the cross-entropy measure for SNSs\(^{41}\), $S_1(A_1, A) = S_2(A_2, A) = 1$ can be obtained, which indicates that $A_1$ is equal to $A_2$. Yet it is not possible to discern which one is the best. Since $T_{A_1}(x) > T_{A_2}(x)$, $I_{A_1}(x) > I_{A_2}(x)$ and $F_{A_1}(x) > F_{A_2}(x)$ for any $x$ in $X$, it is clear that $A_2$ is superior to $A_1$.

(iv) If $I_{A_1}(x) = I_{A_2}(x) = 0$ for any $x$ in $X$, then $A_1$ and $A_2$ are both reduced to IFSs. However, the operations presented in Definition 3 are not in accordance with the operations of two IFSs\(^{6,8,10-21}\).

3. Multi-valued Neutrosophic Sets

In this section, MVNSs are introduced, and the corresponding operations and comparison method are developed in terms of those of IFSs\(^{6,8,10-21}\) and HFSs\(^{22,23}\).

3.1. MVNSs and their Einstein operations

Definition 4. Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. An MVNS $A$ in $X$ is characterized by\(^{48}\):

$$A = \{\langle x, \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\rangle | x \in X\},$$

where $\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three sets of precise values in $[0,1]$, denoting the truth-membership degree, indeterminacy-membership function and falsity-membership degree respectively, satisfying $0 \leq \gamma, \eta, \xi \leq 1, 0 \leq \gamma^* + \eta^* + \xi^* \leq 3$, where $\gamma = \sup \tilde{T}_A(x)$, $\eta = \sup \tilde{I}_A(x)$, $\xi = \sup \tilde{F}_A(x)$, $\gamma^* = \sup \tilde{T}_A(x)$, $\eta^* = \sup \tilde{I}_A(x)$ and $\xi^* = \sup \tilde{F}_A(x)$.

If $X$ has only one element, then $A$ is called a multi-valued neutrosophic number (MVNN), denoted by $A = \{\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\}$. For convenience, an MVNN can be denoted by $A = \{\tilde{T}_A, \tilde{I}_A, \tilde{F}_A\}$. The set of all MVNNs is represented as MVNNS.

Obviously, MVNNs are generally considered as an extension of NSs. If each of $\tilde{T}_A(x)$, $\tilde{I}_A(x)$ and $\tilde{F}_A(x)$ for any $x$ has only one value, i.e. $\gamma, \eta$ and $\xi$, and $0 \leq \gamma + \eta + \xi \leq 3$, then MVNNs are reduced to SNSs; if $\tilde{I}_A(x) = \emptyset$ for any $x$, then MVNNs are reduced to DHFSs; if $\tilde{F}_A(x) = \emptyset$ for any $x$, then MVNNs are reduced to HFSs. In a word, MVNNs are the extensions of SNSs, DHFSs and HFSs.

In the following, the operations of MVNNs can be defined based on the operations of IFSs and HFSs.

Definition 5. Let $A \in$ MVNNS, then the complement of a MVNN can be denoted by $A^c$, which can be defined as follows:

$$A^c = \bigcup_{x \in \tilde{T}_A(x)} \{\bar{\gamma}, \bigcup_{x \in \tilde{I}_A(x)} \{1-\eta\}, \bigcup_{x \in \tilde{F}_A(x)} \{\bar{\xi}\}\}.$$  

It is noted that different aggregation operators are all based on different t-conorms and t-norms and are used to deal with different relationships of the aggregated arguments, which satisfy the requirements of the conjunction and disjunction operators, respectively. Einstein operations include the Einstein sum $a \oplus b = (a+b)/(1+a+b)$ and Einstein product $a \odot b = (a-b)/(1+(1-a)(1-b))$ $(a,b \in [0,1])$, which are examples of t-norms and t-conorms, respectively. In the following, the operations of MVNNs can be defined based on Einstein operations.
**Definition 6.** Let \( A = \{ \tilde{T}_a, \tilde{I}_a, \tilde{F}_a \} \), \( B = \{ \tilde{T}_b, \tilde{I}_b, \tilde{F}_b \} \) be two MVNNs and \( \lambda > 0 \). The operations of MVNNs can be defined as follows:

\[
(1) \quad \lambda \cdot A = \left\{ \frac{(1 + \gamma_a)^\lambda - (1 - \gamma_a)^\lambda}{(1 + \gamma_a)^\lambda + (1 - \gamma_a)^\lambda}, \frac{2 \cdot (\eta_a)^\lambda}{(2 - \eta_a)^\lambda + \eta_a^\lambda}, \frac{2 \cdot (\xi_a)^\lambda}{(2 - \xi_a)^\lambda + (\xi_a)^\lambda} \right\},
\]

\[
(2) \quad A^\lambda = \left\{ \frac{(1 + \gamma_a)^\lambda}{(2 - \gamma_a)^\lambda + \gamma_a^\lambda}, \frac{1}{1 + \eta_a^\lambda}, \frac{1}{1 + \xi_a^\lambda} \right\},
\]

\[
(3) \quad A \oplus B = \left\{ \gamma_a + \gamma_b, \frac{1}{1 + \gamma_a + \gamma_b}, \eta_a + \eta_b, \frac{1}{1 + \eta_a + \eta_b}, \xi_a + \xi_b, \frac{1}{1 + \xi_a + \xi_b} \right\},
\]

\[
(4) \quad A \otimes B = \left\{ \frac{\gamma_a \cdot \gamma_b}{1 + (1 - \gamma_a) \cdot (1 - \gamma_b)}, \frac{\eta_a + \eta_b}{1 + \eta_a \cdot \eta_b}, \frac{\xi_a + \xi_b}{1 + \xi_a \cdot \xi_b} \right\}.
\]

If there is only one specific number in \( \tilde{T}_a, \tilde{I}_a \) and \( \tilde{F}_a \), then the operations in Definition 6 are reduced to the operations of NNNs as follows:

\[
(5) \quad \lambda \cdot A = \left\{ \frac{(1 + \gamma_a)^\lambda - (1 - \gamma_a)^\lambda}{(1 + \gamma_a)^\lambda + (1 - \gamma_a)^\lambda}, \frac{2 \cdot (\eta_a)^\lambda}{(2 - \eta_a)^\lambda + \eta_a^\lambda}, \frac{2 \cdot (\xi_a)^\lambda}{(2 - \xi_a)^\lambda + (\xi_a)^\lambda} \right\}.
\]

\[
(6) \quad A^\lambda = \left\{ \frac{2 \cdot (\gamma_a)^\lambda}{(2 - \gamma_a)^\lambda + (\gamma_a)^\lambda}, \frac{1}{1 + (1 - \gamma_a) \cdot (1 - \gamma_a)^\lambda}, \frac{(1 + \eta_a)^\lambda - (1 - \eta_a)^\lambda}{(1 + \eta_a)^\lambda + (1 - \eta_a)^\lambda}, \frac{(1 + \xi_a)^\lambda - (1 - \xi_a)^\lambda}{(1 + \xi_a)^\lambda + (1 - \xi_a)^\lambda} \right\},
\]

\[
(7) \quad A \oplus B = \left\{ \frac{\gamma_a + \gamma_b}{1 + (1 - \gamma_a) \cdot (1 - \gamma_b)}, \frac{\eta_a + \eta_b}{1 + \eta_a \cdot \eta_b}, \frac{\xi_a + \xi_b}{1 + \xi_a \cdot \xi_b} \right\},
\]

\[
(8) \quad A \otimes B = \left\{ \frac{\gamma_a \cdot \gamma_b}{1 + (1 - \gamma_a) \cdot (1 - \gamma_b)}, \frac{\eta_a + \eta_b}{1 + \eta_a \cdot \eta_b}, \frac{\xi_a + \xi_b}{1 + \xi_a \cdot \xi_b} \right\}.
\]

Note that the operations of MVNNs coincide with the operations of IFSSs.

**Example 4.** Let \( A = \{0.6, 0.1, 0.2\} \) and \( B = \{0.5, 0.3, 0.2, 0.3\} \) be two MVNNs, and \( \lambda = 2 \), then the following results can be achieved.

\[
(1) \quad 2 \cdot A = \{0.8824, 0.1105, 0.2439\},
\]

\[
(2) \quad A^2 = \{1, 0.1980, 0.3846, 0.3846\},
\]

\[
(3) \quad A \oplus B = \{0.8462, 0.0184, 0.0385, 0.0244, 0.0385\},
\]

\[
(4) \quad A \otimes B = \{0.2500, 0.3884, 0.4717, 0.3884, 0.4717\}.
\]

**Theorem 1.** Let \( A = \{\tilde{T}_a, \tilde{I}_a, \tilde{F}_a\} \), \( B = \{\tilde{T}_b, \tilde{I}_b, \tilde{F}_b\} \), and \( C = \{\tilde{T}_c, \tilde{I}_c, \tilde{F}_c\} \) be three MVNNs, then the following equations can be true.

\[
(1) \quad A \oplus A = A \otimes A;
\]

\[
(2) \quad A \oplus A = A \otimes A;
\]

\[
(3) \quad \lambda (A \oplus B) = \lambda A \oplus \lambda B, \lambda > 0;
\]

\[
(4) \quad (A \otimes B)^\lambda = A^\lambda \otimes B^\lambda, \lambda > 0;
\]

\[
(5) \quad \lambda (A \oplus B) = \lambda A + \lambda B, \lambda > 0, \lambda > 0;
\]

\[
(6) \quad A^\lambda \otimes A^{\lambda+i} = A^{\lambda+i}, \lambda > 0, \lambda > 0;
\]

\[
(7) \quad (A \oplus B) \otimes C = A \oplus (B \otimes C);
\]

\[
(8) \quad (A \otimes B) \oplus C = A \otimes (B \oplus C).
\]

**Proof.** (1), (2), (7) and (8) can be easily obtained.

(3) Since \( \lambda > 0 \),
\[ \lambda(A \oplus B) = \left\{ \begin{array}{ll}
\frac{1}{1 + \gamma_A \gamma_B} & - \frac{1}{1 - \gamma_A \gamma_B} \\
\frac{1}{1 + \gamma_A \gamma_B} & + \frac{1}{1 - \gamma_A \gamma_B}
\end{array} \right. \]

Thus, \( \lambda(A \oplus B) = \lambda A \oplus \lambda B \) can be obtained.

Similarly, (4), (5) and (6) can be true.
(1) If \( A = \{(0.6,0.5), (0.3), (0.2)\} \) and \( B = \{(0.5), (0.1,0.2), (0.4)\} \) are two MVNNs, then \( s(A) = 0.017 \) and \( s(B) = -0.017 \). \( s(A) < s(B) \), so \( A > B \).

(2) If \( A = \{(0.6,0.5), (0.4), (0.2)\} \) and \( B = \{(0.5), (0.1,0.2), (0.4)\} \) are two MVNNs, then \( s(A) = s(B) = -0.017 \), \( a(A) = 0.383 \) and \( a(B) = 0.35 \). \( a(A) > a(B) \), so \( A > B \).

(3) If \( A = \{(0.6,0.7), (0.3), (0.2)\} \) and \( B = \{(0.6,0.7), (0.2), (0.3)\} \) are two MVNNs, then \( s(A) = s(B) = 0.05 \) and \( a(A) = a(B) = 0.3833 \). So \( A = B \).

(4) If \( A = \{(0.5), (0.1,0.2), (0.1)\} \) and \( B = \{(0.6), (0.2), (0.1)\} \) are two MVNNs, then \( s(A) = 0.0833 \) and \( s(B) = 0.1 \). \( s(A) < s(B) \), so \( A < B \).

(5) If \( A = \{(0.5), (0.1,0.2), (0.1)\} \) and \( B = \{(0.7), (0.2,0.3), (0.2)\} \) are two MVNNs, then \( s(A) = s(B) = 0.0833 \), \( a(A) = 0.25 \) and \( a(B) = 0.3833 \). \( a(A) < a(B) \), so \( A < B \).

**Definition 9.** Let \( A = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\} \) and \( B = \{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\} \) be two MVNNs, then the Hamming–Hausdorff distance between \( A \) and \( B \) can be defined as follows:

\[
d(A,B) = \frac{1}{6} \left( \max_{\gamma_1, \gamma_2, \eta_1, \eta_2} |\gamma_1 - \gamma_2| + \max_{\gamma_1, \eta_1} |\gamma_2 - \eta_1| \right. \\
\left. + \max_{\gamma_1, \eta_2} |\gamma_1 - \eta_2| + \max_{\eta_1, \eta_2} |\gamma_2 - \eta_1| + \max_{\gamma_1, \gamma_2} |\gamma_1 - \gamma_2| + \max_{\gamma_1, \gamma_2} |\gamma_1 - \gamma_2| \right).
\]

**Example 6.** Let \( A = \{(0.4,0.5), (0.2), (0.3)\} \) and \( B = \{(0.8), (0.8), (0.5)\} \) be two MVNNs, then according to Eq. (5), \( d(A,B) = 0.25 \) can be determined.

### 4. Power Operators and MCGDM Approach

In this section, the power aggregation operators of MVNNs are presented and an approach for MCGDM problems that utilizes these aggregation operators is proposed.

### 4.1. Power aggregation operator

The power average (PA) operator was developed by Yager in the form of nonlinear weighted average aggregation operator\(^1\).

**Definition 10.** The PA operator is the mapping \( PA: R^* \rightarrow R \), which is defined as follows\(^1\):

\[
PA(\alpha_1, \ldots, \alpha_n) = \frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n} (1 + S(\alpha_i))}.
\]

Here \( S(\alpha) = \sum_{i,j=1}^{n} \text{Supp}(\alpha_i, \alpha_j) \) and \( \text{Supp}(\alpha_i, \alpha_j) \) is the support for \( \alpha_i \) from \( \alpha_j \). Then the following properties are true.

1. \( \text{Supp}(\alpha_i, \alpha_j) \in [0,1] \)
2. \( \text{Supp}(\alpha_i, \alpha_j) = \text{Supp}(\alpha_j, \alpha_i) \)
3. \( \text{Supp}(\alpha_i, \alpha_j) \geq \text{Supp}(\alpha_p, \alpha_q) \iff |\alpha_i - \alpha_j| < |\alpha_p - \alpha_q| \)

Apparently, the closer two values get, the more they support each other.

### 4.2. Power weighted average operator

**Definition 11.** Let \( A_j = \{\tilde{a}_{i,j}, \tilde{b}_{i,j}, \tilde{c}_{i,j}\} \) be a collection of MVNNs, and \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector of \( A_j \) \( (j = 1,2,\ldots,n) \), with \( w_j \geq 0 \) \( (j = 1,2,\ldots,n) \) and \( \sum_{j=1}^{n} w_j = 1 \). The multi-valued neutrosophic power weighted average (MVNPWA) operator of dimension \( n \) is the mapping

\[
\text{MVNPWA}_w(A_1, A_2, \ldots, A_n) = \frac{\sum_{j=1}^{n} w_j (1 + S(A_j)) A_j}{\sum_{j=1}^{n} w_j (1 + S(A_j))}.
\]

Here \( S(A_j) = \sum_{i,j=1}^{n} w_{ij} \text{Supp}(A_j, A_i) \) and \( \text{Supp}(A_j, A_i) \) is the support for \( A_j \) from \( A_i \), which satisfies the following conditions:

1. \( \text{Supp}(A_j, A_i) \in [0,1] \)
2. \( \text{Supp}(A_j, A_i) = \text{Supp}(A_i, A_j) \)
3. \( \text{Supp}(A_j, A_i) \geq \text{Supp}(A_p, A_q) \iff d(A_j, A_i) < d(A_j, A_q) \)

where \( d \) is the distance measure as was defined in Definition 9.
Based on the operations in Definition 6 and Eq. (7), Theorem 2 can be derived.

**Theorem 2.** Let \( A_j = \left\{ \tilde{T}_{j}, \tilde{I}_{j}, \tilde{F}_{j} \right\} (j = 1, 2, \ldots , n) \) be a collection of MVNNs, and \( w = (w_1, w_2, \ldots , w_n) \) be the weight vector of \( A_j \) \( (j = 1, 2, \ldots , n) \), with \( w_j \geq 0 \) \( (j = 1, 2, \ldots , n) \) and \( \sum w_j = 1 \). Then their aggregated result using the MVNPWA operator is also an MVNN, and

\[
\text{MVNPWA}_w \left( A_1, A_2, \ldots , A_n \right) = \begin{cases} 
U_{\tilde{T}_1, \tilde{T}_2, \ldots , \tilde{T}_n} \left[ \prod_{j=1}^{n} \left( 1 + \gamma_j \right) \sum_{j=1}^{n} w_j \left( \tilde{I}_{j} \right) - \prod_{j=1}^{n} \left( 1 - \gamma_j \right) \sum_{j=1}^{n} w_j \left( \tilde{F}_{j} \right) \right], \\
U_{\tilde{I}_1, \tilde{I}_2, \ldots , \tilde{I}_n} \left[ \prod_{j=1}^{n} \left( 2 - \eta_j \right) \sum_{j=1}^{n} w_j \left( \tilde{T}_{j} \right) + \prod_{j=1}^{n} \left( \eta_j \right) \sum_{j=1}^{n} w_j \left( \tilde{F}_{j} \right) \right], \\
U_{\tilde{F}_1, \tilde{F}_2, \ldots , \tilde{F}_n} \left[ \prod_{j=1}^{n} \left( 2 - \xi_j \right) \sum_{j=1}^{n} w_j \left( \tilde{T}_{j} \right) + \prod_{j=1}^{n} \left( \xi_j \right) \sum_{j=1}^{n} w_j \left( \tilde{F}_{j} \right) \right]. 
\end{cases}
\]

(8)

Here \( S \left( A_j \right) = \sum_{i=1}^{n} w_i \text{Supp} \left( A_j, A_i \right) \) and satisfies the conditions in Definition 11.

**Proof.** For simplicity, let

\[
\varsigma_j = \frac{w_j \left( 1 + S \left( A_j \right) \right)}{\sum_{j=1}^{n} w_j \left( 1 + S \left( A_j \right) \right)}
\]

in the process of proof. By using the mathematical induction on \( n \).

1. If \( n = 2 \), based on the operations (1) and (3) in Definition 6,

\[
\varsigma_j A \otimes \varsigma_j A_i = \begin{cases} 
\left( 1 + \gamma_j \right)^\varsigma \left( 1 - \gamma_j \right)^\varsigma - \left( 1 + \gamma_j \right)^\varsigma \left( 1 - \gamma_j \right)^\varsigma, \\
\left( 1 + \gamma_j \right)^\varsigma \left( 1 - \gamma_j \right)^\varsigma - \left( 1 + \gamma_j \right)^\varsigma \left( 1 - \gamma_j \right)^\varsigma, \\
\left( 1 + \gamma_j \right)^\varsigma \left( 1 - \gamma_j \right)^\varsigma - \left( 1 + \gamma_j \right)^\varsigma \left( 1 - \gamma_j \right)^\varsigma. 
\end{cases}
\]

(2) If Eq. (8) holds for \( n = k \), then
MVNPWA\(_n\) \((A_1,A_2,\ldots,A_n)\)

\[
\text{MVNPWA}_n \left( A_1, A_2, \ldots, A_n \right) = \left\langle \frac{\prod_{j=1}^{n} (1 + \gamma_j)}{\prod_{j=1}^{n} (1 - \gamma_j)} - \frac{\prod_{j=1}^{n} (1 - \gamma_j)}{\prod_{j=1}^{n} (1 + \gamma_j)} \right\rangle,
\]

If \( n = k + 1 \), by the operations (1) and (3) in Definition 6,

\[
\text{MVNPWA}_{n+1} \left( A_1, A_2, \ldots, A_n, A_{n+1} \right) = \left\langle \frac{\prod_{j=1}^{n+1} (1 + \gamma_j)}{\prod_{j=1}^{n+1} (1 - \gamma_j)} - \frac{\prod_{j=1}^{n+1} (1 - \gamma_j)}{\prod_{j=1}^{n+1} (1 + \gamma_j)} \right\rangle,
\]

i.e., Eq. (8) holds for \( n = k + 1 \). Thus, Eq. (8) holds for all \( n \), then

\[
\text{MVNPWA}_n \left( A_1, A_2, \ldots, A_n \right) = \left\langle \frac{\prod_{j=1}^{n} (1 + \gamma_j)}{\prod_{j=1}^{n} (1 - \gamma_j)} - \frac{\prod_{j=1}^{n} (1 - \gamma_j)}{\prod_{j=1}^{n} (1 + \gamma_j)} \right\rangle.
\]
The MVNPWA operator has the following properties.

**Theorem 3.** Let \( A_j = \left( \tilde{T}_{i_j}, \tilde{I}_{i_j}, \tilde{F}_{i_j} \right) \) \( (j = 1, 2, \ldots, n) \) be a collection of MVNNs. If \( A'_j = \left( \tilde{T}_{i'_j}, \tilde{I}_{i'_j}, \tilde{F}_{i'_j} \right) \) \( (j = 1, 2, \ldots, n) \) is any permutation of \( A_j \), then
\[
\text{MVNPWA}_w (A_j, A_{j'}, \ldots, A_{k'}) = \text{MVNPWA}_w (A_{j'}, A_{j''}, \ldots, A_{k''}).
\]

**Proof.** The process of proof is omitted here. \( \square \)

**Theorem 4.** Let \( A_j = \left( \tilde{T}_{i_j}, \tilde{I}_{i_j}, \tilde{F}_{i_j} \right) \) \( (j = 1, 2, \ldots, n) \) be a collection of MVNNs and \( A = \left( \tilde{T}_{i}, \tilde{I}_{i}, \tilde{F}_{i} \right) \) be an MVNN. If for all \( j \), \( \gamma_j = \gamma, \eta_j = \eta \) and \( \xi_j = \xi \), then
\[
\text{MVNPWA}_w (A_j, A_{j'}, \ldots, A_{k'}) = A.
\]
Where \( \gamma_j, \eta_j \) and \( \xi_j \) are elements of \( \tilde{T}_{i_j}, \tilde{I}_{i_j} \) and \( \tilde{F}_{i_j} \) respectively, \( \gamma, \eta \) and \( \xi \) are elements of \( \tilde{T}_{i}, \tilde{I}_{i} \) and \( \tilde{F}_{i} \) respectively.

**Proof.** The process of proof is omitted here. \( \square \)

**Theorem 5.** Let \( A_j = \left( \tilde{T}_{i_j}, \tilde{I}_{i_j}, \tilde{F}_{i_j} \right) \) \( (j = 1, 2, \ldots, n) \) and \( A'_j = \left( \tilde{T}_{i'_j}, \tilde{I}_{i'_j}, \tilde{F}_{i'_j} \right) \) \( (j = 1, 2, \ldots, n) \) be two collections of MVNNs. If for all \( j \), \( \gamma_j \leq \gamma'_j, \eta_j \geq \eta'_j \) and \( \xi_j \leq \xi'_j \), then
\[
\text{MVNPWA}_w (A_j, A_{j'}, \ldots, A_{k'}) \leq \text{MVNPWA}_w (A'_j, A'_{j'}, \ldots, A'_{k'}).\]
Where \( \gamma_j, \eta_j \) and \( \xi_j \) are elements of \( \tilde{T}_{i_j}, \tilde{I}_{i_j} \) and \( \tilde{F}_{i_j} \) respectively, \( \gamma'_j, \eta'_j \) and \( \xi'_j \) are elements of \( \tilde{T}_{i'_j}, \tilde{I}_{i'_j} \) and \( \tilde{F}_{i'_j} \) respectively.

**Proof.** The process of proof is omitted here. \( \square \)

### 4.3. Power weighted geometric operator

**Definition 12.** Let \( A_j = \left( \tilde{T}_{i_j}, \tilde{I}_{i_j}, \tilde{F}_{i_j} \right) \) \( (j = 1, 2, \ldots, n) \) be a collection of MVNNs, and \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector of \( A_j \) \( (j = 1, 2, \ldots, n) \), with \( w_j \geq 0 \) \( (j = 1, 2, \ldots, n) \) and \( \sum_{j=1}^{n} w_j = 1 \). The multi-valued neutrosophic power weighted geometric (MVNPWG) operator of dimension \( n \) is the mapping MVNPWG: \( \text{MVNN}^n \to \text{MVNN} \), and
\[
\text{MVNPWG}_w (A_j, A_{j'}, \ldots, A_{k'}) = \bigotimes_{j=1}^{n} \left( A_j \right)^{\sum_{j=1}^{n} w_j} \right). \quad (9)
\]
Here \( S(A_j) = \sum_{i=1}^{n} w_i \text{Supp} (A_j, A_{j'}, \ldots, A_{k'}) \) and \( \text{Supp} (A_j, A_{j'}) \) is the support for \( A_j \) from \( A_{j'} \), which satisfies the following conditions:

1. \( \text{Supp} (A_j, A_{j'}) \in [0, 1] \);
2. \( \text{Supp} (A_j, A_{j'}) = \text{Supp} (A_{j'}, A_j) \);
3. \( \text{Supp} (A_j, A_{j'}) \geq \text{Supp} (A_{j'}, A_j) \) iff \( d \left( A_j, A_{j'} \right) < d \left( A_{j'}, A_j \right) \), where \( d \) is the distance measure defined in Definition 9.

Based on the operations in Definition 6 and Eq. (9), Theorem 3 can be derived.

**Theorem 6.** Let \( A_j = \left( \tilde{T}_{i_j}, \tilde{I}_{i_j}, \tilde{F}_{i_j} \right) \) \( (j = 1, 2, \ldots, n) \) be a collection of MVNNs, and \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector of \( A_j \) \( (j = 1, 2, \ldots, n) \), with \( w_j \geq 0 \) \( (j = 1, 2, \ldots, n) \) and \( \sum_{j=1}^{n} w_j = 1 \). Then their aggregated result using the MVNPWG operator is also an MVNN, and
\[
\text{MVNPWG}_w (A_j, A_{j'}, \ldots, A_{k'}) = \left( \bigotimes_{j=1}^{n} \left( A_j \right)^{\sum_{j=1}^{n} w_j} \right) \left( \bigotimes_{j=1}^{n} \left( A_{j'} \right)^{\sum_{j=1}^{n} w_j} \right) \ldots \left( \bigotimes_{j=1}^{n} \left( A_{k'} \right)^{\sum_{j=1}^{n} w_j} \right). \quad (10)
\]
Here \( S(A_j) = \sum_{i=1}^{n} w_i \text{Supp} (A_j, A_{j'}, \ldots, A_{k'}) \) and satisfies the conditions in Definition 11.

**Proof.** Theorem 2 can be proved by the mathematical induction and the process is omitted here. \( \square \)

Similarly, the MVNPWG operator has the following properties.
Theorem 7. Let \( A_j = \left\{ \hat{T}_{a_j}, \tilde{T}_{a_j}, \tilde{F}_{a_j} \right\} \) (\( j = 1, 2, \ldots, n \)) be a collection of MVNNs. If \( A_j' = \left\{ \hat{T}_{a_j'}, \tilde{T}_{a_j'}, \tilde{F}_{a_j'} \right\} \) \((j = 1, 2, \ldots, n)\) is any permutation of \( A_j \), then \( \text{MVNPWG}_r \left( A, A_1, \ldots, A_n \right) = \text{MVNPWG}_r \left( A_1', A_2', \ldots, A_n' \right) \).

Proof. The process of proof is omitted here.

Theorem 8. Let \( A_j = \left\{ \hat{T}_{a_j}, \tilde{T}_{a_j}, \tilde{F}_{a_j} \right\} \) (\( j = 1, 2, \ldots, n \)) be a collection of MVNNs and \( A = \left\{ \hat{T}_a, \tilde{T}_a, \tilde{F}_a \right\} \) be an MVNN. If for all \( j \), \( \gamma_j = \gamma, \eta_j = \eta \) and \( \xi_j = \xi \), then \( \text{MVNPWG}_r \left( A, A_1, \ldots, A_n \right) = A \).

Where \( \gamma, \eta \) and \( \xi \) are elements of \( \hat{T}_a, \tilde{T}_a \) and \( \tilde{F}_a \) respectively, \( \gamma, \eta \) and \( \xi \) are elements of \( \hat{T}_a, \tilde{T}_a \) and \( \tilde{F}_a \) respectively.

Proof. The process of proof is omitted here.

Theorem 9. Let \( A_j = \left\{ \hat{T}_{a_j}, \tilde{T}_{a_j}, \tilde{F}_{a_j} \right\} \) (\( j = 1, 2, \ldots, n \)) and \( A_j' = \left\{ \hat{T}_{a_j'}, \tilde{T}_{a_j'}, \tilde{F}_{a_j'} \right\} \) (\( j = 1, 2, \ldots, n \)) be two collections of MVNNs. If for all \( j \), \( \gamma_j \leq \gamma'_j, \eta_j \geq \eta'_j \) and \( \xi_j \geq \xi'_j \), then \( \text{MVNPWG}_r \left( A, A_1, \ldots, A_n \right) \leq \text{MVNPWG}_r \left( A_1', A_2', \ldots, A_n' \right) \).

Where \( \gamma, \eta \) and \( \xi \) are elements of \( \hat{T}_a, \tilde{T}_a \) and \( \tilde{F}_a \) respectively, \( \gamma', \eta, \) and \( \xi \) are elements of \( \hat{T}_a, \tilde{T}_a \) and \( \tilde{F}_a \) respectively.

Proof. The process of proof is omitted here.

MCGDM approach

Assume there are \( n \) alternatives denoted by \( A = \{ a_1, a_2, \ldots, a_n \} \) and \( m \) criteria denoted by \( C = \{ c_1, c_2, \ldots, c_m \} \), and the weight vector of criteria is \( w = (w_1, w_2, \ldots, w_m) \), where \( w_j \geq 0 \) (\( j = 1, 2, \ldots, m \)) and \( \sum_{j=1}^{m} w_j = 1 \). Suppose that there are \( l \) decision-makers \( D = \{ d_1, d_2, \ldots, d_l \} \) whose corresponding weight vector is \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \), where \( \omega_j \geq 0 \) (\( j = 1, 2, \ldots, m \)) and \( \sum_{j=1}^{n} \omega_j = 1 \). Let \( R^k = \left( \alpha^k_{ij} \right)_{n \times m} \) be the multi-valued neutrosophic decision matrix, and \( \alpha^k_{ij} = \left( \hat{T}_{a^k_{ij}}, \tilde{T}_{a^k_{ij}}, \tilde{F}_{a^k_{ij}} \right) \) be the evaluation value of \( a_i \) for criterion \( c_j \) being in the form of MVNNs provided by the decision-maker \( d_k \in D \), where \( \hat{T}_{a^k_{ij}} \) indicates the truth-membership function, \( \tilde{T}_{a^k_{ij}} \) indicates the indeterminacy-membership function and \( \tilde{F}_{a^k_{ij}} \) indicates the falsity-membership function. This approach is an integration of MVNNs and the aggregation operators, and can be used to solve MCDM problems mentioned above.

In general, there are maximizing criteria and minimizing criteria in MCDM problems. According to the IFSs method proposed by Xu, the minimizing criteria can be transformed into maximizing criteria as follows:

\[
\beta^k_i = \left( \alpha^k_i \right)^c, \quad \text{for maximizing criteria } c_i, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m). \tag{11}
\]

Here \( \left( \alpha^k_i \right)^c \) is the complement of \( \alpha^k_i \) as defined in Definition 5.

In the following, a procedure to rank and select the most desirable alternative(s) is given.

Step 1. Transform the decision matrix.

According to Eq. (11), the MVNN decision matrix \( R^k = \left( \alpha^k_{ij} \right)_{n \times m} \) can be transformed into a normalized MVNN decision matrix \( \tilde{R}^k = \left( \beta^k_{ij} \right)_{n \times m} \).

In order to unify all criteria, we need to transform the minimizing criteria into maximizing criteria (Remark: if all the criteria belong to the maximizing criteria and have the same measurement unit, then there is no need to normalize them). Suppose that the matrix \( R^k = \left( \alpha^k_{ij} \right)_{n \times m} \), where \( \alpha^k_{ij} \) are MVNNs, is normalized into the corresponding matrix \( \tilde{R}^k = \left( \beta^k_{ij} \right)_{n \times m} \).

For the minimizing criteria, the normalization formula is

\[
\beta^k_i = \left( \alpha^k_i \right)^c = \left( \bigcup_{\hat{c} \in \hat{C}} (\tilde{\xi}_i^j, \bigcup_{\eta \in \eta^j} [1-\eta], \bigcup_{\gamma \in \gamma^j} \{ \gamma \} \right), \tag{12}
\]

for the maximizing criteria,

\[
\beta^k_i = \alpha^k_i = \left( \bigcup_{\hat{c} \in \hat{C}} \{ \gamma \}, \bigcup_{\eta \in \eta^j} \eta, \bigcup_{\gamma \in \gamma^j} \{ \tilde{\xi}_i^j \} \right). \tag{13}
\]
Step 2. Calculate the supports $\text{Supp}(\beta^i_y, \beta^j_y)$.

The supports can be obtained by the following formula:
\[
\text{Supp}(\beta^i_y, \beta^j_y) = 1 - d(\beta^i_y, \beta^j_y),
\]
\[i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; k, l = 1, 2, \ldots, l, k \neq l.
\] (14)

Here $\text{Supp}(\beta^i_y, \beta^j_y)$ is the support for $\beta^i_y$ from $\beta^j_y$, and satisfies the three conditions given in Definition 11. $d(\beta^i_y, \beta^j_y)$ is the Hamming-Hausdorff distance between $\beta^i_y$ and $\beta^j_y$ as defined in Definition 9.

Step 3. Calculate the weights $\tau^k_y$ associated with the MVNN $\beta^k_y$.

The weighted support $S(\beta^k_y)$ of the MVNN $\beta^k_y$ by the other MVNNs $\beta^i_y (t = 1, 2, \ldots, l \text{ and } t \neq k)$ can be calculated using the weights $\omega_k (k = 1, 2, \ldots, l)$ of the decision-makers $d_k (k = 1, 2, \ldots, l)$.
\[
S(\beta^k_y) = \sum_{t=1}^{l} \omega_t \text{Supp}(\beta^t_y, \beta^k_y) \quad (k = 1, 2, \ldots, l).
\] (15)

Then, the weights $\tau^k_y (k = 1, 2, \ldots, l)$ associated with the MVNN $\beta^k_y (k = 1, 2, \ldots, l)$ can be obtained:
\[
\tau^k_y = \frac{\omega_k (1 + S(\beta^k_y))}{\sum_{k=1}^{l} \omega_k (1 + S(\beta^k_y))}, \quad k = 1, 2, \ldots, l.
\] (16)

Here $\tau^k_y \geq 0 (k = 1, 2, \ldots, l)$ and $\sum_{k=1}^{l} \tau^k_y = 1$.

Step 4. Aggregate the evaluation information of each expert.

Utilize the MVNPWA operator or MVNPWG operators, Eq. (8) or Eq. (10), to aggregate the MVNNs $\beta^k_y$ for all decision-makers:
\[
\beta_y = \text{MVNPWA}_{\omega_y} (\beta^1_y, \beta^2_y, \ldots, \beta^n_y)
\]
\[= \bigcup_{\tau^k_y \in \tau^k_y} \left[ \frac{\prod_{k=1}^{l} (1 + \gamma^k_y)^{\tau^k_y} - \prod_{k=1}^{l} (1 - \gamma^k_y)^{\tau^k_y}}{\prod_{k=1}^{l} (1 + \gamma^k_y)^{\tau^k_y} + \prod_{k=1}^{l} (1 - \gamma^k_y)^{\tau^k_y}} \right],
\]
\[\bigcup_{\tau^k_y \in \tau^k_y} \left[ \frac{2 \prod_{k=1}^{l} (\eta^k_y)^{\tau^k_y}}{\prod_{k=1}^{l} (2 - \eta^k_y)^{\tau^k_y} + \prod_{k=1}^{l} (\eta^k_y)^{\tau^k_y}} \right],
\]
\[\bigcup_{\tau^k_y \in \tau^k_y} \left[ \frac{2 \prod_{k=1}^{l} (\xi^k_y)^{\tau^k_y}}{\prod_{k=1}^{l} (2 - \xi^k_y)^{\tau^k_y} + \prod_{k=1}^{l} (\xi^k_y)^{\tau^k_y}} \right].
\] (17)

Or
\[
\beta_y = \text{MVNPWG}_{\omega_y} (\beta^1_y, \beta^2_y, \ldots, \beta^n_y)
\]
\[= \bigcup_{\tau^k_y \in \tau^k_y} \left[ \frac{2 \prod_{k=1}^{l} (\gamma^k_y)^{\tau^k_y}}{\prod_{k=1}^{l} (2 - \gamma^k_y)^{\tau^k_y} + \prod_{k=1}^{l} (\gamma^k_y)^{\tau^k_y}} \right],
\]
\[\bigcup_{\tau^k_y \in \tau^k_y} \left[ \frac{\prod_{k=1}^{l} (1 + \eta^k_y)^{\tau^k_y} - \prod_{k=1}^{l} (1 - \eta^k_y)^{\tau^k_y}}{\prod_{k=1}^{l} (1 + \eta^k_y)^{\tau^k_y} + \prod_{k=1}^{l} (1 - \eta^k_y)^{\tau^k_y}} \right],
\]
\[\bigcup_{\tau^k_y \in \tau^k_y} \left[ \frac{\prod_{k=1}^{l} (1 + \xi^k_y)^{\tau^k_y} - \prod_{k=1}^{l} (1 - \xi^k_y)^{\tau^k_y}}{\prod_{k=1}^{l} (1 + \xi^k_y)^{\tau^k_y} + \prod_{k=1}^{l} (1 - \xi^k_y)^{\tau^k_y}} \right].
\] (18)

Step 5. Calculate the supports $\text{Supp}(\beta^i_y, \beta^j_y)$.

The supports can be obtained by the following formula:
\[
\text{Supp}(\beta^i_y, \beta^j_y) = 1 - d(\beta^i_y, \beta^j_y).
\] (19)

Here $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; j \neq p, \text{ Supp}(\beta^i_y, \beta^j_y)$ is the support for $\beta^i_y$ from $\beta^j_y$, and satisfies the three conditions given in Definition 11. $d(\beta^i_y, \beta^j_y)$ is the Hamming-Hausdorff distance between $\beta^i_y$ and $\beta^j_y$ as defined in Definition 9.

Step 6. Calculate the weights $\rho^j_y$ associated with the MVNN $\beta^j_y$.

The weighted support $S(\beta^j_y)$ of the MVNN $\beta^j_y$ by the other MVNNs $\beta^p_y (p = 1, 2, \ldots, m \text{ and } p \neq j)$ can be calculated using the weights $w^p_j (j = 1, 2, \ldots, m)$ of the criteria $c_j (j = 1, 2, \ldots, m)$.
\[
S(\beta^j_y) = \frac{\sum_{p=1, p\neq j}^{m} \omega^p_j \text{Supp}(\beta^i_y, \beta^p_y)}{p = 1, 2, \ldots, m}.
\] (20)

Then, the weights $\rho^j_y (j = 1, 2, \ldots, m)$ associated with the MVNN $\beta^j_y (j = 1, 2, \ldots, m)$ can be obtained as follows:
\[
\rho^j_y = \omega^j_y \frac{\sum_{p=1}^{m} \omega^p_j S(\beta^p_y)}{\sum_{p=1, p\neq j}^{m} \omega^p_j \sum_{p=1}^{m} \omega^p_j S(\beta^p_y)}, j = 1, 2, \ldots, m.
\] (21)

Here $\rho^j_y \geq 0 (j = 1, 2, \ldots, m)$ and $\sum_{j=1}^{m} \rho^j_y = 1$. 

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356
Step 7. Calculate the comprehensive evaluation value of each alternative.

Utilize the MVNPWA operator or MVNPWG operators, Eq. (8) or Eq. (10), to aggregate all the preference values $\beta_j (j = 1, 2, \ldots, m)$ of each alternative, then the comprehensive evaluation value $\beta_i (i = 1, 2, \ldots, n)$ of alternative $\alpha_i (i = 1, 2, \ldots, n)$ can be calculated:

$$\beta_i = \text{MVNPWA}_n (\beta_1, \beta_2, \ldots, \beta_m)$$
$$= \left( \bigcup_{\gamma \in T^\alpha} \left[ \frac{\prod_{j=1}^{m} (1+\gamma_j)^{\alpha_j} - \prod_{j=1}^{m} (1-\gamma_j)^{\alpha_j}}{\prod_{j=1}^{m} (1+\gamma_j)^{\alpha_j} + \prod_{j=1}^{m} (1-\gamma_j)^{\alpha_j}} \right] \right),$$
$$\bigcup_{\nu \in T^\alpha} \left[ \frac{2 \prod_{j=1}^{m} (\nu_j)^{\alpha_j}}{\prod_{j=1}^{m} (2-\nu_j)^{\alpha_j} + \prod_{j=1}^{m} (\nu_j)^{\alpha_j}} \right]$$
$$\bigcup_{\xi \in T^\alpha} \left[ \frac{2 \prod_{j=1}^{m} (\xi_j)^{\alpha_j}}{\prod_{j=1}^{m} (2-\xi_j)^{\alpha_j} + \prod_{j=1}^{m} (\xi_j)^{\alpha_j}} \right],$$

or

$$\beta_i = \text{MVNPWG}_n (\beta_1, \beta_2, \ldots, \beta_m)$$
$$= \left( \bigcup_{\gamma \in T^\alpha} \left[ \frac{2 \prod_{j=1}^{m} (\gamma_j)^{\alpha_j}}{\prod_{j=1}^{m} (2-\gamma_j)^{\alpha_j} + \prod_{j=1}^{m} (\gamma_j)^{\alpha_j}} \right] \right),$$
$$\bigcup_{\nu \in T^\alpha} \left[ \frac{\prod_{j=1}^{m} (1+\nu_j)^{\alpha_j} - \prod_{j=1}^{m} (1-\nu_j)^{\alpha_j}}{\prod_{j=1}^{m} (1+\nu_j)^{\alpha_j} + \prod_{j=1}^{m} (1-\nu_j)^{\alpha_j}} \right]$$
$$\bigcup_{\xi \in T^\alpha} \left[ \frac{\prod_{j=1}^{m} (1+\xi_j)^{\alpha_j} - \prod_{j=1}^{m} (1-\xi_j)^{\alpha_j}}{\prod_{j=1}^{m} (1+\xi_j)^{\alpha_j} + \prod_{j=1}^{m} (1-\xi_j)^{\alpha_j}} \right],$$

Step 8. Calculate the score function value and the accuracy function value.

Based on Definition 7, the score function value $s(\beta)$ and the accuracy function value $a(\beta)$ of $\alpha_i (i = 1, 2, \ldots, n)$ can be obtained.

Step 9. Rank the alternatives.

According to Definition 8, all alternatives $\alpha_i (i = 1, 2, \ldots, n)$ can be ranked with respect to superiority and finally the best one(s) can be chosen.

5. Illustrative Example

In this section, an example of MCDM problems is used to demonstrate the application and effectiveness of the proposed decision-making approach.

There is an investment company, which wants to invest a sum of money in the best option (adapted from Ref. 37). The company has set up a panel which has to choose between four possible alternatives for investing the money: (1) $\alpha_1$ is a car company; (2) $\alpha_2$ is a food company; (3) $\alpha_3$ is a computer company; (4) $\alpha_4$ is an arms company. Each company is evaluated based on three criteria, which are denoted by $c_j (j = 1, 2, 3)$: $c_1$ is the risk analysis, $c_2$ is the growth analysis and $c_3$ is the environmental impact analysis, where $c_1$ and $c_2$ are of the maximizing type, and $c_3$ is of the minimizing type. The weight vector of criteria is represented by $\omega = (0.35, 0.25, 0.4)$. There are three decision-makers to make decisions on this investment and the weight vector of them is $\omega = (0.3, 0.5, 0.2)$. They could evaluate these criteria based on their knowledge and experience. Moreover, the $k$-th decision-maker can provide their evaluations about the project $\alpha_i$ under the criterion $c_j$ in the form of MVNNs and denoted by $\alpha^c_{ij} = \{ \tilde{F}_{ij}, \tilde{I}_{ij}, \tilde{T}_{ij} \}$, $i = 1, 2, 3, 4; j = 1, 2, 3; k = 1, 2, 3$.

$\tilde{F}_{ij}$ and $\tilde{I}_{ij}$ are in the form of HFNs, which represents their degrees of satisfaction, uncertainty and dissatisfaction regarding an alternative by using the concept of “excellent” against each criterion. It is noted that one decision-maker could give several evaluation values for the degree of satisfaction, uncertainty and dissatisfaction regarding an alternative respectively. All of the possible values for an alternative under a criterion are collected, and each value provided only means that it is a possible value. So in the case where the decision-maker gives two same value for one degree, it is counted only once, and $\alpha^c_{ij}$ is the set of evaluation values for the decision-maker. Then the multi-valued neutrosophic decision matrix $R^k = (\alpha^c_{ij})_{4 \times 3}$ can be found as follows:
5.1. Decision-making procedure based on MVNNs

Step 1. Transform the decision matrix.

Since criteria $c_i$ and $c_j$ are of the maximizing type, and criterion $c_3$ is of the minimizing type, so according to Eqs. (12) and (13), the normalized MVNN decision matrix $\mathbf{R}^\dagger = \left( \beta_{ij}^k \right)_{4 \times 3}$ can be obtained as follows:

\[
\mathbf{R}^\dagger = \begin{bmatrix}
(0.4,0.1,0.2) & (0.5,0.2,0.1) & (0.4,0.0,0.1) \\
(0.7,0.1,0.2) & (0.6,0.2,0.1) & (0.4,0.2,0.1) \\
(0.4,0.5,0.1) & (0.5,0.2,0.1) & (0.4,0.5,0.1) \\
(0.6,0.1,0.2) & (0.5,0.2,0.1) & (0.4,0.5,0.1)
\end{bmatrix};
\]

Then the supports $\text{Supp} (\beta_{ij}^k)$ can be calculated and denoted $\text{Supp}^\dagger$. For simplicity, we denote $\text{Supp} (\beta_{ij}^k, \beta_{ij}^k)$ with $\text{Supp}^\dagger$. According to Eq. (14) and Definition 9, the supports $\text{Supp}^\dagger (k,t=1,2,3;k \neq t)$ can be obtained. As an example, $\text{Supp}^\dagger_{12}$ can be calculated as follows:

\[
\text{Supp}^\dagger_{12} = 1 - d\left( \beta_{11}^1, \beta_{21}^1 \right)
= 1 - d\left( \langle 0.4,0.1,0.2 \rangle, \langle 0.6,0.1,0.2 \rangle \right)
= 0.9167.
\]

Then the $\text{Supp}^\dagger (k,t=1,2,3;k \neq t)$ can be achieved:

\[
\begin{align*}
\text{Supp}^\dagger_{12} &= \begin{bmatrix} 0.9167 & 0.9667 & 0.9000 \\
0.9167 & 0.9667 & 0.9000 \\
0.9000 & 0.9000 & 0.9333 \\
0.9333 & 0.9000 & 0.9667 \\
\end{bmatrix}, \\
\text{Supp}^\dagger_{13} &= \begin{bmatrix} 0.9167 & 0.8833 & 0.9167 \\
0.9667 & 0.9667 & 0.9167 \\
0.9333 & 0.9333 & 0.9667 \\
0.9167 & 0.9167 & 0.9000 \\
\end{bmatrix}, \\
\text{Supp}^\dagger_{23} &= \begin{bmatrix} 0.8667 & 0.9167 & 0.8167 \\
0.9500 & 0.9500 & 0.9667 \\
0.9000 & 0.9667 & 0.9333 \\
0.8500 & 0.8833 & 0.9333 \\
\end{bmatrix}.
\end{align*}
\]

Step 3. Calculate the weights $\tau_i^k$ associated with the MVNN $\beta_{ij}^k$.

According to Eq. (15), the weighted supports $S (\beta_{ij}^k)$ can be obtained. As an example, $S (\beta_{11}^1)$ can be calculated as follows:

\[
S (\beta_{11}^1) = \frac{3}{3} \omega_k \text{Supp} (\beta_{11}^1, \beta_{11}^1) = 0.6417.
\]

Then the $\left( S (\beta_{ij}^k) \right)_{4 \times 3}$ can be calculated and denoted with $\left( S (k=1,2,3) \right)$ in the following:

\[
\begin{align*}
S^1 &= \begin{bmatrix} 0.6417 & 0.6600 & 0.6333 \\
0.6517 & 0.6734 & 0.6333 \\
0.6367 & 0.6367 & 0.6600 \\
0.6500 & 0.6333 & 0.6634 \\
\end{bmatrix}, \\
S^2 &= \begin{bmatrix} 0.4484 & 0.4734 & 0.6784 \\
0.7500 & 0.4800 & 0.4633 \\
0.4500 & 0.4633 & 0.4667 \\
0.4500 & 0.4467 & 0.4767 \\
\end{bmatrix}, \\
S^3 &= \begin{bmatrix} 0.7084 & 0.7233 & 0.6834 \\
0.7650 & 0.7600 & 0.7584 \\
0.7300 & 0.7633 & 0.7567 \\
0.7000 & 0.7167 & 0.7367 \\
\end{bmatrix}.
\end{align*}
\]

Based on Eq. (16), the weights $\tau_i^k (i=1,2,3,4;k=1,2,\ldots,l)$ associated with the MVNN $\beta_{ij}^k (k=1,2,\ldots,l)$ can be obtained using the weights $\omega_k (k=1,2,\ldots,l)$ of the decision-makers $d_k (k=1,2,\ldots,l)$. $\tau_i^k$ is denoted by $\tau^k (k=1,2,3)$ as follows:
**Step 4.** Aggregate the evaluation information of each expert.

According to MVNPWA operator, i.e., Eq. (17), the collective multi-valued neutrosophic decision matrix \( \tilde{R} = (\tilde{r}_{ij})_{n \times m} \) can be obtained. For example, \( \beta_1 \) can be calculated as follows:

\[
\beta_1 = \text{MVNPWA}_{\alpha}(\beta_1^1, \beta_1^2, \beta_1^3) = \left\{ [0.4996, 0.5201], [0.1168], [0.1599, 0.2190] \right\}.
\]

Then the other collective values can be obtained:

\[
\begin{bmatrix}
0.4996 & 0.5201 & 0.1168 & 0.3727 & 0.8500 & 0.8796 & 0.4792 & 0.2000 & 0.2189 & 0.1770 \\
0.4672 & 0.3785 & 0.4000 & 0.4273 & 0.4669 & 0.2000 & 0.2470 & 0.8229 & 0.8459 & 0.4118 & 0.4326 & 0.4591 & 0.4788 & 0.5000 & 0.1456 & 0.2159 & 0.4668 & 0.5000 \\
0.3759 & 0.4118 & 0.4326 & 0.4591 & 0.4788 & 0.1666 & 0.2278 & 0.7529 & 0.2388 & 0.2752 & 0.6839 & 0.7072 & 0.1965 & 0.2381 & 0.1387 & 0.2000 & 0.2085 & 0.4767 \\
0.5871 & 0.5182 & 0.3862 & 0.3598 & 0.3215 & 0.3626 & 0.4668 & 0.5000 & 0.1456 & 0.2159 & 0.4668 & 0.5000 & 0.1616 & 0.1829 & 0.1387 & 0.1456 & 0.2159 & 0.5000 \\
\end{bmatrix}
\]

**Step 5.** Calculate the supports \( \text{Supp} \beta_{ij, \alpha} \).

According to Eq. (19),

\[
\text{Supp} \beta_{ij, \alpha} = \left\{ [0.4996, 0.5201], [0.1168], [0.1599, 0.2190] \right\} = \left\{ [0.4792], [0.2000, 0.2189], [0.1770] \right\} = \left\{ [0.3727], [0.8500, 0.8796], [0.3215, 0.3626] \right\}.
\]

Then the other collective values can be obtained:

\[
\begin{bmatrix}
0.4996 & 0.5201 & 0.1168 & 0.3727 & 0.8500 & 0.8796 & 0.4792 & 0.2000 & 0.2189 & 0.1770 \\
0.4672 & 0.3785 & 0.4000 & 0.4273 & 0.4669 & 0.2000 & 0.2470 & 0.8229 & 0.8459 & 0.4118 & 0.4326 & 0.4591 & 0.4788 & 0.5000 & 0.1456 & 0.2159 & 0.4668 & 0.5000 \\
0.3759 & 0.4118 & 0.4326 & 0.4591 & 0.4788 & 0.1666 & 0.2278 & 0.7529 & 0.2388 & 0.2752 & 0.6839 & 0.7072 & 0.1965 & 0.2381 & 0.1387 & 0.2000 & 0.2085 & 0.4767 \\
0.5871 & 0.5182 & 0.3862 & 0.3598 & 0.3215 & 0.3626 & 0.4668 & 0.5000 & 0.1456 & 0.2159 & 0.4668 & 0.5000 & 0.1616 & 0.1829 & 0.1387 & 0.1456 & 0.2159 & 0.5000 \\
\end{bmatrix}
\]

**Step 6.** Calculate the weights \( \rho_{ij} \) associated with the MVNN \( \beta_{ij} \).

According to Eq. (20), the weighted support \( S(\beta_{ij, \alpha}) \) of the MVNN \( \beta_{ij} \) by the other MVNNs \( \beta_p (p = 1, 2, ... , m \text{ and } p \neq j) \) can be calculated.

\[
S(\beta_{ij, \alpha}) = \begin{bmatrix}
0.4989 & 0.6086 & 0.4017 \\
0.4835 & 0.5816 & 0.3551 \\
0.4829 & 0.5788 & 0.3598 \\
0.4627 & 0.5431 & 0.3382 \\
\end{bmatrix}.
\]

So the weights \( \rho_{ij} (j = 1, 2, ..., m) \) associated with the MVNN \( \beta_{ij} (j = 1, 2, ..., m) \) can be obtained using the weights \( w_j (j = 1, 2, ..., m) \) of the criteria \( c_j (j = 1, 2, ..., m) \) and Eq. (21).

\[
\rho_{ij} = \begin{bmatrix}
0.3527 & 0.2704 & 0.3769 \\
0.3564 & 0.2714 & 0.3721 \\
0.3561 & 0.2708 & 0.3732 \\
0.3573 & 0.2692 & 0.3735 \\
\end{bmatrix}.
\]
Peng et al.

**Step 7.** Calculate the comprehensive evaluation value of each alternative.
Utilize the MVNPWA operator i.e., Eq. (22), to aggregate all the preference values \( \beta_j (j = 1, 2, \ldots, m) \) of each alternative, then the comprehensive value \( \beta_i (i = 1, 2, \ldots, n) \) of the alternative \( \alpha_i \) can be calculated:

\[
\beta_i = \left\{ \left[ 0.4481, 0.4558 \right], \left[ 0.3119, 0.3179, 0.3192, 0.3253 \right], \left[ 0.2153, 0.2260 \right] \right\};
\]

\[
\beta_2 = \left\{ \left[ 0.4670, 0.4737 \right], \left[ 0.2978, 0.3119, 0.3178, 0.3326 \right], \left[ 0.2567, 0.2785, 0.2657, 0.2881 \right] \right\};
\]

\[
\beta_3 = \left\{ \left[ 0.3507, 0.3668, 0.3746, 0.3586, 0.3688, 0.3846, 0.3995, 0.3839, 0.3630, 0.3789, 0.3866, 0.3708, 0.3809, 0.3966, 0.4041, 0.3886 \right], \left[ 0.3119, 0.3166 \right], \left[ 0.2757, 0.2838 \right] \right\};
\]

\[
\beta_4 = \left\{ \left[ 0.4916, 0.5008 \right], \left[ 0.3783, 0.3964, 0.3965, 0.4152 \right], \left[ 0.2484 \right] \right\};
\]

**Step 8.** Calculate the score function value and the accuracy function value.

Based on Definition 7, \( s(\beta_j) \) can be obtained:

\[
s(\beta_1) = 0.0291; s(\beta_2) = 0.0390; s(\beta_3) = 0.0718; s(\beta_4) = 0.0496
\]

The score values are different. Therefore there is no need to compute the values of the accuracy function value.

**Step 9:** Rank the alternatives.

According to Definition 8 and the results in Step 8, \( s(\beta_2) > s(\beta_3) > s(\beta_4) > s(\beta_1) \) can be obtained. So for MVNPWA operator, the final ranking is \( \alpha_1 \succ \alpha_2 \succ \alpha_3 \succ \alpha_4 \). Clearly, the best alternative is \( \alpha_1 \) while the worst alternative is \( \alpha_4 \).

If the MVNPWG operator is utilized in Step 4 and Step 7, then the score function value \( s(\beta_j) \) can be obtained:

\[
s(\beta_1) = 0.0301; s(\beta_2) = 0.0259; s(\beta_3) = 0.0860; s(\beta_4) = 0.0572
\]

Since \( s(\beta_2) > s(\beta_3) > s(\beta_4) > s(\beta_1) \) and the score values are different. Therefore, for MVNPWG operator, the final ranking is \( \alpha_2 \succ \alpha_3 \succ \alpha_4 \succ \alpha_1 \), and the best alternative is \( \alpha_2 \) while the worst alternative is \( \alpha_1 \).

From the results given above, the best one is \( \alpha_1 \) or \( \alpha_2 \), and the worst one is \( \alpha_4 \). In most cases, in order to calculate the actual aggregation values of the alternatives, different aggregation operators can be used. Moreover, we can find that two aggregation operators mentioned in the manuscript, the MVNPWA operator or the MVNPWG operator, are all used to deal with different relationships of the aggregated arguments, which can provide more choices for decision-makers. They can choose different aggregation operator according to their preference.

### 5.2. Comparison analysis

In order to verify the feasibility of the proposed decision-making approach based on the MVNNs power aggregation operators, a comparison analysis based on the same illustrative example is conducted here.

The comparison analysis includes two cases. One is the other methods that were outlined in Ye36, 37, 41, which are compared to the proposed method using single-valued neutrosophic information. In the other, the method that was introduced in Wang and Li 48 are compared with the proposed approach using multi-valued neutrosophic information.

The proposed approach is compared with some methods using single-valued neutrosophic information.

- The proposed approach is compared with some methods using single-valued neutrosophic information.

With regard to the three methods in Ye 36–37, 41, all multi-valued neutrosophic evaluation values are translated into single-valued neutrosophic values by using the mean values of truth-membership, indeterminacy-membership and falsity-membership respectively. Then two aggregation operators were used to aggregate the single-valued neutrosophic information first; and the correlation coefficient and weighted cross-entropy between each alternative and the ideal alternative were calculated and used to determine the final ranking order of all the alternatives. If the methods in Ye 36–37, 41 and the proposed method are utilized to solve the same MCDM problem, then the results can be obtained and are shown in Table 1.
Table 1. The compared results utilizing the different methods with SNSs

<table>
<thead>
<tr>
<th>Methods</th>
<th>The final ranking</th>
<th>The best alternative(s)</th>
<th>The worst alternative(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ye [36]</td>
<td>$\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$</td>
<td>$\alpha_1$</td>
<td>$\alpha_3$</td>
</tr>
<tr>
<td>Ye [37]</td>
<td>$\alpha_4 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$ or $\alpha_4 \succ \alpha_3 \succ \alpha_1$</td>
<td>$\alpha_4$ or $\alpha_1$</td>
<td>$\alpha_3$</td>
</tr>
<tr>
<td>Ye [41]</td>
<td>$\alpha_4 \succ \alpha_2 \succ \alpha_1 \succ \alpha_3$</td>
<td>$\alpha_4$</td>
<td>$\alpha_3$</td>
</tr>
<tr>
<td>The proposed method</td>
<td>$\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$ or $\alpha_2 \succ \alpha_1 \succ \alpha_4 \succ \alpha_3$</td>
<td>$\alpha_1$ or $\alpha_2$</td>
<td>$\alpha_3$</td>
</tr>
</tbody>
</table>

If the aggregation operators proposed by Ye [37] are used, for the weighted average operator, the final ranking is $\alpha_4 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$. Clearly, the best alternative is $\alpha_4$, while the worst alternative is $\alpha_3$. For the weighted geometric operator, the final ranking is $\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$, and the best alternative is $\alpha_1$, while the worst alternative is $\alpha_3$. However, if the methods of Ye [36, 41] are used, then the final ranking is $\alpha_4 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$ or $\alpha_4 \succ \alpha_2 \succ \alpha_1 \succ \alpha_3$, and the best alternative is $\alpha_4$, while the worst alternative is $\alpha_3$. It can be seen that the results of the proposed approach are different from those that use the earlier methods of Ye [36-37, 41].

There are three reasons why differences exist in the final rankings of all the compared methods and the proposed approach. Firstly, the aggregation operators that are involved in the method of Ye [37] are related to some impractical operations as was discussed in Examples 1-3. Secondly, if the correlation coefficient and cross-entropy proposed in [36, 41], proposed on the basis of the operations in [37], are extend to MVNNs, the shortcomings discussed in Section 2 would still exist. Finally, the aggregation values, correlation coefficients and cross-entropy measures of SNSs were obtained firstly in Ye [36-37, 41] and the differences were amplified in the final results due to the use of criteria weights.

- The proposed approach is compared with the method using multi-valued neutrosophic information.

If the method in Wang and Li [48] is utilized to solve the same MCDM problem, then the MVNPWA and MVNPWG operators were used to aggregate the evaluation information of each expert, respectively; and the final ranking can be determined by using the TODIM method in Ref. 48. If the MVNPWA operator is used first, then the final ranking is $\alpha_1 \succ \alpha_2 \succ \alpha_4 \succ \alpha_3$, and the best alternative is $\alpha_1$, while the worst alternative is always $\alpha_3$; if the MVNPWG operator is used, then the final ranking is $\alpha_2 \succ \alpha_1 \succ \alpha_4 \succ \alpha_3$, and the best alternative is $\alpha_2$. Apparently, the result of the proposed approach is the same as that using Wang and Li's method [48], and the best alternative is always $\alpha_1$ or $\alpha_2$ while the worst alternative is always $\alpha_3$.

From the analysis presented above, it can be concluded that the main advantages of the approach developed in this paper over the other methods are not only due to its ability to effectively overcome the shortcomings of the compared methods, but also due to its ability to relieve the influence of unfair assessments provided by different decision-makers on the final aggregated results. This means that it can avoid losing and distorting the preference information provided which makes the final results more precise and reliable correspond with real-life decision-making problems.

6. Conclusions

MVNSs can be applied in solving problems with uncertain, imprecise, incomplete and inconsistent information that exist in scientific and engineering situations. Based on the related research of IFSs and HFSs, the operations of MVNNs were defined in this paper and the comparison method was also developed. Furthermore, two aggregation operators, namely the MVNPWA operator and MVNPWG operator, were provided. Thus, a MCGDM approach was established that was based on proposed operators. An illustrative example demonstrated the application of the proposed decision-making approach. Moreover, the comparison analysis showed that the final result produced by the
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