Multi-valued Neutrosophic Sets and Power Aggregation Operators with Their Applications in Multi-criteria Group Decision-making Problems

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Abstract

In recent years, hesitant fuzzy sets (HFSs) and neutrosophic sets (NSs) have become a subject of great interest for researchers and have been widely applied to multi-criteria group decision-making (MCGDM) problems. In this paper, multi-valued neutrosophic sets (MVNSs) are introduced, which allow the truth-membership, indeterminacy-membership and falsity-membership degree have a set of crisp values between zero and one, respectively. Then the operations of multi-valued neutrosophic numbers (MVNNs) based on Einstein operations are defined, and a comparison method for MVNNs is developed depending on the related research of HFSs and Atanassov's intuitionistic fuzzy sets (IFSs). Furthermore, the multi-valued neutrosophic power weighted average (MVNPWA) operator and the multi-valued neutrosophic power weighted geometric (MVNPWG) operator are proposed and the desirable properties of two operators are also discussed. Finally, an approach for solving MCGDM problems is explored by applying the power aggregation operators, and an example is provided to illustrate the application of the proposed method, together with a comparison analysis.

Keywords: Multi-criteria group decision-making, multi-valued neutrosophic sets, power aggregation operators.

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1. Introduction

In many cases, it is difficult for decision-makers to precisely express a preference when solving multi-criteria decision-making (MCDM) and multi-criteria group decision-making (MCGDM) problems with inaccurate, uncertain or incomplete information. Under these circumstances, Zadeh's fuzzy sets (FSs)[1], where the membership degree is represented by a real number between zero and one, are regarded as an important tool for solving MCDM and MCGDM problems[2–3], fuzzy logic and approximate reasoning[4], and pattern recognition[5].

However, FSs can not handle certain cases where it is hard to define the membership degree using one specific value. In order to overcome the lack of knowledge of non-membership degrees, Atanassov introduced intuitionistic fuzzy sets (IFSs) [6], an extension of Zadeh's FSs. Furthermore, Gau and Buehrer defined vague sets[7] and subsequently Bustince pointed out that the vague sets and IFSs are mathematically equivalent objects[8]. IFSs simultaneously take into account the membership degree, non-membership degree and degree of hesitation. Therefore, they are more flexible and practical when addressing fuzziness and uncertainty than FSs. Moreover, in some actual cases, the membership degree, non-membership degree and hesitation degree of an element in IFSs may not be a specific number; hence, they were extended to the interval-valued intuitionistic fuzzy sets (IVIFSs)[9]. To date, IFSs and IVIFSs have been widely applied in solving MCDM and MCGDM problems[10–21]. In order to handle situations where people are hesitant in expressing their preference regarding objects in a decision-making process, hesitant fuzzy sets (HFSs) were introduced by Torra and Narukawa[22–23]. Then some work on HFSs and their extensions have been undertaken, including the aggregation operators, the correlation coefficient, distance, correlation measures and outranking relations for HFSs[24–30].

Although the theory of FSs has been developed and generalized, it can not deal with all types of uncertainties in different real-world problems. Types of uncertainties, such as the indeterminate information and inconsistent information, cannot be managed. For example, when an expert is asked for their opinion about a certain statement, he or she may say the possibility that the statement is true is 0.5, the possibility that the statement is false is 0.6 and the degree that he or she is not sure is 0.2[31]. This issue is beyond the scope of the FSs and IFSs. Then Smarandache proposed neutrosophic logic and neutrosophic sets (NSs)[32–33] and subsequently Rivieccio pointed out that an NS is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and it lies in $]0^-, 1^+[$, the non-standard unit interval[34]. Clearly, this is the extension of the standard interval $[0, 1]$. Furthermore, the uncertainty presented here, i.e. indeterminacy factor, is dependent on of truth and falsity values, whereas the incorporated uncertainty is dependent on the degrees of belongingness and degree of non-belongingness of IFSs[35]. Additionally, the aforementioned example of NSs can be expressed as $x(0.5, 0.2, 0.6)$. However, without specific description, NSs are difficult to apply to real-life situations. Therefore, single-valued neutrosophic sets (SVNSs) were proposed, which are an extension of NSs[31,35]. Majumdar et al introduced a measure of entropy of SVNSs[35]. Furthermore, the correlation coefficients of SVNSs as well as a decision-making method using SVNSs were introduced[36]. In addition, Ye also introduced the concept of simplified neutrosophic sets (SNSs), which can be described by three real numbers in the real unit interval [0,1], and proposed an MCDM method using the aggregation operators of SNSs[37]. Wang et al and Lupiáñez proposed the concept of interval neutrosophic sets (INSs) and provided the set-theoretic operators of INSs[38,39]. Broumi and Smarandache discussed the correlation coefficient of INSs[40]. Furthermore, Ye proposed the cross-entropy of SVNSs and similarity of INSs respectively[41–42]. However, in certain cases, the operations of SNSs provided by Ye may be unreasonable[37]. For example, the sum of any element and the maximum value should be equal to the maximum value, but this is not always the case during operations. The similarity measures and distances of SVNSs that are based on those operations may also be unrealistic. Peng et al developed novel operations, outranking relations and aggregation operators of SNSs[43–44], which were based on the operations in Ye[37] and applied them to MCGDM problems. Zhang et al introduced a MCDM method with INSs[45]. Liu and Wang investigated single-valued neutrosophic normalized weighted Bonferroni mean and applied it to MCDM problems[46]. Liu et al developed some Hamacher aggregation operators with NSs[47]. Tian et al developed simplified neutrosophic linguistic normalized weighted Bonferroni mean operator and applied it to MCDM problems[48].

However, decision-makers can also be hesitant when expressing their evaluation values for each parameter in SNSs. For example, if the possibility of a statement being true is 0.6 or 0.7, the possibility of it being false is 0.2 or 0.3 and the degree that he or she is not sure is 0.1 or 0.2, this will be beyond the capability of SNSs. If the operations and comparison method of SNSs were extended to multiple values, the shortcomings discussed earlier would still exist. Therefore, Wang and Li developed the definition of multi-valued neutrosophic sets (MVNSs)[49], based on which, the Einstein operations and comparison method, and power aggregation operators for multi-valued neutrosophic numbers (MVNNs) are defined in this paper. Consequently, a MCGDM method is established based on the proposed operators. An illustrative example is also given to demonstrate the applicability of the proposed method.

The rest of paper is organized as follows. In Section 2 some basic concepts and operations of SNSs are briefly reviewed. Then the definition of MVNSs is introduced, and the operations, a comparison method and distance of MVNNs are defined in Section 3. Section 4 contains two MVNN power aggregation operators and a MCGDM approach with MVNNs. In Section 5, an illustrative example and a comparison analysis are presented to verify the proposed approach. Finally, the conclusions are drawn in Section 6.

2. Preliminaries

In this section, the definitions and operations of NSs and SNSs are introduced, which will be utilized in the latter analysis.

Definition 1. Let X be a space of points (objects), with a generic element in X denoted by x. An NS A in X is characterized by a truth-membership function $T_A(x)$, a indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$ as follows[32]:

$$A=\left\{\left\langle x,T_A(x),I_A(x),F_A(x)\right\rangle \middle| x\in X\right\}, \qquad (1)$$

$T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, that is, $T_A(x): X \to]0^-, 1^+[$, $I_A(x): X \to]0^-, 1^+[$, and $F_A(x): X \to]0^-, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, therefore $0^- \le \sup T_A(x)+\sup I_A(x)+\sup F_A(x) \le 3^+$.

Considering the applicability of NSs, Ye reduced NSs of nonstandard intervals into SNSs of standard intervals[37], which can preserve the operations of NSs properly.

Definition 2. Let X be a space of points (objects), with a generic element in X denoted by x. An NS A in X is characterized by $T_A(x)$, $I_A(x)$ and $F_A(x)$, which are singleton subintervals/subsets in the real standard [0, 1], that is $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$, and $F_A(x): X \to [0,1]$. Then, a simplification of A is denoted by[37]:

$$A=\left\{\left\langle x,T_A(x),I_A(x),F_A(x)\right\rangle \mid x\in X\right\}, \qquad (2)$$

which is called an SNS and is a subclass of NSs. For convenience, the SNSs is denoted by the simplified symbol $A=\left\{\left\langle T_A(x),I_A(x),F_A(x)\right\rangle\right\}$. The set of all SNSs is represented as SNSS.
The operations of SNSs are also defined by Ye[37].

Definition 3. Let A, A_1 and A_2 be three SNSs. For any $x \in X$, the following operations can be true[37].

(1) $A_1+A_2=\left\langle T_{A_1}(x)+T_{A_2}(x)-T_{A_1}(x)\cdot T_{A_2}(x),\ I_{A_1}(x)+I_{A_2}(x)-I_{A_1}(x)\cdot I_{A_2}(x),\ F_{A_1}(x)+F_{A_2}(x)-F_{A_1}(x)\cdot F_{A_2}(x)\right\rangle$;

(2) $A_1\cdot A_2=\left\langle T_{A_1}(x)\cdot T_{A_2}(x), I_{A_1}(x)\cdot I_{A_2}(x), F_{A_1}(x)\cdot F_{A_2}(x)\right\rangle$;

(3) $\lambda\cdot A=\left\langle 1-\left(1-T_A(x)\right)^\lambda, 1-\left(1-I_A(x)\right)^\lambda, 1-\left(1-F_A(x)\right)^\lambda\right\rangle, \lambda>0$;

(4) $A^\lambda=\left\langle T_A^\lambda(x), I_A^\lambda(x), F_A^\lambda(x)\right\rangle, \lambda>0$.

There are some limitations related to Definition 3 and these are now outlined.

(i) In some situations, operations such as A_1+A_2 and $A_1\cdot A_2$ might be impractical. This is demonstrated in Example 1.

Example 1. Let $A_1 = \{\langle x, 0.5, 0.5, 0.5\rangle\}$ and $A_2 = \{\langle x, 1, 0, 0\rangle\}$ be two SNSs. Clearly, $A_2 = \{\langle x, 1, 0, 0\rangle\}$ can be the larger of these SNSs. Theoretically, the sum of any number and the maximum number should be equal to the maximum one. However, according to Definition 3, $A_1 + A_2 = \{\langle x, 1, 0.5, 0.5\rangle\} \neq A_2$, therefore the operation "+" cannot be accepted. Similar contradictions exist in other operations of Definition 3, and thus those defined above are incorrect.

(ii) The correlation coefficient of SNSs[36], which is based on the operations of Definition 3, cannot be accepted in some specific cases.

Example 2. Let $A_1 = \{\langle x, 0.8, 0, 0\rangle\}$ and $A_2 = \{\langle x, 0.7, 0, 0\rangle\}$ be two SNSs, and $A = \{\langle x, 1, 0, 0\rangle\}$ be the largest one of the SNSs. According to the correlation coefficient of SNSs[36], $W(A_1, A) = W_2(A_2, A) = 1$ can be obtained, but this does not indicate which one is the best. However, it is clear that A_1 is superior to A_2.

(iii) In addition, the cross-entropy measure for SNSs[41], which is based on the operations of Definition 3, cannot be accepted in special cases.

Example 3. Let $A_1 = \{\langle x, 0.1, 0, 0\rangle\}$ and $A_2 = \{\langle x, 0.9, 0, 0\rangle\}$ be two SNSs, and $A = \{\langle x, 1, 0, 0\rangle\}$ be the largest one of the SNSs. According to the cross-entropy measure for SNSs[41], $S_1(A_1, A) = S_2(A_2, A) = 1$ can be obtained, which indicates that A_1 is equal to A_2. Yet it is not possible to discern which one is the best. Since $T_{A_2}(x) > T_{A_1}(x)$, $I_{A_2}(x) > I_{A_1}(x)$ and $F_{A_2}(x) > F_{A_1}(x)$ for any x in X, it is clear that A_2 is superior to A_1.

(iv) If $I_{A_1}(x) = I_{A_2}(x) = 0$ for any x in X, then A_1 and A_2 are both reduced to IFSs. However, the operations presented in Definition 3 are not in accordance with the operations of two IFSs[6, 8, 10-21].

3. Multi-valued Neutrosophic Sets

In this section, MVNSs are introduced, and the corresponding operations and comparison method are developed in terms of those of IFSs[6, 8, 10-21] and HFSs[22, 23].

3.1. *MVNSs and theirs Einstein operations*

Definition 4. Let X be a space of points (objects), with a generic element in X denoted by x. An MVNSs A in X is characterized by[48]:

$$A = \left\{\left\langle x, \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right\rangle \middle| x \in X\right\}, \quad (3)$$

where $\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three sets of precise values in [0,1], denoting the truth-membership degree, indeterminacy-membership function and falsity-membership degree respectively, satisfying $0 \le \gamma, \eta, \xi \le 1, 0 \le \gamma^+ + \eta^+ + \xi^+ \le 3$, where $\gamma \in \tilde{T}_A(x), \eta \in \tilde{I}_A(x), \xi \in \tilde{F}_A(x)$, $\gamma^+ = \sup \tilde{T}_A(x)$, $\eta^+ = \sup \tilde{I}_A(x)$ and $\xi^+ = \sup \tilde{F}_A(x)$.

If X has only one element, then A is called a multi-valued neutrosophic number (MVNN), denoted by $A = \left\langle \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right\rangle$. For convenience, an MVNN can be denoted by $A = \left\langle \tilde{T}_A, \tilde{I}_A, \tilde{F}_A\right\rangle$. The set of all MVNNs is represented as MVNNS.

Obviously, MVNSs are generally considered as an extension of NSs. If each of $\tilde{T}_A(x), \tilde{I}_A(x)$ and $\tilde{F}_A(x)$ for any x has only one value, i.e. γ, η and ξ, and $0 \le \gamma + \eta + \xi \le 3$, then MVNSs are reduced to SNSs; if $\tilde{I}_A(x) = \varnothing$ for any x, then MVNSs are reduced to DHFSs; if $\tilde{I}_A(x) = \tilde{F}_A(x) = \varnothing$ for any x, then MVNSs are reduced to HFSs. In a word, MVNSs are the extensions of SNSs, DHFSs and HFSs.

In the following, the operations of MVNNs can be defined based on the operations of IFSs and HFSs.

Definition 5. Let $A \in MVNNS$, then the complement of a MVNN can be denoted by A^C, which can be defined as follows:

$$A^C = \left\langle \cup_{\xi \in \tilde{F}_A}\{\xi\}, \cup_{\eta \in \tilde{I}_A}\{1-\eta\}, \cup_{\gamma \in \tilde{T}_A}\{\gamma\}\right\rangle. \quad (4)$$

It is noted that different aggregation operators are all based on different t-conorms and t-norms and are used to deal with different relationships of the aggregated arguments, which satisfy the requirements of the conjunction and disjunction operators, respectively. Einstein operations include the Einstein sum $a \oplus b = (a+b)/(1+a\cdot b)$ and Einstein product $a \otimes b = (a\cdot b)/(1+(1-a)\cdot(1-b))$ $(a, b \in [0,1])$[50], which are examples of t-norms and t-conorms, respectively. In the following, the operations of MVNNs can be defined based on Einstein operations.

Definition 6. Let $A=\left\langle \tilde{T}_A,\tilde{I}_A,\tilde{F}_A\right\rangle$, $B=\left\langle \tilde{T}_B,\tilde{I}_B,\tilde{F}_B\right\rangle$ be two MVNNs and $\lambda>0$. The operations of MVNNs can be defined as follows:

(1) $$\lambda A=\left\langle \bigcup_{\gamma_A\in\tilde{T}_A}\left\{\frac{(1+\gamma_A)^\lambda-(1-\gamma_A)^\lambda}{(1+\gamma_A)^\lambda+(1-\gamma_A)^\lambda}\right\}, \bigcup_{\eta_A\in\tilde{I}_A}\left\{\frac{2\cdot(\eta_A)^\lambda}{(2-\eta_A)^\lambda+(\eta_A)^\lambda}\right\}, \bigcup_{\xi_A\in\tilde{F}_A}\left\{\frac{2\cdot(\xi_A)^\lambda}{(2-\xi_A)^\lambda+(\xi_A)^\lambda}\right\}\right\rangle;$$

(2) $$A^\lambda=\left\langle \bigcup_{\gamma_A\in\tilde{T}_A}\left\{\frac{2\cdot(\gamma_A)^\lambda}{(2-\gamma_A)^\lambda+(\gamma_A)^\lambda}\right\}, \bigcup_{\eta_A\in\tilde{I}_A}\left\{\frac{(1+\eta_A)^\lambda-(1-\eta_A)^\lambda}{(1+\eta_A)^\lambda+(1-\eta_A)^\lambda}\right\}, \bigcup_{\xi_A\in\tilde{F}_A}\left\{\frac{(1+\xi_A)^\lambda-(1-\xi_A)^\lambda}{(1+\xi_A)^\lambda+(1-\xi_A)^\lambda}\right\}\right\rangle;$$

(3) $$A\oplus B=\left\langle \bigcup_{\gamma_A\in\tilde{T}_A,\gamma_B\in\tilde{T}_B}\left\{\frac{\gamma_A+\gamma_B}{1+\gamma_A\cdot\gamma_B}\right\}, \bigcup_{\eta_A\in\tilde{I}_A,\eta_B\in\tilde{I}_B}\left\{\frac{\eta_A\cdot\eta_B}{1+(1-\eta_A)\cdot(1-\eta_B)}\right\}, \bigcup_{\xi_A\in\tilde{F}_A,\xi_B\in\tilde{F}_B}\left\{\frac{\xi_A\cdot\xi_B}{1+(1-\xi_A)\cdot(1-\xi_B)}\right\}\right\rangle;$$

(4) $$A\otimes B=\left\langle \bigcup_{\gamma_A\in\tilde{T}_A,\gamma_B\in\tilde{T}_B}\left\{\frac{\gamma_A\cdot\gamma_B}{1+(1-\gamma_A)\cdot(1-\gamma_B)}\right\}, \bigcup_{\eta_A\in\tilde{I}_A,\eta_B\in\tilde{I}_B}\left\{\frac{\eta_A+\eta_B}{1+\eta_A\cdot\eta_B}\right\}, \bigcup_{\xi_A\in\tilde{F}_A,\xi_B\in\tilde{F}_B}\left\{\frac{\xi_A+\xi_B}{1+\xi_A\cdot\xi_B}\right\}\right\rangle.$$

If there is only one specific number in $\tilde{T}_A,\tilde{I}_A$ and $\tilde{F}_A$, then the operations in Definition 6 are reduced to the operations of SNNs as follows:

(5) $$\lambda A=\left\langle \frac{(1+\gamma_A)^\lambda-(1-\gamma_A)^\lambda}{(1+\gamma_A)^\lambda+(1-\gamma_A)^\lambda}, \frac{2\cdot(\eta_A)^\lambda}{(2-\eta_A)^\lambda+(\eta_A)^\lambda}, \frac{2\cdot(\xi_A)^\lambda}{(2-\xi_A)^\lambda+(\xi_A)^\lambda}\right\rangle;$$

(6) $$A^\lambda=\left\langle \frac{2\cdot(\gamma_A)^\lambda}{(2-\gamma_A)^\lambda+(\gamma_A)^\lambda}, \frac{(1+\eta_A)^\lambda-(1-\eta_A)^\lambda}{(1+\eta_A)^\lambda+(1-\eta_A)^\lambda}, \frac{(1+\xi_A)^\lambda-(1-\xi_A)^\lambda}{(1+\xi_A)^\lambda+(1-\xi_A)^\lambda}\right\rangle;$$

(7) $$A\oplus B=\left\langle \frac{\gamma_A+\gamma_B}{1+\gamma_A\cdot\gamma_B}, \frac{\eta_A\cdot\eta_B}{1+(1-\eta_A)\cdot(1-\eta_B)}, \frac{\xi_A\cdot\xi_B}{1+(1-\xi_A)\cdot(1-\xi_B)}\right\rangle;$$

(8) $$A\otimes B=\left\langle \frac{\gamma_A\cdot\gamma_B}{1+(1-\gamma_A)\cdot(1-\gamma_B)}, \frac{\eta_A+\eta_B}{1+\eta_A\cdot\eta_B}, \frac{\xi_A+\xi_B}{1+\xi_A\cdot\xi_B}\right\rangle.$$

Note that the operations of MVNNs coincide with the operations of IFSs[6, 8, 10-21].

Example 4. Let $A=\langle\{0.6\},\{0.1,0.2\},\{0.2\}\rangle$ and $B=\langle\{0.5\},\{0.3\},\{0.2,0.3\}\rangle$ be two MVNNs, and $\lambda=2$, then the following results can be achieved.

(1) $2\cdot A=\langle\{0.8824\},\{0.1105,0.2439\},\{0.2439\}\rangle$;

(2) $A^2=\langle\{1\},\{0.1980,0.3846\},\{0.3846\}\rangle$;

(3) $A\oplus B=\langle\{0.8462\},\{0.0184,0.0385\},\{0.0244,0.0385\}\rangle$;

(4) $A\otimes B=\langle\{0.2500\},\{0.3884,0.4717\},\{0.3884,0.4717\}\rangle$.

Theorem 1. *Let $A=\left\langle \tilde{T}_A,\tilde{I}_A,\tilde{F}_A\right\rangle$, $B=\left\langle \tilde{T}_B,\tilde{I}_B,\tilde{F}_B\right\rangle$, and $C=\left\langle \tilde{T}_C,\tilde{I}_C,\tilde{F}_C\right\rangle$ be three MVNNs, then the following equations can be true.*

(1) $A_1\oplus A_2=A_2\oplus A_1$;

(2) $A_1\otimes A_2=A_2\otimes A_1$;

(3) $\lambda(A\oplus B)=\lambda A\oplus\lambda B,\lambda>0$;

(4) $(A\otimes B)^\lambda=A^\lambda\otimes B^\lambda,\lambda>0$;

(5) $\lambda_1 A\oplus\lambda_2 A=(\lambda_1+\lambda_2)A,\lambda_1>0,\lambda_2>0$;

(6) $A^{\lambda_1}\otimes A^{\lambda_2}=A^{\lambda_1+\lambda_2},\lambda_1>0,\lambda_2>0$;

(7) $(A\oplus B)\oplus C=A\oplus(B\oplus C)$;

(8) $(A\otimes B)\otimes C=A\otimes(B\otimes C)$.

Proof. (1), (2), (7) and (8) can be easily obtained.
(3) Since $\lambda>0$,

$$\lambda(A \oplus B)$$

$$= \left\langle \bigcup_{\gamma_A \in \tilde{T}_A, \gamma_B \in \tilde{T}_B} \left\{ \frac{\left(1+\frac{\gamma_A+\gamma_B}{1+\gamma_A\cdot\gamma_B}\right)^\lambda - \left(1-\frac{\gamma_A+\gamma_B}{1+\gamma_A\cdot\gamma_B}\right)^\lambda}{\left(1+\frac{\gamma_A+\gamma_B}{1+\gamma_A\cdot\gamma_B}\right)^\lambda + \left(1-\frac{\gamma_A+\gamma_B}{1+\gamma_A\cdot\gamma_B}\right)^\lambda} \right\}, \right.$$

$$\bigcup_{\eta_A \in \tilde{I}_A, \eta_B \in \tilde{I}_B} \left\{ \frac{2\left(\frac{\eta_A\cdot\eta_B}{1+(1-\eta_A)\cdot(1-\eta_B)}\right)^\lambda}{\left(2-\frac{\eta_A\cdot\eta_B}{1+(1-\eta_A)\cdot(1-\eta_B)}\right)^\lambda + \left(\frac{\eta_A\cdot\eta_B}{1+(1-\eta_A)\cdot(1-\eta_B)}\right)^\lambda} \right\},$$

$$\left. \bigcup_{\xi_A \in \tilde{F}_A, \xi_B \in \tilde{F}_B} \left\{ \frac{2\left(\frac{\xi_A\cdot\xi_B}{1+(1-\xi_A)\cdot(1-\xi_B)}\right)^\lambda}{\left(2-\frac{\xi_A\cdot\xi_B}{1+(1-\xi_A)\cdot(1-\xi_B)}\right)^\lambda + \left(\frac{\xi_A\cdot\xi_B}{1+(1-\xi_A)\cdot(1-\xi_B)}\right)^\lambda} \right\} \right\rangle.$$

$$= \left\langle \bigcup_{\gamma_A \in \tilde{T}_A, \gamma_B \in \tilde{T}_B} \left\{ \frac{(1+\gamma_A)^\lambda(1+\gamma_B)^\lambda - (1-\gamma_A)^\lambda(1-\gamma_B)^\lambda}{(1+\gamma_A)^\lambda(1+\gamma_B)^\lambda + (1-\gamma_A)^\lambda(1-\gamma_B)^\lambda} \right\}, \right.$$

$$\bigcup_{\eta_A \in \tilde{I}_A, \eta_B \in \tilde{I}_B} \left\{ \frac{2(\eta_A)^\lambda(\eta_B)^\lambda}{(2-\eta_A)^\lambda(2-\eta_B)^\lambda + (\eta_A)^\lambda(\eta_B)^\lambda} \right\},$$

$$\left. \bigcup_{\xi_A \in \tilde{F}_A, \xi_B \in \tilde{F}_B} \left\{ \frac{2(\xi_A)^\lambda(\xi_B)^\lambda}{(2-\xi_A)^\lambda(2-\xi_B)^\lambda + (\xi_A)^\lambda(\xi_B)^\lambda} \right\} \right\rangle$$

and

$$\lambda A \oplus \lambda B$$

$$= \left\langle \bigcup_{\gamma_A \in \tilde{T}_A, \gamma_B \in \tilde{T}_B} \left\{ \frac{\frac{(1+\gamma_A)^\lambda-(1-\gamma_A)^\lambda}{(1+\gamma_A)^\lambda+(1-\gamma_A)^\lambda} + \frac{(1+\gamma_B)^\lambda-(1-\gamma_B)^\lambda}{(1+\gamma_B)^\lambda+(1-\gamma_B)^\lambda}}{1+\frac{(1+\gamma_A)^\lambda-(1-\gamma_A)^\lambda}{(1+\gamma_A)^\lambda+(1-\gamma_A)^\lambda}\cdot\frac{(1+\gamma_B)^\lambda-(1-\gamma_B)^\lambda}{(1+\gamma_B)^\lambda+(1-\gamma_B)^\lambda}} \right\}, \right.$$

$$\bigcup_{\eta_A \in \tilde{I}_A, \eta_B \in \tilde{I}_B} \left\{ \frac{\frac{2\cdot\eta_A^\lambda}{(2-\eta_A)^\lambda+(\eta_A)^\lambda}\cdot\frac{2\cdot\eta_B^\lambda}{(2-\eta_B)^\lambda+(\eta_B)^\lambda}}{1+\left(1-\frac{2\cdot\eta_A^\lambda}{(2-\eta_A)^\lambda+(\eta_A)^\lambda}\right)\cdot\left(1-\frac{2\cdot\eta_B^\lambda}{(2-\eta_B)^\lambda+(\eta_B)^\lambda}\right)} \right\},$$

$$\left. \bigcup_{\xi_A \in \tilde{F}_A, \xi_B \in \tilde{F}_B} \left\{ \frac{\frac{2\cdot\xi_A^\lambda}{(2-\xi_A)^\lambda+(\xi_A)^\lambda}\cdot\frac{2\cdot\xi_B^\lambda}{(2-\xi_B)^\lambda+(\xi_B)^\lambda}}{1+\left(1-\frac{2\cdot\xi_A^\lambda}{(2-\xi_A)^\lambda+(\xi_A)^\lambda}\right)\cdot\left(1-\frac{2\cdot\xi_B^\lambda}{(2-\xi_B)^\lambda+(\xi_B)^\lambda}\right)} \right\} \right\rangle.$$

Thus, $\lambda(A \oplus B) = \lambda A \oplus \lambda B$ can be obtained.

Similarly, (4), (5) and (6) can be true. □

3.2. *Comparison method*

Based on the score function and accuracy function of IFSs[10-21], the score function and accuracy function of a MVNN can be provided below.

Definition 7. Let $A = \langle \tilde{T}_A, \tilde{I}_A, \tilde{F}_A \rangle$ be an MVNN, and then score function $s(A)$ and accuracy function $a(A)$ of an MVNN can be defined as follows:

(1) $s(A) = \frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{I}_A}\cdot l_{\tilde{F}_A}} \sum_{\gamma_A \in \tilde{T}_A, \eta_A \in \tilde{I}_A, \xi_A \in \tilde{F}_A} (\gamma_A - \eta_A - \xi_A)/3$;

(2) $a(A) = \frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{I}_A}\cdot l_{\tilde{F}_A}} \sum_{\gamma_A \in \tilde{T}_A, \eta_A \in \tilde{I}_A, \xi_A \in \tilde{F}_A} (\gamma_A + \eta_A + \xi_A)/3$.

Here $\gamma_A \in \tilde{T}_A$, $\eta_A \in \tilde{I}_A$ and $\xi_A \in \tilde{F}_A$; $l_{\tilde{T}_A}, l_{\tilde{I}_A}$ and $l_{\tilde{F}_A}$ denote the number of element in $\tilde{T}_A, \tilde{I}_A$ and $\tilde{F}_A$, respectively.

The score function is an important index in ranking MVNNs. For an MVNN *A*, the bigger the truth-membership $\tilde{T}_A$ is, the greater the MVNN will be; the smaller the indeterminacy-membership $\tilde{I}_A$ is, the greater the MVNN will be; similarly, the smaller the false-membership $\tilde{F}_A$ is, the greater the MVNN will be. For the score function, if the greater the result of $\gamma_A - \eta_A - \xi_A$ is, the more affirmative the statement will be. For the accuracy function, the bigger the sum of the truth, indeterminacy and falsity, the more affirmative the statement will be.

On the basis of Definition 7, the method for comparing MVNNs can be defined as follows.

Definition 8. Let A and B be two MVNNs. The comparision method can be defined as follows:

(1)If $s(A) > s(B)$ or $s(A) = s(B)$ and $a(A) > a(B)$, then A is superior to B, denoted by $A \succ B$;

(2)If $s(A) = s(B)$ and $a(A) = a(B)$, then A is indifferent to B, denoted by $A \sim B$.

(3)If $s(A) = s(B)$ and $a(A) < a(B)$ or $s(A) < s(B)$, then A is inferior to B, denoted by $A \prec B$;

Example 5. Let A and B be two MVNNs, and according to Definition 8, the following can be obtained:

(1) If $A=\langle\{0.6,0.5\},\{0.3\},\{0.2\}\rangle$ and $B=\langle\{0.5\},\{0.1,0.2\},\{0.4\}\rangle$ are two MVNNs, then $s(A)=0.017$ and $s(B)=-0.017$. $s(A)>s(B)$, so $A\succ B$.

(2) If $A=\langle\{0.6,0.5\},\{0.4\},\{0.2\}\rangle$ and $B=\langle\{0.5\},\{0.1,0.2\},\{0.4\}\rangle$ are two MVNNs, then $s(A)=s(B)=-0.017$, $a(A)=0.383$ and $a(B)=0.35$, $a(A)>a(B)$, so $A\succ B$.

(3) If $A=\langle\{0.6,0.7\},\{0.3\},\{0.2\}\rangle$ and $B=\langle\{0.6,0.7\}$, $\{0.2\},\{0.3\}\rangle$ are two MVNNs, then $s(A)=s(B)$ $=0.05$ and $a(A)=a(B)=0.3833$. So $A\sim B$.

(4) If $A=\langle\{0.5\},\{0.1,0.2\},\{0.1\}\rangle$ and $B=\langle\{0.6\},\{0.2\},\{0.1\}\rangle$ are two MVNNs, then $s(A)=0.0833$ and $s(B)=0.1$. $s(A)<s(B)$, so $A\prec B$.

(5) If $A=\langle\{0.5\},\{0.1,0.2\},\{0.1\}\rangle$ and $B=\langle\{0.7\},\{0.2,0.3\},\{0.2\}\rangle$ are two MVNNs, then $s(A)=s(B)$ $=0.0833$, $a(A)=0.25$ and $a(B)=0.3833$. $a(A)<a(B)$, so $A\prec B$.

Definition 9. Let $A=\langle\tilde{T}_A,\tilde{I}_A,\tilde{F}_A\rangle$ and $B=\langle\tilde{T}_B,\tilde{I}_B,\tilde{F}_B\rangle$ be two MVNNs, then the Hamming–Hausdorff distance between *A* and *B* can be defined as follows:

$$\begin{aligned}d(A,B)=\frac{1}{6}\Big(&\max_{\gamma_A\in\tilde{T}_A}\min_{\gamma_B\in\tilde{T}_B}|\gamma_A-\gamma_B|+\max_{\gamma_B\in\tilde{T}_B}\min_{\gamma_A\in\tilde{T}_A}|\gamma_B-\gamma_A|\\&+\max_{\eta_A\in\tilde{I}_A}\min_{\eta_B\in\tilde{I}_B}|\eta_A-\eta_B|+\max_{\eta_B\in\tilde{I}_B}\min_{\eta_A\in\tilde{I}_A}|\eta_B-\eta_A|\\&+\max_{\xi_A\in\tilde{F}_A}\min_{\xi_B\in\tilde{F}_B}|\xi_A-\xi_B|+\max_{\xi_B\in\tilde{F}_B}\min_{\xi_A\in\tilde{F}_A}|\xi_B-\xi_A|\Big).\end{aligned}\tag{5}$$

Example 6. Let $A=\langle\{0.4,0.5\},\{0.2\},\{0.3\}\rangle$ and $B=\langle\{0.8\},\{0.8\},\{0.5\}\rangle$ be two MVNNs, then according to Eq. (5), $d(A,B)=0.25$ can be determined.

4. Power Operators and MCGDM Approach

In this section, the power aggregation operators of MVNNs are presented and an approach for MCGDM problems that utilizes these aggregation operators is proposed.

4.1. *Power aggregation operator*

The power average (PA) operator was developed by Yager in the form of nonlinear weighted average aggregation operator[51].

Definition 10. The PA operator is the mapping PA: $R^n\to R$, which is defined as follows[51]:

$$PA(\alpha_1,\alpha_2,\ldots,\alpha_n)=\frac{\sum_{i=1}^n(1+S(\alpha_i))\alpha_i}{\sum_{i=1}^n(1+S(\alpha_i))}.\tag{6}$$

Here $S(\alpha_i)=\sum_{i=1,j\neq i}^n Supp(\alpha_i,\alpha_j)$, and $Supp(\alpha_i,\alpha_j)$ is the support for α_i from α_j . Then the following properties are true.

(1) $Supp(\alpha_i,\alpha_j)\in[0,1]$;

(2) $Supp(\alpha_i,\alpha_j)=Supp(\alpha_j,\alpha_i)$;

(3) $Supp(\alpha_i,\alpha_j)\geq Supp(\alpha_p,\alpha_q)$ *iff* $|\alpha_i-\alpha_j|<|\alpha_p-\alpha_q|$.

Apparently, the closer two values get, the more they support each other.

4.2. *Power weighted average operator*

Definition 11. Let $A_j=\langle\tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\rangle$ $(j=1,2,\ldots,n)$ be a collection of MVNNs, and $w=(w_1,w_2,\ldots,w_n)$ be the weight vector of A_j $(j=1,2,\ldots,n)$, with $w_j\geq 0$ $(j=1,2,\ldots,n)$ and $\sum_{j=1}^n w_j=1$. The multi-valued neutrosophic power weighted average (MVNPWA) operator of dimension n is the mapping $\mathrm{MVNPWA}:\mathrm{MVNN}^n\to\mathrm{MVNN}$, and

$$\mathrm{MVNPWA}_w(A_1,A_2,\ldots,A_n)=\frac{\bigoplus_{j=1}^n w_j\left(1+S(A_j)\right)A_j}{\sum_{j=1}^n w_j\left(1+S(A_j)\right)}.\tag{7}$$

Here $S(A_j)=\sum_{i=1,j\neq i}^n w_j Supp(A_j,A_i)$ and $Supp(A_j,A_i)$ is the support for A_j from A_i , which satisfies the following conditions:

(1) $Supp(A_i,A_j)\in[0,1]$;

(2) $Supp(A_i,A_j)=Supp(A_j,A_i)$;

(3) $Supp(A_i,A_j)\geq Supp(A_p,A_q)$ iff $d(A_i,A_j)<d(A_p,A_q)$, where *d* is the distance measure as was defined in Definition 9.

Based on the operations in Definition 6 and Eq. (7), Theorem 2 can be derived.

Theorem 2. *Let* $A_j=\left\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\right\rangle$ $(j=1,2,\ldots,n)$ *be a collection of MVNNs, and* $w=(w_1,w_2,\ldots,w_n)$ *be the weight vector of* A_j $(j=1,2,\ldots,n)$ *, with* $w_j\geq 0$ $(j=1,2,\ldots,n)$ *and* $\sum_{j=1}^{n}w_j=1$ *. Then their aggregated result using the MVNPWA operator is also an MVNN, and*

$$\mathrm{MVNPWA}_w\left(A_1,A_2,\ldots,A_n\right)$$
$$=\left\langle \bigcup_{\gamma_j\in\tilde{T}_{A_j}}\left\{\frac{\prod_{j=1}^{n}\left(1+\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}-\prod_{j=1}^{n}\left(1-\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}{\prod_{j=1}^{n}\left(1+\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}+\prod_{j=1}^{n}\left(1-\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}\right\},\right.$$
$$\bigcup_{\eta_j\in\tilde{I}_{A_j}}\left\{\frac{2\prod_{j=1}^{n}\left(\eta_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}{\prod_{j=1}^{n}\left(2-\eta_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}+\prod_{j=1}^{n}\left(\eta_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}\right\}, \qquad (8)$$
$$\left.\bigcup_{\xi_j\in\tilde{F}_{A_j}}\left\{\frac{2\prod_{j=1}^{n}\left(\xi_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}{\prod_{j=1}^{n}\left(2-\xi_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}+\prod_{i=1}^{n}\left(\xi_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}\right\}\right\rangle.$$

Here $S\left(A_j\right)=\sum_{i=1,j\neq i}^{n}w_j Supp\left(A_j,A_i\right)$ and satisfies the conditions in Definition 11.

Proof. For simplicity, let

$$\varsigma_j=\frac{w_j\left(1+S\left(A_j\right)\right)}{\sum_{j=1}^{n}w_j\left(1+S\left(A_j\right)\right)}$$

in the process of proof. By using the mathematical induction on n.

(1) If $n=2$, based on the operations (1) and (3) in Definition 6,

$$\varsigma_1 A_1\oplus\varsigma_2 A_2$$
$$=\left\langle \bigcup_{\gamma_1\in\tilde{T}_{A_1},\gamma_2\in\tilde{T}_{A_2}}\left\{\frac{\frac{(1+\gamma_1)^{\varsigma_1}-(1-\gamma_1)^{\varsigma_1}}{(1+\gamma_1)^{\varsigma_1}+(1-\gamma_1)^{\varsigma_1}}+\frac{(1+\gamma_2)^{\varsigma_2}-(1-\gamma_2)^{\varsigma_2}}{(1+\gamma_2)^{\varsigma_2}+(1-\gamma_2)^{\varsigma_2}}}{1+\frac{(1+\gamma_1)^{\varsigma_1}-(1-\gamma_1)^{\varsigma_1}}{(1+\gamma_1)^{\varsigma_1}+(1-\gamma_1)^{\varsigma_1}}\cdot\frac{(1+\gamma_2)^{\varsigma_2}-(1-\gamma_2)^{\varsigma_2}}{(1+\gamma_2)^{\varsigma_2}+(1-\gamma_2)^{\varsigma_2}}}\right\},\right.$$
$$\bigcup_{\eta_1\in\tilde{I}_{A_1},\eta_2\in\tilde{I}_{A_2}}\left\{\frac{\frac{2\cdot\eta_1^{\varsigma_1}}{(2-\eta_1)^{\lambda}+(\eta_1)^{\varsigma_1}}\cdot\frac{2\cdot\eta_2^{\varsigma_2}}{(2-\eta_2)^{\varsigma_2}+(\eta_2)^{\varsigma_2}}}{1+\left(1-\frac{2\cdot\eta_1^{\varsigma_1}}{(2-\eta_1)^{\varsigma_1}+(\eta_1)^{\varsigma_1}}\right)\cdot\left(1-\frac{2\cdot\eta_2^{\varsigma_2}}{(2-\eta_2)^{\varsigma_2}+(\eta_2)^{\varsigma_2}}\right)}\right\},$$
$$\left.\bigcup_{\xi_1\in\tilde{F}_{A},\xi_2\in\tilde{F}_{A_2}}\left\{\frac{\frac{2\cdot\xi_1^{\varsigma_1}}{(2-\xi_1)^{\varsigma_1}+(\xi_1)^{\varsigma_1}}\cdot\frac{2\cdot\xi_2^{\varsigma_2}}{(2-\xi_2)^{\varsigma_2}+(\xi_2)^{\varsigma_2}}}{1+\left(1-\frac{2\cdot\xi_1^{\varsigma_1}}{(2-\xi_1)^{\varsigma_1}+(\xi_1)^{\varsigma_1}}\right)\cdot\left(1-\frac{2\cdot\xi_2^{\varsigma_2}}{(2-\xi_2)^{\varsigma_2}+(\xi_2)^{\varsigma_2}}\right)}\right\}\right\rangle$$

$$=\left\langle \bigcup_{\gamma_1\in\tilde{T}_{A_1},\gamma_2\in\tilde{T}_{A_2}}\left\{\frac{(1+\gamma_1)^{\varsigma_1}(1+\gamma_2)^{\varsigma_2}-(1-\gamma_1)^{\varsigma_1}(1-\gamma_2)^{\varsigma_2}}{(1+\gamma_1)^{\varsigma_1}(1+\gamma_2)^{\varsigma_2}+(1-\gamma_1)^{\varsigma_1}(1-\gamma_2)^{\varsigma_2}}\right\},\right.$$
$$\bigcup_{\eta_1\in\tilde{I}_{A_1},\eta_2\in\tilde{I}_{A_2}}\left\{\frac{2(\eta_1)^{\varsigma_1}(\eta_2)^{\varsigma_2}}{(2-\eta_1)^{\varsigma_1}\left(2-\eta_{A_2}\right)^{\varsigma_2}+(\eta_1)^{\varsigma_1}(\eta_2)^{\varsigma_2}}\right\},$$
$$\left.\bigcup_{\xi_1\in\tilde{F}_{A_1},\xi_2\in\tilde{F}_{A_2}}\left\{\frac{2(\xi_1)^{\varsigma_1}(\xi_2)^{\varsigma_2}}{(2-\xi_1)^{\varsigma_1}(2-\xi_2)^{\varsigma_2}+(\xi_1)^{\varsigma_1}(\xi_2)^{\varsigma_2}}\right\}\right\rangle.$$

So

$$\mathrm{MVNPWA}_w\left(A_1,A_2\right)$$
$$=\left\langle \bigcup_{\gamma_1\in\tilde{T}_{A_1},\gamma_2\in\tilde{T}_{A_2}}\left\{\frac{(1+\gamma_1)^{\varsigma_1}(1+\gamma_2)^{\varsigma_2}-(1-\gamma_1)^{\varsigma_1}(1-\gamma_2)^{\varsigma_2}}{(1+\gamma_1)^{\varsigma_1}(1+\gamma_2)^{\varsigma_2}+(1-\gamma_1)^{\varsigma_1}(1-\gamma_2)^{\varsigma_2}}\right\},\right.$$
$$\bigcup_{\eta_1\in\tilde{I}_{A_1},\eta_2\in\tilde{I}_{A_2}}\left\{\frac{2(\eta_1)^{\varsigma_1}(\eta_2)^{\varsigma_2}}{(2-\eta_1)^{\varsigma_1}\left(2-\eta_{A_2}\right)^{\varsigma_2}+(\eta_1)^{\varsigma_1}(\eta_2)^{\varsigma_2}}\right\},$$
$$\left.\bigcup_{\xi_1\in\tilde{F}_{A_1},\xi_2\in\tilde{F}_{A_2}}\left\{\frac{2(\xi_1)^{\varsigma_1}(\xi_2)^{\varsigma_2}}{(2-\xi_1)^{\varsigma_1}(2-\xi_2)^{\varsigma_2}+(\xi_1)^{\varsigma_1}(\xi_2)^{\varsigma_2}}\right\}\right\rangle.$$

(2) If Eq. (8) holds for $n=k$, then

$$\mathrm{MVNPWA}_w\left(A_1, A_2, \dots, A_k\right) = \left\langle \bigcup_{\gamma_j \in \tilde{T}_{A_j}} \left\{ \frac{\prod_{j=1}^{k}\left(1+\gamma_j\right)^{\varsigma_j} - \prod_{j=1}^{k}\left(1-\gamma_j\right)^{\varsigma_j}}{\prod_{j=1}^{k}\left(1+\gamma_j\right)^{\varsigma_j} + \prod_{j=1}^{k}\left(1-\gamma_j\right)^{\varsigma_j}} \right\}, \bigcup_{\eta_j \in \tilde{I}_{A_j}} \left\{ \frac{2\prod_{j=1}^{k}\left(\eta_j\right)^{\varsigma_j}}{\prod_{j=1}^{k}\left(2-\eta_j\right)^{\varsigma_j} + \prod_{j=1}^{k}\left(\eta_j\right)^{\varsigma_j}} \right\}, \bigcup_{\xi_j \in \tilde{F}_{A_j}} \left\{ \frac{2\prod_{j=1}^{k}\left(\xi_j\right)^{\varsigma_j}}{\prod_{j=1}^{k}\left(2-\xi_j\right)^{\varsigma_j} + \prod_{i=1}^{k}\left(\xi_j\right)^{\varsigma_j}} \right\} \right\rangle.$$

If $n = k+1$, by the operations (1) and (3) in Definition 6,

$$\mathrm{MVNPWA}_w\left(A_1, A_2, \dots, A_k, A_{k+1}\right) = \left\langle \bigcup_{\gamma_j \in \tilde{T}_{A_j}} \left\{ \frac{\frac{\prod_{j=1}^{k}(1+\gamma_j)^{\varsigma_j} - \prod_{j=1}^{k}(1-\gamma_j)^{\varsigma_j}}{\prod_{j=1}^{k}(1+\gamma_j)^{\varsigma_j} + \prod_{j=1}^{k}(1-\gamma_j)^{\varsigma_j}} + \frac{(1+\gamma_{k+1})^{\varsigma_{k+1}} - (1-\gamma_{k+1})^{\varsigma_{k+1}}}{(1+\gamma_{k+1})^{\varsigma_{k+1}} + (1-\gamma_{k+1})^{\varsigma_{k+1}}}}{1+\frac{\prod_{j=1}^{k}(1+\gamma_j)^{\varsigma_j} - \prod_{j=1}^{k}(1-\gamma_j)^{\varsigma_j}}{\prod_{j=1}^{k}(1+\gamma_j)^{\varsigma_j} + \prod_{j=1}^{k}(1-\gamma_j)^{\varsigma_j}} \cdot \frac{(1+\gamma_{k+1})^{\varsigma_{k+1}} - (1-\gamma_{k+1})^{\varsigma_{k+1}}}{(1+\gamma_{k+1})^{\varsigma_{k+1}} + (1-\gamma_{k+1})^{\varsigma_{k+1}}}} \right\},$$

$$\bigcup_{\eta_j \in \tilde{I}_{A_j}} \left\{ \frac{\frac{2\prod_{j=1}^{k}(\eta_j)^{\varsigma_j}}{\prod_{j=1}^{k}(2-\eta_j)^{\varsigma_j} + \prod_{j=1}^{k}(\eta_j)^{\varsigma_j}} \cdot \frac{2(\eta_{k+1})^{\varsigma_{k+1}}}{(2-\eta_{k+1})^{\varsigma_{k+1}} + (\eta_j)^{\varsigma_{k+1}}}}{1+\left(1-\frac{2\prod_{j=1}^{k}(\eta_j)^{\varsigma_j}}{\prod_{j=1}^{k}(2-\eta_j)^{\varsigma_j} + \prod_{j=1}^{k}(\eta_j)^{\varsigma_j}}\right)\cdot\left(1-\frac{2(\eta_{k+1})^{\varsigma_{k+1}}}{(2-\eta_{k+1})^{\varsigma_{k+1}} + (\eta_j)^{\varsigma_{k+1}}}\right)} \right\},$$

$$\bigcup_{\xi_j \in \tilde{F}_{A_j}} \left\{ \frac{\frac{2\prod_{j=1}^{k}(\xi_j)^{\varsigma_j}}{\prod_{j=1}^{k}(2-\xi_j)^{\varsigma_j} + \prod_{j=1}^{k}(\xi_j)^{\varsigma_j}} \cdot \frac{2(\xi_{k+1})^{\varsigma_{k+1}}}{(2-\xi_{k+1})^{\varsigma_{k+1}} + (\xi_j)^{\varsigma_{k+1}}}}{1+\left(1-\frac{2\prod_{j=1}^{k}(\xi_j)^{\varsigma_j}}{\prod_{j=1}^{k}(2-\xi_j)^{\varsigma_j} + \prod_{j=1}^{k}(\xi_j)^{\varsigma_j}}\right)\cdot\left(1-\frac{2(\xi_{k+1})^{\varsigma_{k+1}}}{(2-\xi_{k+1})^{\varsigma_{k+1}} + (\xi_j)^{\varsigma_{k+1}}}\right)} \right\} \right\rangle.$$

$$= \left\langle \bigcup_{\gamma_j \in \tilde{T}_{A_j}} \left\{ \frac{\prod_{j=1}^{k}(1+\gamma_j)^{\varsigma_j}\cdot(1+\gamma_{k+1})^{\varsigma_{k+1}} - \prod_{j=1}^{k}(1-\gamma_j)^{\varsigma_j}\cdot(1-\gamma_{k+1})^{\varsigma_{k+1}}}{\prod_{j=1}^{k}(1+\gamma_j)^{\varsigma_j}\cdot(1+\gamma_{k+1})^{\varsigma_{k+1}} + \prod_{j=1}^{k}(1-\gamma_j)^{\varsigma_j}\cdot(1-\gamma_{k+1})^{\varsigma_{k+1}}} \right\}, \right.$$

$$\bigcup_{\eta_j \in \tilde{I}_{A_j}} \left\{ \frac{4\prod_{j=1}^{k}(\eta_j)^{\varsigma_j}\cdot(\eta_{k+1})^{\varsigma_{k+1}}}{2\prod_{j=1}^{k}(2-\eta_j)^{\varsigma_j}\cdot(2-\eta_{k+1})^{\varsigma_{k+1}} + 2\prod_{i=1}^{k}(\eta_j)^{\varsigma_j}\cdot(\eta_{k+1})^{\varsigma_{k+1}}} \right\},$$

$$\left. \bigcup_{\xi_j \in \tilde{F}_{A_j}} \left\{ \frac{4\prod_{j=1}^{k}(\xi_j)^{\varsigma_j}\cdot(\xi_{k+1})^{\varsigma_{k+1}}}{2\prod_{j=1}^{k}(2-\xi_j)^{\varsigma_j}\cdot(2-\xi_{k+1})^{\varsigma_{k+1}} + 2\prod_{i=1}^{k}(\xi_j)^{\varsigma_j}\cdot(\xi_{k+1})^{\varsigma_{k+1}}} \right\} \right\rangle$$

$$= \left\langle \bigcup_{\gamma_j \in \tilde{T}_{A_j}} \left\{ \frac{\prod_{j=1}^{k+1}\left(1+\gamma_j\right)^{\varsigma_j} - \prod_{j=1}^{k+1}\left(1-\gamma_j\right)^{\varsigma_j}}{\prod_{j=1}^{k+1}\left(1+\gamma_j\right)^{\varsigma_j} + \prod_{j=1}^{k+1}\left(1-\gamma_j\right)^{\varsigma_j}} \right\}, \bigcup_{\eta_j \in \tilde{I}_{A_j}} \left\{ \frac{2\prod_{j=1}^{k+1}\left(\eta_j\right)^{\varsigma_j}}{\prod_{j=1}^{k+1}\left(2-\eta_j\right)^{\varsigma_j} + \prod_{j=1}^{k+1}\left(\eta_j\right)^{\varsigma_j}} \right\}, \bigcup_{\xi_j \in \tilde{F}_{A_j}} \left\{ \frac{2\prod_{j=1}^{k+1}\left(\xi_j\right)^{\varsigma_j}}{\prod_{j=1}^{k+1}\left(2-\xi_j\right)^{\varsigma_j} + \prod_{i=1}^{k+1}\left(\xi_j\right)^{\varsigma_j}} \right\} \right\rangle.$$

i.e., Eq. (8) holds for $n = k+1$. Thus, Eq. (8) holds for all n, then

$$\mathrm{MVNPWA}_w\left(A_1, A_2, \dots, A_n\right) = \left\langle \bigcup_{\gamma_j \in \tilde{T}_{A_j}} \left\{ \frac{\prod_{j=1}^{n}\left(1+\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}} - \prod_{j=1}^{n}\left(1-\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}}}{\prod_{j=1}^{n}\left(1+\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}} + \prod_{j=1}^{n}\left(1-\gamma_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}}} \right\}, \right.$$

$$\bigcup_{\eta_j \in \tilde{I}_{A_j}} \left\{ \frac{2\prod_{j=1}^{n}\left(\eta_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}}}{\prod_{j=1}^{n}\left(2-\eta_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}} + \prod_{j=1}^{n}\left(\eta_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}}} \right\},$$

$$\left. \bigcup_{\xi_j \in \tilde{F}_{A_j}} \left\{ \frac{2\prod_{j=1}^{n}\left(\xi_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}}}{\prod_{j=1}^{n}\left(2-\xi_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}} + \prod_{i=1}^{n}\left(\xi_j\right)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n} w_j(1+S(A_j))}}} \right\} \right\rangle.$$

The MVNPWA operator has the following properties.

Theorem 3. *Let* $A_j=\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\rangle$ $(j=1,2,\ldots,n)$ *be a collection of MVNNs. If* A_j' $(j=1,2,\ldots,n)$ *is any permutation of* A_j $(j=1,2,\ldots,n)$, *then*
$\mathrm{MVNPWA}_w(A_1,A_2,\ldots,A_n)=\mathrm{MVNPWA}_w(A_1',A_2',\ldots,A_n')$.

Proof. The process of proof is omitted here. □

Theorem 4. *Let* $A_j=\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\rangle$ $(j=1,2,\ldots,n)$ *be a collection of MVNNs and* $A=\langle \tilde{T}_A,\tilde{I}_A,\tilde{F}_A\rangle$ *be an MVNN. If for all* j, $\gamma_j=\gamma,\eta_j=\eta$ *and* $\xi_j=\xi$, *then*

$$\mathrm{MVNPWA}_w(A_1,A_2,\ldots,A_n)=A.$$

Where γ_j,η_j *and* ξ_j *are elements of* $\tilde{T}_{A_j},\tilde{I}_{A_j}$ *and* $\tilde{F}_{A_j}$ *respectively,* γ,η *and* ξ *are elements of* $\tilde{T}_A,\tilde{I}_A$ *and* $\tilde{F}_A$ *respectively.*

Proof. The process of proof is omitted here. □

Theorem 5. *Let* $A_j=\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\rangle$ $(j=1,2,\ldots,n)$ *and* $A_j^*=\langle \tilde{T}_{A_j^*},\tilde{I}_{A_j^*},\tilde{F}_{A_j^*}\rangle$ $(j=1,2,\ldots,n)$ *be two collections of MVNNs. If for all* j, $\gamma_j\le\gamma_j^*,\eta_j\ge\eta_j^*$ *and* $\xi_j\ge\xi_j^*$, *then*
$\mathrm{MVNPWA}_w(A_1,A_2,\ldots,A_n)\le\mathrm{MVNPWA}_w(A_1^*,A_2^*,\ldots,A_n^*)$.

Where γ_j,η_j and ξ_j are elements of $\tilde{T}_{A_j},\tilde{I}_{A_j}$ and $\tilde{F}_{A_j}$ respectively, γ_j^*,η_j^* and ξ_j^* are elements of $\tilde{T}_{A_j^*},\tilde{I}_{A_j^*}$ and $\tilde{F}_{A_j^*}$ respectively.

Proof. The process of proof is omitted here. □

4.3. *Power weighted geometric operator*

Definition 12. Let $A_j=\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\rangle$ $(j=1,2,\ldots,n)$ be a collection of MVNNs, and $w=(w_1,w_2,\ldots,w_n)$ be the weight vector of A_j $(j=1,2,\ldots,n)$, with $w_j\ge 0$ $(j=1,2,\ldots,n)$ and $\sum_{j=1}^{n}w_j=1$. The multi-valued neutrosophic power weighted geometric (MVNPWG) operator of dimension n is the mapping MVNPWG: $\mathrm{MVNN}^n\to\mathrm{MVNN}$, and

$$\mathrm{MVNPWG}_w(A_1,A_2,\ldots,A_n)=\bigotimes_{j=1}^{n}(A_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}. \quad (9)$$

Here $S(A_j)=\sum_{i=1,j\ne i}^{n}w_j Supp(A_j,A_i)$ and $Supp(A_j,A_i)$ is the support for A_j from A_i, which satisfies the following conditions:

(1) $Supp(A_i,A_j)\in[0,1]$;

(2) $Supp(A_i,A_j)=Supp(A_j,A_i)$;

(3) $Supp(A_i,A_j)\ge Supp(A_p,A_q)$ *iff* $d(A_i,A_j)<d(A_p,A_q)$, where *d* is the distance measure defined in Definition 9.

Based on the operations in Definition 6 and Eq. (9), Theorem 3 can be derived.

Theorem 6. *Let* $A_j=\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\rangle$ $(j=1,2,\ldots,n)$ *be a collection of MVNNs, and* $w=(w_1,w_2,\ldots,w_n)$ *be the weight vector of* A_j $(j=1,2,\ldots,n)$, *with* $w_j\ge 0$ $(j=1,2,\ldots,n)$ *and* $\sum_{j=1}^{n}w_j=1$. *Then their aggregated result using the MVNPWG operator is also an MVNN, and*

$$\mathrm{MVNPWG}_w(A_1,A_2,\ldots,A_n)$$
$$=\left\langle \bigcup_{\gamma_j\in\tilde{T}_{A_j}}\left\{\frac{2\prod_{j=1}^{n}(\gamma_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}{\prod_{j=1}^{n}(2-\gamma_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}+\prod_{j=1}^{n}(\gamma_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}\right\},\right.$$
$$\bigcup_{\eta_j\in\tilde{I}_{A_j}}\left\{\frac{\prod_{j=1}^{n}(1+\eta_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}-\prod_{j=1}^{n}(1-\eta_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}{\prod_{j=1}^{n}(1+\eta_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}+\prod_{j=1}^{n}(1-\eta_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}\right\},$$
$$\left.\bigcup_{\xi_j\in\tilde{F}_{A_j}}\left\{\frac{\prod_{j=1}^{n}(1+\xi_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}-\prod_{j=1}^{n}(1-\xi_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}{\prod_{j=1}^{n}(1+\xi_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}+\prod_{j=1}^{n}(1-\xi_j)^{\frac{w_j(1+S(A_j))}{\sum_{j=1}^{n}w_j(1+S(A_j))}}}\right\}\right\rangle. \quad (10)$$

Here $S(A_j)=\sum_{i=1,j\ne i}^{n}w_j Supp(A_j,A_i)$ and satisfies the conditions in Definition 11.

Proof. Theorem 2 can be proved by the mathematical induction and the process is omitted here. □

Similarly, the MVNPWG operator has the following properties.

Theorem 7. *Let* $A_j=\left\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\right\rangle$ $(j=1,2,\ldots,n)$ *be a collection of MVNNs. If* A_j' $(j=1,2,\ldots,n)$ *is any permutation of* A_j $(j=1,2,\ldots,n)$, *then*

$$\mathrm{MVNPWG}_w(A_1,A_2,\ldots,A_n)=\mathrm{MVNPWG}_w\left(A_1',A_2',\ldots,A_n'\right).$$

Proof. The process of proof is omitted here. □

Theorem 8. *Let* $A_j=\left\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\right\rangle$ $(j=1,2,\ldots,n)$ *be a collection of MVNNs and* $A=\left\langle \tilde{T}_A,\tilde{I}_A,\tilde{F}_A\right\rangle$ *be an MVNN. If for all* j, $\gamma_j=\gamma,\eta_j=\eta$ *and* $\xi_j=\xi$, *then*

$$\mathrm{MVNPWG}_w(A_1,A_2,\ldots,A_n)=A.$$

Where γ_j,η_j *and* ξ_j *are elements of* $\tilde{T}_{A_j},\tilde{I}_{A_j}$ *and* $\tilde{F}_{A_j}$ *respectively,* γ,η *and* ξ *are elements of* $\tilde{T}_A,\tilde{I}_A$ *and* $\tilde{F}_A$ *respectively.*

Proof. The process of proof is omitted here. □

Theorem 9. *Let* $A_j=\left\langle \tilde{T}_{A_j},\tilde{I}_{A_j},\tilde{F}_{A_j}\right\rangle$ $(j=1,2,\ldots,n)$ *and* $A_j^*=\left\langle \tilde{T}_{A_j^*},\tilde{I}_{A_j^*},\tilde{F}_{A_j^*}\right\rangle$ $(j=1,2,\ldots,n)$ *be two collections of MVNNs. If for all* j, $\gamma_j\le\gamma_j^*,\eta_j\ge\eta_j^*$ *and* $\xi_j\ge\xi_j^*$, *then*

$$\mathrm{MVNPWG}_w(A_1,A_2,\ldots,A_n)\le\mathrm{MVNPWG}_w\left(A_1^*,A_2^*,\ldots,A_n^*\right).$$

Where γ_j,η_j and ξ_j are elements of $\tilde{T}_{A_j},\tilde{I}_{A_j}$ and $\tilde{F}_{A_j}$ respectively, γ_j^*,η_j^* and ξ_j^* are elements of $\tilde{T}_{A_j^*},\tilde{I}_{A_j^*}$ and $\tilde{F}_{A_j^*}$ respectively.

Proof. The process of proof is omitted here. □

4.4. *MCGDM approach*

Assume there are n alternatives denoted by $A=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ and m criteria denoted by $C=\{c_1,c_2,\ldots,c_m\}$, and the weight vector of criteria is $w=(w_1,w_2,\ldots,w_m)$, where $w_j\ge 0$ $(j=1,2,\ldots,m)$ and $\sum_{j=1}^{m}w_j=1$. Suppose that there are l decision-makers $D=\{d_1,d_2,\ldots,d_l\}$ whose corresponding weight vector is $\omega=(\omega_1,\omega_2,\ldots,\omega_m)$, where $\omega_j\ge 0$ ($j=1,2,\ldots,m$) and $\sum_{k=1}^{l}\omega_k=1$. Let $R^k=\left(\alpha_{ij}^k\right)_{n\times m}$ be the multi-valued neutrosophic decision matrix, and $\alpha_{ij}^k=\langle\tilde{T}_{\alpha_{ij}^k},\tilde{I}_{\alpha_{ij}^k},\tilde{F}_{\alpha_{ij}^k}\rangle$ be the evaluation value of α_i for criterion c_j being in the form of MVNNs provided by the decision-maker $d_k\in D$, where $\tilde{T}_{\alpha_{ij}^k}$ indicates the truth-membership function, $\tilde{I}_{\alpha_{ij}^k}$ indicates the indeterminacy-membership function and $\tilde{F}_{\alpha_{ij}^k}$ indicates the falsity-membership function. This approach is an integration of MVNNs and the aggregation operators, and can be used to solve MCDM problems mentioned above.

In general, there are maximizing criteria and minimizing criteria in MCDM problems. According to the IFSs method proposed by Xu[12], the minimizing criteria can be transformed into maximizing criteria as follows:

$$\beta_{ij}^k=\begin{cases}\alpha_{ij}^k, & \text{for maximizing criteria } c_j\\ \left(\alpha_{ij}^k\right)^c, & \text{for minimizing criteria } c_j\end{cases},\ (i=1,2,\ldots,n;j=1,2,\ldots,m). \quad (11)$$

Here $\left(\alpha_{ij}^k\right)^c$ is the complement of α_{ij}^k as defined in Definition 5.

In the following, a procedure to rank and select the most desirable alternative(s) is given.

Step 1. Transform the decision matrix.

According to Eq. (11), the MVNN decision matrix $R^k=\left(\alpha_{ij}^k\right)_{n\times m}$ can be transformed into a normalized MVNN decision matrix $\tilde{R}^k=\left(\beta_{ij}^k\right)_{n\times m}$.

In order to unify all criteria, we need to transform the minimizing criteria into maximizing criteria (Remark: if all the criteria belong to the maximizing criteria and have the same measurement unit, then there is no need to normalize them). Suppose that the matrix $R^k=\left(\alpha_{ij}^k\right)_{n\times m}$, where α_{ij}^k are MVNNs, is normalized into the corresponding matrix $\tilde{R}^k=\left(\beta_{ij}^k\right)_{n\times m}$.

For the minimizing criteria, the normalization formula is

$$\beta_{ij}^k=\left(\alpha_{ij}^k\right)^c=\left\langle \cup_{\xi\in\tilde{F}_{\alpha_{ij}^k}}\{\xi\},\cup_{\eta\in\tilde{I}_{\alpha_{ij}^k}}\{1-\eta\},\cup_{\gamma\in\tilde{T}_{\alpha_{ij}^k}}\{\gamma\}\right\rangle;(12)$$

for the maximizing criteria,

$$\beta_{ij}^k=\alpha_{ij}^k=\left\langle \cup_{\gamma\in\tilde{T}_{\alpha_{ij}^k}}\{\gamma\},\cup_{\eta\in\tilde{I}_{\alpha_{ij}^k}}\{\eta\},\cup_{\xi\in\tilde{F}_{\alpha_{ij}^k}}\{\xi\}\right\rangle. \quad (13)$$

Step 2. Calculate the supports $Supp\left(\beta_{ij}^k,\beta_{ij}^t\right)$.

The supports can be obtained by the following formula:

$$Supp\left(\beta_{ij}^k,\beta_{ij}^t\right)=1-d\left(\beta_{ij}^k,\beta_{ij}^t\right),$$
$$i=1,2,\ldots,n;\ j=1,2,\ldots,m;\ k,t=1,2,\ldots,l,k\neq t. \quad (14)$$

Here $Supp\left(\beta_{ij}^k,\beta_{ij}^t\right)$ is the support for β_{ij}^k from β_{ij}^t, and satisfies the three conditions given in Definition 11. $d\left(\beta_{ij}^k,\beta_{ij}^t\right)$ is the Hamming-Hausdorff distance between β_{ij}^k and β_{ij}^t as defined in Definition 9.

Step 3. Calculate the weights τ_{ij}^k associated with the MVNN β_{ij}^k.

The weighted support $S\left(\beta_{ij}^k\right)$ of the MVNN β_{ij}^k by the other MVNNs $\beta_{ij}^t\left(t=1,2,\ldots,l \text{ and } t\neq k\right)$ can be calculated using the weights $\omega_k\left(k=1,2,\ldots,l\right)$ of the decision-makers $d_k\left(k=1,2,\ldots,l\right)$.

$$S\left(\beta_{ij}^k\right)=\sum_{t=1,t\neq k}^{l}\omega_t Supp\left(\beta_{ij}^k,\beta_{ij}^t\right)\ \left(k=1,2,\ldots,l\right). \quad (15)$$

Then, the weights $\tau_{ij}^k\left(k=1,2,\ldots,l\right)$ associated with the MVNN $\beta_{ij}^k\left(k=1,2,\ldots,l\right)$ can be obtained:

$$\tau_{ij}^k=\frac{\omega_k\left(1+S\left(\beta_{ij}^k\right)\right)}{\sum_{k=1}^{l}\omega_k\left(1+S\left(\beta_{ij}^k\right)\right)},k=1,2,\ldots,l. \quad (16)$$

Here $\tau_{ij}^k\geq 0\left(k=1,2,\ldots,l\right)$ and $\sum_{k=1}^{l}\tau_{ij}^k=1$.

Step 4. Aggregate the evaluation information of each expert.

Utilize the MVNPWA operator or MVNPWG operators, Eq. (8) or Eq. (10), to aggregate the MVNNs β_{ij}^k for all decision-makers:

$$\beta_{ij}=\mathrm{MVNPWA}_{\omega}\left(\beta_{ij}^1,\beta_{ij}^2,\ldots,\beta_{ij}^l\right)$$
$$=\left\langle \bigcup_{\gamma_{ij}^k\in\tilde{T}_{\beta_{ij}^k}}\left\{\frac{\prod_{k=1}^{l}\left(1+\gamma_{ij}^k\right)^{\tau_{ij}^k}-\prod_{k=1}^{l}\left(1-\gamma_{ij}^k\right)^{\tau_{ij}^k}}{\prod_{k=1}^{l}\left(1+\gamma_{ij}^k\right)^{\tau_{ij}^k}+\prod_{k=1}^{l}\left(1-\gamma_{ij}^k\right)^{\tau_{ij}^k}}\right\},\right.$$
$$\bigcup_{\eta_{ij}^k\in\tilde{I}_{\beta_{ij}^k}}\left\{\frac{2\prod_{k=1}^{l}\left(\eta_{ij}^k\right)^{\tau_{ij}^k}}{\prod_{k=1}^{l}\left(2-\eta_{ij}^k\right)^{\tau_{ij}^k}+\prod_{k=1}^{l}\left(\eta_{ij}^k\right)^{\tau_{ij}^k}}\right\},$$
$$\left.\bigcup_{\xi_{ij}^k\in\tilde{F}_{\beta_{ij}^k}}\left\{\frac{2\prod_{k=1}^{l}\left(\xi_{ij}^k\right)^{\tau_{ij}^k}}{\prod_{k=1}^{l}\left(2-\xi_{ij}^k\right)^{\tau_{ij}^k}+\prod_{k=1}^{l}\left(\xi_{ij}^k\right)^{\tau_{ij}^k}}\right\}\right\rangle. \quad (17)$$

Or

$$\beta_{ij}=\mathrm{MVNPWG}_{\omega}\left(\beta_{ij}^1,\beta_{ij}^2,\ldots,\beta_{ij}^l\right)$$
$$=\left\langle \bigcup_{\gamma_{ij}^k\in\tilde{T}_{\beta_{ij}^k}}\left\{\frac{2\prod_{k=1}^{l}\left(\gamma_{ij}^k\right)^{\tau_{ij}^k}}{\prod_{k=1}^{l}\left(2-\gamma_{ij}^k\right)^{\tau_{ij}^k}+\prod_{k=1}^{l}\left(\gamma_{ij}^k\right)^{\tau_{ij}^k}}\right\},\right.$$
$$\bigcup_{\eta_{ij}^k\in\tilde{I}_{\beta_{ij}^k}}\left\{\frac{\prod_{k=1}^{l}\left(1+\eta_{ij}^k\right)^{\tau_{ij}^k}-\prod_{k=1}^{l}\left(1-\eta_{ij}^k\right)^{\tau_{ij}^k}}{\prod_{k=1}^{l}\left(1+\eta_{ij}^k\right)^{\tau_{ij}^k}+\prod_{k=1}^{l}\left(1-\eta_{ij}^k\right)^{\tau_{ij}^k}}\right\},$$
$$\left.\bigcup_{\xi_{ij}^k\in\tilde{F}_{\beta_{ij}^k}}\left\{\frac{\prod_{k=1}^{l}\left(1+\xi_{ij}^k\right)^{\tau_{ij}^k}-\prod_{k=1}^{l}\left(1-\xi_{ij}^k\right)^{\tau_{ij}^k}}{\prod_{k=1}^{l}\left(1+\xi_{ij}^k\right)^{\tau_{ij}^k}+\prod_{k=1}^{l}\left(1-\xi_{ij}^k\right)^{\tau_{ij}^k}}\right\}\right\rangle. \quad (18)$$

Step 5. Calculate the supports $Supp\left(\beta_{ij},\beta_{ip}\right)$.

The supports can be obtained by the following formula:

$$Supp\left(\beta_{ij},\beta_{ip}\right)=1-d\left(\beta_{ij},\beta_{ip}\right). \quad (19)$$

Here $i=1,2,\ldots,n;\ j,p=1,2,\ldots,m;\ j\neq p$, $Supp\left(\beta_{ij},\beta_{ip}\right)$ is the support for β_{ij} from β_{ip}, and satisfies the three conditions given in Definition 11. $d\left(\beta_{ij},\beta_{ip}\right)$ is the Hamming-Hausdorff distance between β_{ij} and β_{ip} as defined in Definition 9.

Step 6. Calculate the weights ρ_{ij} associated with the MVNN β_{ij}.

The weighted support $S\left(\beta_{ij}\right)$ of the MVNN β_{ij} by the other MVNNs $\beta_{ip}\left(p=1,2,\ldots,m \text{ and } p\neq j\right)$ can be calculated using the weights $w_j\left(j=1,2,\ldots,m\right)$ of the criteria $c_j\left(j=1,2,\ldots,m\right)$.

$$S\left(\beta_{ij}\right)=\sum_{p=1,p\neq j}^{m}w_p Supp\left(\beta_{ij},\beta_{ip}\right)\ \left(p=1,2,\ldots,m\right). \quad (20)$$

Then, the weights $\rho_{ij}\left(j=1,2,\ldots,m\right)$ associated with the MVNN $\beta_{ij}\left(j=1,2,\ldots,m\right)$ can be obtained as follows:

$$\rho_{ij}=\frac{w_j\left(1+S\left(\beta_{ij}\right)\right)}{\sum_{j=1}^{m}w_j\left(1+S\left(\beta_{ij}\right)\right)},j=1,2,\ldots,m. \quad (21)$$

Here $\rho_{ij}\geq 0\left(j=1,2,\ldots,m\right)$ and $\sum_{j=1}^{m}\rho_{ij}=1$.

Step 7. Calculate the comprehensive evaluation value of each alternative.

Utilize the MVNPWA operator or MVNPWG operators, Eq. (8) or Eq. (10), to aggregate all the preference values $\beta_{ij}\left(j=1,2,\ldots,m\right)$ of each alternative, then the comprehensive evaluation value $\beta_i\left(i=1,2,\ldots,n\right)$ of alternative $\alpha_i\left(i=1,2,\ldots,n\right)$ can be calculated:

$$\begin{aligned}\beta_i &= \mathrm{MVNPWA}_w\left(\beta_{i1},\beta_{i2},\ldots,\beta_{im}\right)\\ &= \left\langle \bigcup_{\gamma_{ij}\in\tilde{T}_{\beta_{ij}}}\left\{\frac{\prod_{j=1}^{m}\left(1+\gamma_{ij}\right)^{\rho_{ij}}-\prod_{j=1}^{m}\left(1-\gamma_{ij}\right)^{\rho_{ij}}}{\prod_{j=1}^{m}\left(1+\gamma_{ij}\right)^{\rho_{ij}}+\prod_{j=1}^{m}\left(1-\gamma_{ij}\right)^{\rho_{ij}}}\right\},\right.\\ &\qquad \bigcup_{\eta_{ij}\in\tilde{I}_{\beta_{ij}}}\left\{\frac{2\prod_{j=1}^{m}\left(\eta_{ij}\right)^{\rho_{ij}}}{\prod_{j=1}^{m}\left(2-\eta_{ij}\right)^{\rho_{ij}}+\prod_{j=1}^{m}\left(\eta_{ij}\right)^{\rho_{ij}}}\right\},\\ &\qquad \left.\bigcup_{\xi_{ij}\in\tilde{F}_{\beta_{ij}}}\left\{\frac{2\prod_{j=1}^{m}\left(\xi_{ij}\right)^{\rho_{ij}}}{\prod_{j=1}^{m}\left(2-\xi_{ij}\right)^{\rho_{ij}}+\prod_{j=1}^{m}\left(\xi_{ij}\right)^{\rho_{ij}}}\right\}\right\rangle. \end{aligned} \tag{22}$$

or

$$\begin{aligned}\beta_i &= \mathrm{MVNPWG}_w\left(\beta_{i1},\beta_{i2},\ldots,\beta_{im}\right)\\ &= \left\langle \bigcup_{\gamma_{ij}\in\tilde{T}_{\beta_{ij}}}\left\{\frac{2\prod_{j=1}^{m}\left(\gamma_{ij}\right)^{\rho_{ij}}}{\prod_{j=1}^{m}\left(2-\gamma_{ij}\right)^{\rho_{ij}}+\prod_{j=1}^{m}\left(\gamma_{ij}\right)^{\rho_{ij}}}\right\},\right.\\ &\qquad \bigcup_{\eta_{ij}\in\tilde{I}_{\beta_{ij}}}\left\{\frac{\prod_{j=1}^{m}\left(1+\eta_{ij}\right)^{\rho_{ij}}-\prod_{j=1}^{m}\left(1-\eta_{ij}\right)^{\rho_{ij}}}{\prod_{j=1}^{m}\left(1+\eta_{ij}\right)^{\rho_{ij}}+\prod_{j=1}^{m}\left(1-\eta_{ij}\right)^{\rho_{ij}}}\right\},\\ &\qquad \left.\bigcup_{\xi_{ij}\in\tilde{F}_{\beta_{ij}}}\left\{\frac{\prod_{j=1}^{m}\left(1+\xi_{ij}\right)^{\rho_{ij}}-\prod_{j=1}^{m}\left(1-\xi_{ij}\right)^{\rho_{ij}}}{\prod_{j=1}^{m}\left(1+\xi_{ij}\right)^{\rho_{ij}}+\prod_{j=1}^{m}\left(1-\xi_{ij}\right)^{\rho_{ij}}}\right\}\right\rangle. \end{aligned} \tag{23}$$

Step 8. Calculate the score function value and the accuracy function value.

Based on Definition 7, the score function value $s\left(\beta_i\right)$ and the accuracy function value $a\left(\beta_i\right)$ of α_i ($i=1,2,\ldots,n$) can be obtained.

Step 9. Rank the alternatives.

According to Definition 8, all alternatives α_i $\left(i=1,2,\ldots,n\right)$ can be ranked with respect to superiority and finally the best one(s) can be chosen.

5. Illustrative Example

In this section, an example of MCDM problems is used to demonstrate the application and effectiveness of the proposed decision-making approach.

There is an investment company, which wants to invest a sum of money in the best option (adapted from Ref. 37). The company has set up a panel which has to choose between four possible alternatives for investing the money: (1) α_1 is a car company; (2) α_2 is a food company; (3) α_3 is a computer company; (4) α_4 is an arms company. Each company is evaluated based on three criteria, which are denoted by $c_j\left(j=1,2,3\right)$: c_1 is the risk analysis, c_2 is the growth analysis and c_3 is the environmental impact analysis, where c_1 and c_2 are of the maximizing type, and c_3 is of the minimizing type. The weight vector of criteria is represented by $w=\left(0.35,0.25,0.4\right)$. There are three decision-makers to make decisions on this investment and the weight vector of them is $\omega=\left(0.3,\,0.5,\,0.2\right)$. They could evaluate these criteria based on their knowledge and experience. Moreover, the k-th decision-maker can provide their evaluations about the project α_i under the criterion c_j in the form of MVNNs and denoted by $\alpha_{ij}^k=\left\langle \tilde{T}_{\alpha_{ij}^k},\tilde{I}_{\alpha_{ij}^k},\tilde{F}_{\alpha_{ij}^k}\right\rangle$ $\left(i=1,2,3,4;j=1,2,3,4;k=1,2,3\right)$. $\tilde{T}_{\alpha_{ij}^k}$, $\tilde{I}_{\alpha_{ij}^k}$ and $\tilde{F}_{\alpha_{ij}^k}$ are in the form of HFNs, which represents their degrees of satisfaction, uncertainty and dissatisfaction regarding an alternative by using the concept of "excellent" against each criterion. It is noted that one decision-maker could give several evaluation values for the degree of satisfaction, uncertainty and dissatisfaction regarding an alternative respectively. All of the possible values for an alternative under a criterion are collected, and each value provided only means that it is a possible value. So in the case where the decision-maker gives two same value for one degree, it is counted only once, and α_{ij}^k is the set of evaluation values for the decision-maker. Then the multi-valued neutrosophic decision matrix $R^k=\left(\alpha_{ij}^k\right)_{4\times 3}$ can be found as follows:

$$R^1 = \begin{pmatrix} \langle\{0.4\},\{0.1\},\{0.2\}\rangle & \langle\{0.5\},\{0.2\},\{0.1\}\rangle & \langle\{0.3\},\{0.1,0.2\},\{0.4\}\rangle \\ \langle\{0.7\},\{0.1,0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.2\},\{0.2,0.3\}\rangle & \langle\{0.4\},\{0.2\},\{0.3\}\rangle \\ \langle\{0.4,0.5\},\{0.1\},\{0.3\}\rangle & \langle\{0.5\},\{0.2\},\{0.1\}\rangle & \langle\{0.4,0.5\},\{0.2\},\{0.2\}\rangle \\ \langle\{0.6\},\{0.3\},\{0.1\}\rangle & \langle\{0.5,0.6\},\{0.3\},\{0.2\}\rangle & \langle\{0.5\},\{0.1\},\{0.2\}\rangle \end{pmatrix};$$

$$R^2 = \begin{pmatrix} \langle\{0.6\},\{0.1\},\{0.1,0.2\}\rangle & \langle\{0.5\},\{0.2\},\{0.2\}\rangle & \langle\{0.4,0.5\},\{0.1\},\{0.3\}\rangle \\ \langle\{0.5\},\{0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.2\},\{0.1,0.2\}\rangle & \langle\{0.5\},\{0.3\},\{0.2\}\rangle \\ \langle\{0.4,0.5\},\{0.2\},\{0.1\}\rangle & \langle\{0.5\},\{0.1\},\{0.3\}\rangle & \langle\{0.5\},\{0.1\},\{0.2,0.3\}\rangle \\ \langle\{0.5\},\{0.3\},\{0.2\}\rangle & \langle\{0.8\},\{0.2,0.3\},\{0.2\}\rangle & \langle\{0.5\},\{0.2\},\{0.2\}\rangle \end{pmatrix};$$

$$R^3 = \begin{pmatrix} \langle\{0.4,0.5\},\{0.2\},\{0.3\}\rangle & \langle\{0.4\},\{0.2,0.3\},\{0.3\}\rangle & \langle\{0.2\},\{0.2\},\{0.5\}\rangle \\ \langle\{0.6\},\{0.1,0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.1\},\{0.2\}\rangle & \langle\{0.5\},\{0.2\},\{0.1,0.2\}\rangle \\ \langle\{0.3,0.4\},\{0.2\},\{0.3\}\rangle & \langle\{0.5\},\{0.2\},\{0.3\}\rangle & \langle\{0.5\},\{0.2,0.3\},\{0.2\}\rangle \\ \langle\{0.7\},\{0.1,0.2\},\{0.1\}\rangle & \langle\{0.6\},\{0.1\},\{0.2\}\rangle & \langle\{0.4\},\{0.3\},\{0.2\}\rangle \end{pmatrix}.$$

5.1. *Decision-making procedure based on MVNNs*

Step 1. Transform the decision matrix.

Since criteria c_1 and c_2 are of the maximizing type, and criterion c_3 is of the minimizing type, so according to Eqs. (12) and (13), the normalized MVNN decision matrix $\tilde{R}^k = \left(\beta_{ij}^k\right)_{4\times 3}$ can be obtained as follows:

$$\tilde{R}^1 = \begin{pmatrix} \langle\{0.4\},\{0.1\},\{0.2\}\rangle & \langle\{0.5\},\{0.2\},\{0.1\}\rangle & \langle\{0.4\},\{0.8,0.9\},\{0.3\}\rangle \\ \langle\{0.7\},\{0.1,0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.2\},\{0.2,0.3\}\rangle & \langle\{0.3\},\{0.8\},\{0.4\}\rangle \\ \langle\{0.4,0.5\},\{0.1\},\{0.3\}\rangle & \langle\{0.5\},\{0.2\},\{0.1\}\rangle & \langle\{0.2\},\{0.8\},\{0.4,0.5\}\rangle \\ \langle\{0.6\},\{0.3\},\{0.1\}\rangle & \langle\{0.5,0.6\},\{0.3\},\{0.2\}\rangle & \langle\{0.2\},\{0.9\},\{0.5\}\rangle \end{pmatrix};$$

$$\tilde{R}^2 = \begin{pmatrix} \langle\{0.6\},\{0.1\},\{0.1,0.2\}\rangle & \langle\{0.5\},\{0.2\},\{0.2\}\rangle & \langle\{0.3\},\{0.9\},\{0.4,0.5\}\rangle \\ \langle\{0.5\},\{0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.2\},\{0.1,0.2\}\rangle & \langle\{0.2\},\{0.7\},\{0.5\}\rangle \\ \langle\{0.4,0.5\},\{0.2\},\{0.1\}\rangle & \langle\{0.5\},\{0.1\},\{0.3\}\rangle & \langle\{0.2,0.3\},\{0.9\},\{0.5\}\rangle \\ \langle\{0.5\},\{0.3\},\{0.2\}\rangle & \langle\{0.8\},\{0.2,0.3\},\{0.2\}\rangle & \langle\{0.2\},\{0.8\},\{0.5\}\rangle \end{pmatrix};$$

$$\tilde{R}^3 = \begin{bmatrix} \langle\{0.4,0.5\},\{0.2\},\{0.3\}\rangle & \langle\{0.4\},\{0.2,0.3\},\{0.3\}\rangle & \langle\{0.5\},\{0.8\},\{0.2\}\rangle \\ \langle\{0.6\},\{0.1,0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.1\},\{0.2\}\rangle & \langle\{0.1,0.2\},\{0.8\},\{0.5\}\rangle \\ \langle\{0.3,0.4\},\{0.2\},\{0.3\}\rangle & \langle\{0.5\},\{0.2\},\{0.3\}\rangle & \langle\{0.2\},\{0.7,0.8\},\{0.5\}\rangle \\ \langle\{0.7\},\{0.1,0.2\},\{0.1\}\rangle & \langle\{0.6\},\{0.1\},\{0.2\}\rangle & \langle\{0.2\},\{0.7\},\{0.4\}\rangle \end{bmatrix}.$$

Step 2. Calculate the supports $Supp\left(\beta_{ij}^k, \beta_{ij}^t\right)$.

For simplicity, we denote $\left(Supp\left(\beta_{ij}^k, \beta_{ij}^t\right)\right)_{4\times 3}$ with $Supp^{kt}$. According to Eq. (14) and Definition 9, the supports $Supp^{kt}\left(k,t=1,2,3;k\neq t\right)$ can be obtained. As an example, $Supp_{11}^{12}$ can be calculated as follows:

$$\begin{aligned} & Supp_{11}^{12} \\ & = 1-d\left(\beta_{11}^1, \beta_{11}^2\right) \\ & = 1-d\left(\langle\{0.4\},\{0.1\},\{0.2\}\rangle, \langle\{0.6\},\{0.1\},\{0.1,0.2\}\rangle\right) \\ & = 0.9167. \end{aligned}$$

Then the $Supp^{kt}\left(k,t=1,2,3;k\neq t\right)$ can be achieved:

$$Supp^{12} = Supp^{21} = \begin{pmatrix} 0.9167 & 0.9667 & 0.9000 \\ 0.9167 & 0.9667 & 0.9000 \\ 0.9000 & 0.9000 & 0.9333 \\ 0.9333 & 0.9000 & 0.9667 \end{pmatrix},$$

$$Supp^{13} = Supp^{31} = \begin{pmatrix} 0.9167 & 0.8833 & 0.9167 \\ 0.9667 & 0.9667 & 0.9167 \\ 0.9333 & 0.9333 & 0.9667 \\ 0.9167 & 0.9167 & 0.9000 \end{pmatrix},$$

$$Supp^{23} = Supp^{32} = \begin{pmatrix} 0.8667 & 0.9167 & 0.8167 \\ 0.9500 & 0.9500 & 0.9667 \\ 0.9000 & 0.9667 & 0.9333 \\ 0.8500 & 0.8833 & 0.9333 \end{pmatrix}.$$

Step 3. Calculate the weights τ_{ij}^k associated with the MVNN β_{ij}^k.

According to Eq. (15), the weighted supports $S\left(\beta_{ij}^k\right)$ can be obtained. As an example, $S(\beta_{11}^1)$ can be calculated as follows:

$$S\left(\beta_{11}^1\right) = \sum_{t=1,t\neq 1}^{3} \omega_t Supp\left(\beta_{11}^1, \beta_{11}^t\right) = 0.6417.$$

Then the $\left(S\left(\beta_{ij}^k\right)\right)_{4\times 3}$ can be calculated and denoted with $S^k\left(k=1,2,3\right)$ in the following:

$$S^1 = \begin{pmatrix} 0.6417 & 0.6600 & 0.6333 \\ 0.6517 & 0.6734 & 0.6333 \\ 0.6367 & 0.6367 & 0.6600 \\ 0.6500 & 0.6333 & 0.6634 \end{pmatrix},$$

$$S^2 = \begin{pmatrix} 0.4484 & 0.4734 & 0.6784 \\ 0.7500 & 0.4800 & 0.4633 \\ 0.4500 & 0.4633 & 0.4667 \\ 0.4500 & 0.4467 & 0.4767 \end{pmatrix},$$

$$S^3 = \begin{pmatrix} 0.7084 & 0.7233 & 0.6834 \\ 0.7650 & 0.7600 & 0.7584 \\ 0.7300 & 0.7633 & 0.7567 \\ 0.7000 & 0.7167 & 0.7367 \end{pmatrix}.$$

Based on Eq. (16), the weights $\tau_{ij}^k\left(i,j=1,2,3,4;k=1,2,\ldots,l\right)$ associated with the MVNN $\beta_{ij}^k\left(k=1,2,\ldots,l\right)$ can be obtained using the weights $\omega_k\left(k=1,2,\ldots,l\right)$ of the decision-makers $d_k\left(k=1,2,\ldots,l\right)$. $\left(\tau_{ij}^k\right)_{4\times 3}$ is denoted by $\tau^k\left(k=1,2,3\right)$ as follows:

$$\tau^1 = \begin{pmatrix} 0.3160 & 0.3153 & 0.2941 \\ 0.2875 & 0.3149 & 0.3114 \\ 0.3143 & 0.3117 & 0.3147 \\ 0.3173 & 0.3148 & 0.3149 \end{pmatrix},$$

$$\tau^2 = \begin{pmatrix} 0.4647 & 0.4665 & 0.5038 \\ 0.5077 & 0.4642 & 0.4650 \\ 0.4641 & 0.4644 & 0.4634 \\ 0.4647 & 0.4647 & 0.4659 \end{pmatrix},$$

$$\tau^3 = \begin{pmatrix} 0.2193 & 0.2182 & 0.2021 \\ 0.2048 & 0.2208 & 0.2235 \\ 0.2215 & 0.2239 & 0.2220 \\ 0.2179 & 0.2206 & 0.2192 \end{pmatrix}.$$

Step 4. Aggregate the evaluation information of each expert.

According to MVNPWA operator, i.e., Eq. (17), the collective multi-valued neutrosophic decision matrix $\tilde{R} = \left(\beta_{ij}\right)_{n\times m}$ can be obtained. For example, β_{11} can be calculated as follows:

$$\beta_{11} = \text{MVNPWA}_{\omega}\left(\beta_{11}^1, \beta_{11}^2, \beta_{11}^3\right) = \left\langle\{0.4996, 0.5201\}, \{0.1168\}, \{0.1599, 0.2190\}\right\rangle.$$

Then the other collective values can be obtained:

$$\tilde{R} = \begin{pmatrix} \langle\{0.4996,0.5201\},\{0.1168\},\{0.1599, 0.2190\}\rangle & \langle\{0.4792\},\{0.2000,0.2189\},\{0.1770\}\rangle & \langle\{0.3727\},\{0.8500, 0.8796\},\{0.3215, 0.3626\}\rangle \\ \langle\{0.5851\},\{0.1428,0.1645, 0.1741,0.2000\},\{0.2000\}\rangle & \langle\{0.6000\},\{0.1722\},\{0.1457,0.2000, 0.1666,0.2278\}\rangle & \langle\{0.2099,0.2317\},\{0.7529\},\{0.4672\}\rangle \\ \langle\{0.3785,0.4000,0.4273,0.4669, 0.4118,0.4326,0.4591,0.4788\},\{0.1616\},\{0.1829\}\rangle & \langle\{0.5000\},\{0.1456\},\{0.2159\}\rangle & \langle\{0.2000,0.2470\},\{0.8229,0.8459\},\{0.4668, 0.5000\}\rangle \\ \langle\{0.5811\},\{0.2388, 0.2752\},\{0.1387\}\rangle & \langle\{0.6839,0.7072\},\{0.1965,0.2381\},\{0.2000\}\rangle & \langle\{0.2000\},\{0.8085\},\{0.4767\}\rangle \end{pmatrix}$$

Step 5. Calculate the supports $Supp\left(\beta_{ij}, \beta_{ip}\right)$.

According to Eq. (19),

$Supp\left(\beta_{ij}, \beta_{ip}\right)$ $(i = 1,2,\ldots,n; j, p = 1,2,\ldots,m;\ j \neq p)$ can be calculated as follows:

$Supp\left(\beta_{11}, \beta_{12}\right) = Supp\left(\beta_{12}, \beta_{11}\right) = 0.9491;$
$Supp\left(\beta_{11}, \beta_{13}\right) = Supp\left(\beta_{13}, \beta_{11}\right) = 0.6541;$
$Supp\left(\beta_{12}, \beta_{13}\right) = Supp\left(\beta_{13}, \beta_{12}\right) = 0.6910;$

$Supp\left(\beta_{21}, \beta_{22}\right) = Supp\left(\beta_{22}, \beta_{21}\right) = 0.9808;$
$Supp\left(\beta_{21}, \beta_{23}\right) = Supp\left(\beta_{23}, \beta_{21}\right) = 0.5957;$

$Supp\left(\beta_{22}, \beta_{23}\right) = Supp\left(\beta_{23}, \beta_{22}\right) = 0.5866;$
$Supp\left(\beta_{31}, \beta_{32}\right) = Supp\left(\beta_{32}, \beta_{31}\right) = 0.9599;$
$Supp\left(\beta_{31}, \beta_{33}\right) = Supp\left(\beta_{33}, \beta_{31}\right) = 0.6072;$
$Supp\left(\beta_{32}, \beta_{33}\right) = Supp\left(\beta_{33}, \beta_{32}\right) = 0.5891;$

$Supp\left(\beta_{41}, \beta_{42}\right) = Supp\left(\beta_{42}, \beta_{41}\right) = 0.9282;$

$Supp\left(\beta_{41}, \beta_{43}\right) = Supp\left(\beta_{43}, \beta_{41}\right) = 0.5765;$

$Supp\left(\beta_{42}, \beta_{43}\right) = Supp\left(\beta_{43}, \beta_{42}\right) = 0.5455.$

Step 6. Calculate the weights ρ_{ij} associated with the MVNN β_{ij}.

According to Eq. (20), the weighted support $\left(S\left(\beta_{ij}\right)\right)_{4\times 3}$ of the MVNN β_{ij} by the other MVNNs β_{ip} $(p = 1,2,\ldots,m$ and $p \neq j)$ can be calculated.

$$\left(S\left(\beta_{ij}\right)\right)_{4\times 3} = \begin{pmatrix} 0.4989 & 0.6086 & 0.4017 \\ 0.4835 & 0.5816 & 0.3551 \\ 0.4829 & 0.5788 & 0.3598 \\ 0.4627 & 0.5431 & 0.3382 \end{pmatrix}.$$

So the weights ρ_{ij} $(j = 1,2,\ldots,m)$ associated with the MVNN β_{ij} $(j = 1,2,\ldots,m)$ can be obtained using the weights w_j $(j = 1,2,\ldots,m)$ of the criteria c_j $(j = 1,2,\ldots,m)$ and Eq. (21).

$$\left(\rho_{ij}\right)_{4\times 3} = \begin{pmatrix} 0.3527 & 0.2704 & 0.3769 \\ 0.3564 & 0.2714 & 0.3721 \\ 0.3561 & 0.2708 & 0.3732 \\ 0.3573 & 0.2692 & 0.3735 \end{pmatrix}.$$

Step 7. Calculate the comprehensive evaluation value of each alternative.
Utilize the MVNPWA operator i.e., Eq. (22), to aggregate all the preference values $\beta_{ij}\left(j=1,2,\ldots,m\right)$ of each alternative, then the comprehensive value $\beta_i\left(i=1,2,\ldots,n\right)$ of the alternative $\alpha_i\left(i=1,2,\ldots,n\right)$ can be calculated:

$$\beta_1=\left\langle\{0.4481,0.4558\},\{0.3119,0.3179,0.3192,0.3253\},\{0.2153,0.2260\}\right\rangle;$$

$$\beta_2=\left\langle\{0.4670,0.4737\},\{0.2978,0.3119,0.3178,0.3326\},\{0.2567,0.2785,0.2657,0.2881\}\right\rangle;$$

$$\beta_3=\left\langle\{0.3507,0.3668,0.3746,0.3586,0.3688,0.3846,0.3995,0.3839,0.3630,0.3789,0.3866,0.3708,0.3809,0.3966,0.4041,0.3886\},\{0.3119,0.3166\},\{0.2757,0.2838\}\right\rangle;$$

$$\beta_4=\left\langle\{0.4916,0.5008\},\{0.3783,0.3964,0.3965,0.4152\},\{0.2484\}\right\rangle.$$

Step 8. Calculate the score function value and the accuracy function value.

Based on Definition 7, $s\left(\beta_i\right)$ can be obtained:

$$s\left(\beta_1\right)=-0.0291;s\left(\beta_2\right)=-0.0390;s\left(\beta_3\right)=-0.0718;s\left(\beta_4\right)=-0.0496$$

The score values are different. Therefore there is no need to compute the values of the accuracy function value.
Step 9: Rank the alternatives.

According to Definition 8 and the results in Step 8, $s\left(\beta_1\right)>s\left(\beta_2\right)>s\left(\beta_4\right)>s\left(\beta_3\right)$ can be obtained. So for MVNPWA operator, the final ranking is $\alpha_1\succ\alpha_2\succ\alpha_4\succ\alpha_3$. Clearly, the best alternative is α_1 while the worst alternative is α_3.

If the MVNPWG operator is utilized in Step 4 and Step 7, then the score function value $s\left(\beta_i\right)$ can be obtained:

$$s\left(\beta_1\right)=-0.0301;s\left(\beta_2\right)=-0.0259;s\left(\beta_3\right)=-0.0860;s\left(\beta_4\right)=-0.0572$$

Since $s\left(\beta_2\right)>s\left(\beta_1\right)>s\left(\beta_4\right)>s\left(\beta_3\right)$ and the score values are different. Therefore, for MVNPWG operator, the final ranking is $\alpha_2\succ\alpha_1\succ\alpha_4\succ\alpha_3$, and the best alternative is α_2 while the worst alternative is α_3.

From the results given above, the best one is α_1 or α_2, and the worst one is α_3. In most cases, in order to calculate the actual aggregation values of the alternatives, different aggregation operators can be used. Moreover, we can find that two aggregation operators mentioned in the manuscript, the MVNPWA operator or the MVNPWG operator, are all used to deal with different relationships of the aggregated arguments, which can provide more choices for decision-makers. They can choose different aggregation operator according to their preference.

5.2. *Comparison analysis*

In order to verify the feasibility of the proposed decision-making approach based on the MVNNs power aggregation operators, a comparison analysis based on the same illustrative example is conducted here.

The comparison analysis includes two cases. One is the other methods that were outlined in Ye[36, 37, 41], which are compared to the proposed method using single-valued neutrosophic information. In the other, the method that was introduced in Wang and Li[48] are compared with the proposed approach using multi-valued neutrosophic information.

The proposed approach is compared with some methods using single-valued neutrosophic information.

- The proposed approach is compared with some methods using single-valued neutrosophic information.

With regard to the three methods in Ye[36–37, 41], all multi-valued neutrosophic evaluation values are translated into single-valued neutrosophic values by using the mean values of truth-membership, indeterminacy-membership and falsity-membership respectively. Then two aggregation operators were used to aggregate the single-valued neutrosophic information first; and the correlation coefficient and weighted cross-entropy between each alternative and the ideal alternative were calculated and used to determine the final ranking order of all the alternatives. If the methods in Ye[36–37, 41] and the proposed method are utilized to solve the same MCDM problem, then the results can be obtained and are shown in Table 1.

Table 1. The compared results utilizing the different methods with SNSs

Methods	The final ranking	The best alternative(s)	The worst alternative(s)
Ye [36]	$\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$	α_1	α_3
Ye [37]	$\alpha_4 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$ or $\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$	α_4 or α_1	α_3
Ye [41]	$\alpha_4 \succ \alpha_2 \succ \alpha_1 \succ \alpha_3$	α_4	α_3
The proposed method	$\alpha_1 \succ \alpha_2 \succ \alpha_4 \succ \alpha_3$ or $\alpha_2 \succ \alpha_1 \succ \alpha_4 \succ \alpha_3$	α_1 or α_2	α_3

If the aggregation operators proposed by Ye[37] are used, for the weighted average operator, the final ranking is $\alpha_4 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$. Clearly, the best alternative is α_4 while the worst alternative is α_3. For the weighted geometric operator, the final ranking is $\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$, and the best alternative is α_1 while the worst alternative is α_3. However, if the methods of Ye[36,41] are used, then the final ranking is $\alpha_1 \succ \alpha_4 \succ \alpha_2 \succ \alpha_3$ or $\alpha_4 \succ \alpha_2 \succ \alpha_1 \succ \alpha_3$ and the best alternative is α_1 or a_4. It can be seen that the results of the proposed approach are different from those that use the earlier methods of Ye[36–37, 41].

There are three reasons why differences exist in the final rankings of all the compared methods and the proposed approach. Firstly, the aggregation operators that are involved in the method of Ye[37] are related to some impractical operations as was discussed in Examples 1-3. Secondly, if the correlation coefficient and cross-entropy proposed[36, 41], proposed on the basis of the operations[37], are extend to MVNNs, the shortcomings discussed in Section 2 would still exist. Finally, the aggregation values, correlation coefficients and cross-entropy measures of SNSs were obtained firstly in Ye[36–37, 41] and the differences were amplified in the final results due to the use of criteria weights.

- The proposed approach is compared with the method using multi-valued neutrosophic information.

If the method in Wang and Li[48] is utilized to solve the same MCDM problem, then the MVNPWA and MVNPWG operators were used to aggregate the evaluation information of each expert respectively; and the final ranking can be determined by using the TODIM method in Ref. 48. If the MVNPWA operator is used first, then the final ranking is $\alpha_1 \succ \alpha_2 \succ \alpha_4 \succ \alpha_3$, and the best alternative is α_1 while the worst alternative is always α_3; if the MVNPWG operator is used, then the final ranking is $\alpha_2 \succ \alpha_1 \succ \alpha_4 \succ \alpha_3$, and the best alternative is α_2. Apparently, the result of the proposed approach is the same as that using Wang and Li's method[48], and the best alternative is always α_1 or α_2 while the worst alternative is always α_3.

From the analysis presented above, it can be concluded that the main advantages of the approach developed in this paper over the other methods are not only due to its ability to effectively overcome the shortcomings of the compared methods, but also due to its ability to relieve the influence of unfair assessments provided by different decision-makers on the final aggregated results. This means that it can avoid losing and distorting the preference information provided which makes the final results more precise and reliable correspond with real life decision-making problems.

6. Conclusions

MVNSs can be applied in solving problems with uncertain, imprecise, incomplete and inconsistent information that exist in scientific and engineering situations. Based on the related research of IFSs and HFSs, the operations of MVNNs were defined in this paper and the comparison method was also developed. Furthermore, two aggregation operators, namely the MVNPWA operator and MVNPWG operator, were provided. Thus, a MCGDM approach was established that was based on proposed operators. An illustrative example demonstrated the application of the proposed decision-making approach. Moreover, the comparison analysis showed that the final result produced by the proposed method is more precise and reliable than the results produced by the existing methods. The contribution of this study is that the proposed approach for MCDM problems with MVNNs could overcome the shortcomings of the existing methods as was discussed earlier and relieves the influence of unfair assessments provided by different decision-makers on the final aggregated results. In future research, the authors will continue to study the related measures of MVNNs and applied them to solve more decision-making problems.

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References

1. L.A. Zadeh, Fuzzy sets, *Inf. Control* **8** (1965) 338–356.
2. R. Bellman and L.A. Zadeh, Decision making in a fuzzy environment, *Manage. Sci.* **17** (1970) 141–164.
3. R.R. Yager, Multiple objective decision-making using fuzzy sets, Int. J. Man-Mach. Stud. **9** (1997) 375–382.
4. L.A. Zadeh, Fuzzy logic and approximate reasoning, *Synthese* **30** (1975) 407–428.
5. W. Pedrycz, Fuzzy sets in pattern recognition: methodology and methods, *Pattern Recogn.* **23** (1990) 121–146.
6. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.* **20** (1986) 87–96.
7. W.L. Gau and D.J. Buehrer, Vague sets, *IEEE T. Syst. Man Cy. B.* **23** (1993) 610–614.
8. H. Bustince and P. Burillo, Vague sets are intuitionistic fuzzy sets, *Fuzzy Set. Syst.* **79** (1996) 403–405.
9. K. T. Atanassov and G. Gargov, Interval valued intuitionistic fuzzy sets, *Fuzzy Set. Syst.* **31** (1989) 343–349.
10. X.H.Yu and Z.S. Xu, Prioritized intuitionistic fuzzy aggregation operators, *Inf. Fusion* **14** (2013) 108–116.
11. Y.T. Chen, A outcome-oriented approach to multicriteria decision analysis with intuitionistic fuzzy optimistic/pessimistic operators, *Expert Syst. Appl.* **37** (2010) 7762–7774.
12. Z.S. Xu and H. Hu, Projection models for intuitionistic fuzzy multiple attribute decision making, *Int. J. Inf. Tech. Decis.* **9** (2010) 267–280.
13. T. Chaira, Intuitionistic fuzzy set approach for color region extraction, *J. Sci. Ind. Res.* **69** (2010) 426–432.
14. S.Z. Zeng and W.H. Su, Intuitionistic fuzzy ordered weighted distance operator, *Knowl.-Based Syst.* **24** (2011) 1224–1232.
15. G.W. Wei, Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making, *Expert Syst. Appl.* **38** (2011) 11671–11677.
16. Z. Pei and L. Zheng, A novel approach to multi-attribute decision making based on intuitionistc fuzzy sets, *Expert Syst. Appl.* **39** (2012) 2560–2566.
17. J.Q. Wang, R.R. Nie, H.Y. Zhang and X.H. Chen, Intuitionistic fuzzy multi-criteria decision-making method based on evidential reasoning, *Appl. Soft Comput.* **13** (2013) 1823–1831.
18. J.Q. Wang and H.Y. Zhang, Multi-criteria decision-making approach based on Atanassov's intuitionistic fuzzy sets with incomplete certain information on weights, *IEEE T. Fuzzy Syst.* **21** (3) (2013) 510–515.
19. J.Q. Wang, R.R. Nie, H.Y. Zhang and X.H. Chen, New operators on triangular intuitionistic fuzzy numbers and their applications in system fault analysis, *Inform. Sci.* **251** (2013) 79–95.
20. J.Q. Wang, P. Zhou, K.J. Li, H.Y. Zhang and X.H. Chen, Multi-criteria decision-making method based on normal intuitionistic fuzzy-induced generalized aggregation operator, *TOP* **22** (2014) 1103–1122.
21. J.Q. Wang , Z.Q. Han and H.Y. Zhang, Multi-criteria group decision-making method based on intuitionistic interval fuzzy information, *Group Decis. Negot.* **23** (2014) 715–733.
22. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.* **25** (2010) 529–539.
23. V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, 18th IEEE Int. *Conf. Fuzzy Syst.* (Jeju Island, Korea, 2009), pp. 1378–1382.
24. G.W. Wei, Hesitant fuzzy prioritized operators and their application to multiple attribute decision making, *Knowl.-Based Syst.* **31** (2012) 176–182.
25. Z.M. Zhang, Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making, *Inform. Sci.* **234** (2013) 150–181.
26. B. Farhadinia, Distance and similarity measures for higher order hesitant fuzzy sets, *Knowl.-Based Syst.* 55 (2014) 43–48.
27. J.Q. Wang, D.D. Wang, H.Y. Zhang and X.H. Chen, Multi-criteria outranking approach with hesitant fuzzy sets, *OR Spectrum* **36** (2014) 1001–1019.
28. G. Qian, H. Wang, and X.Q. Feng, Generalized hesitant fuzzy sets and their application in decision support system, *Knowl.-Based Syst.* **37** (2013) 357–365.
29. J.J. Peng, J.Q. Wang, H. Zhou, X.H. Chen, A multi-criteria decision-making approach based on TODIM and Choquet integral within a multiset hesitant fuzzy environment, *Appl. Math. Inform. Sci.* **9** (4) (2015) 1–11.
30. J.J. Peng, J.Q. Wang, J. Wang, X.H. Chen, Multi-criteria decision-making approach with hesitant interval-valued

intuitionistic fuzzy set, *Sci. World J.* (volume 2014, Article ID 868515, 22 pages).
31. H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, Single valued neutrosophic sets, *Multispace Multistructure* **4** (2010) 410–413.
32. F. Smarandache, *A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic* (American Research Press, Rehoboth, 1999).
33. F. Smarandache, *A unifying field in logics neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*, 3rd edn. (Xiquan, Phoenix, 2003).
34. U. Rivieccio, Neutrosophic logics: prospects and problems, *Fuzzy Set. Syst.* **159** (2008) 1860–1868.
35. P. Majumdar and S.K. Samant, On similarity and entropy of neutrosophic sets, *J. Intell. Fuzzy Syst.* **26** (3) (2014) 1245–1252.
36. J. Ye, Multicriteria decision-making method using the correlation coefficient under single-value neutrosophic environment, *Int. J. Gen. Syst.* **42** (4) (2013) 386–394.
37. J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, *J. Intell. Fuzzy Syst.* **26** (5) (2014) 2459–2466.
38. H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, *Interval neutrosophic sets and logic: theory and applications in computing* (Hexis, Phoenix, AZ, 2005).
39. F. G. Lupiáñez, Interval neutrosophic sets and topology, *Kybernetes* **38** (3–4) (2009) 621–624.
40. S. Broumi and F. Smarandache, Correlation coefficient of interval neutrosophic set, *Appl. Mech. Mater.* **436** (2013) 511–517.
41. J. Ye, Single valued neutrosophic cross-entropy for multicriteria decision making problems, *Appl. Math. Model.* **38** (3) (2014) 1170–1175.
42. J. Ye, Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making, *J. Intell. Fuzzy Syst.* **26** (1) (2014) 165–172.
43. J.J. Peng, J.Q. Wang, J. Wang, H.Y. Zhang and X.H. Chen, Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, *Int. J. Syst. Sci.*, doi: 10.1080/00207721.2014.994050, 2015.
44. J.J. Peng, J.Q. Wang, H.Y. Zhang and X.H. Chen, An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets, *Appl. Soft Comput.* **25** (2014) 336–346.
45. H.Y. Zhang, J.Q. Wang and X.H. Chen, Interval neutrosophic sets and their application in multicriteria decision making problems, *Sci. World J.* (Volume 2014, Article ID 645953, 15 pages).
46. P.D. Liu and Y.M. Wang, Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean, *Neural Comput. Appl.*, doi: 10.1007/s00521-014-1688-8, 2014.
47. P.D. Liu, Y.C. Chu, Y.W. Li and Y.B. Chen, Some generalized neutrosophic number Hamacher aggregation operators and their application to group decision making, *Int. J. Fuzzy Syst.* **16** (2) (2014) 242–255.
48. Z.P. Tian, J. Wang, H.Y. Zhang, X.H. Chen and J.Q. Wang, Simplified neutrosophic linguistic normalized weighted Bonferroni meanto multi-criteria decision-making problems, *Filomat*, In Press, 2015.
49. J.Q. Wang and X.E. Li, An application of the TODIM method with multi-valued neutrosophic set, Control Decis., In Press, 2015.
50. E.P. Klement, R. Mesiar and E. Pap, Triangular norms. position paper I: Basic analytical and algebraic properties, *Fuzzy Set. Syst.* **143** (2004) 5–26.
51. R.R. Yager, The power average operator, Man and Cybemetics, Part A: Systems and Humans, IEEE T. Syst. **31** (6) (2001) 724–731.