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A Further Research on S-core for Interval Cooperative Games

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Abstract

Interval cooperative game is a kind of model focusing on how to distribute the profit reasonably when payoffs of any alliance are interval numbers. In recent years, the existence and reasonableness of its solution have aroused widespread concern. In this paper, based on the conceptual analyses of various solutions of interval cooperative games, S-core is further researched. The concepts of weak balanced interval cooperative games and minimal weak balanced interval cooperative games are firstly proposed. The necessary and sufficient condition which guarantees S-core is nonempty is proven, furthermore, the inequalities can be simplified on the condition that the left endpoints of interval numbers satisfy the superadditivity. Then this paper analyzes the whole solution space of S-core and the solution method of S-core is converted into the method solving a linear programming problem. After that the concept of S-dominance core is put forward and the equivalent conditions of S-core and S-dominance core are proved. Finally, the reasonableness and validity of S-core are verified through a specific example.

Keywords: Interval cooperative game; S-core; balanced collection; minimal weak balanced interval cooperative game.

1. Introduction

Interval cooperative games [1], introduced by Branzei in the context of bankruptcy situations, mainly solve the profit distribution problems when the profit belong to interval numbers. In recent years, many scholars have conducted extensive researches focusing on its solution concepts, properties and relations among so many solutions. Much success has been achieved, for example, in [2], Alparslan-GöK derived a solution concept of interval cooperative game and obtained a sufficient condition for its non-emptiness. In [3], under the condition that all imputations are n-dimensional vectors, Nishizaki and Sakawa gave the definition of α-*core* of interval cooperative games. In [4], considering that all imputations are interval numbers, Alparslan-GöK defined a new interval solution and some relevant properties were discussed. In [5], Chen analyzed the interval Shapley value based on the element games and then proved three axioms of Shapley value. In [6-10], many interval solution concepts such as interval dominance core, square interval core and interval nondominated core were proposed and the relations among them were analyzed. In this paper, we mainly have a further discussion on the interval solution

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presented in [2]. (In order to facilitate the description below, we define the new interval solution as called S-core). By analyzing S-core, we can clearly get the main advantage of S-core which can solve the profit distribution problems under interval conditions by using concepts and methods in classical cooperative game, thus effectively avoiding the order and subtraction operations of interval numbers. Furthermore, S-core is intuitive and easy to understand and accept. However, there are still some disadvantages: 1) In [2], the sufficient condition for the nonempty S-core is given on the condition that it is a strongly balanced interval game. However, this condition is so strict that it requires a large number of inequalities to be verified, thus affecting the practicability of S-core. Therefore, it is particularly necessary to give a weakening sufficient condition for the nonempty S-core which can also simplifies the inequalities; 2) S-core is not easy to be solved. How to get an easier calculation method is conducive to our understanding of S-core; 3) Many properties of S-core are not discussed in detail, further analyses of S-core can provide a theoretical basis for its practical application.

The main contributions of this paper are motivated by the above presentation and can be summarized in the following way: 1) The concepts of the weak balanced and minimal weak balanced interval cooperative games are defined, the necessary and sufficient condition for a nonempty S-core is proved and the inequalities needed to be verified are simplified under the condition that the left endpoints satisfy the superadditivity; 2) The entire solution space of S-core is discussed, the solution method of S-core is transformed into the method solving a linear programming problem, the concept of S-dominance core is put forward and the equivalent condition of S-core and S-dominance core is proved.

The rest of the paper is structured as follows. Preliminaries are introduced in Section 2. In Section 3, the existence theorem of S-core is further discussed, the concepts of weak balanced and minimal weak balanced interval cooperative game are defined and the necessary and sufficient condition for a nonempty S-core is given. In Section 4, the entire solution space of S-core is discussed. The solution method of S-core is transformed into the method solving a linear programming problem. The concept of S-dominance core is put forward and the equivalence of S-core and S-dominance core is proved as well. A Case-based example is shown in Section 5. The conclusions and further studies are discussed in Section 6.

2. Preliminaries

We start this section with some basic definitions from classical cooperative game theory which are used in the following sections.

2.1. *Concepts of classical cooperative games*

A classical cooperative game in coalition form is an ordered pair (N,v), where $N=\{1,2,\cdots,n\}$ is the set of players, and $v:2^N\rightarrow R$ is a map, assigning to each coalition $S\in 2^N$ a real number $v(S)$, such that $v(\varnothing)=0$. This function v is called the characteristic function of the game and $v(S)$ is called the worth (or value) of coalition *S*.

On the research of classical cooperative games, how to allocate the payoff value $v(S)$ reasonably among each player is the most important issue. Following, we will firstly introduce the definition of imputation.

Definition 1.[11] An imputation for the game (N,v) is a vector $x\in R^n$ satisfing

$$x_i\geq v(i),\forall i\in N \tag{1}$$

$$\sum_{i=1}^{n}x_i=v(N) \tag{2}$$

we shall use the notation $E(v)$ for the set of all imputations.

Definition 2.[12] The core of a game (N,v) is the set

$$C(v)=\{x\in E(v)\,|\,x(S)\geq v(S),\forall S\neq\varnothing\subseteq N\} \tag{3}$$

Core is one of the most important solution concepts. From Definition 2, we can easily see that sometimes there may not exist one solution $x=(x_1,x_2,\cdots,x_n)\in R^n$ which satisfies (3), that is to say, core is usually an empty set, in order to give some theorems to discuss the existence of core, we will introduce some relevant definitions.

Definition 3.[11] Let $\zeta=\{R_1,R_2,\cdots,R_m\}$ be a collection of nonempty subsets of $N=\{1,2,\cdots,n\}$, we say that ζ is balanced if there exits positive numbers $y_1,y_2,\cdots,y_m$ such that, for each $\forall i\in N$, $\sum_{j:i\in R_j}y_j=1, j=1,2,\cdots,m.$

Then we call $y=(y_1,y_2,\cdots,y_m)$ the balancing vector for ζ, y_j the balancing coefficients.

A minimal balanced collection is a balanced collection includes no other balanced collections. We call a proper minimal balanced collection if no two of its elements are disjoint.

2.2. *Concepts of interval cooperative games and S-core*

In order to introduce the concept of interval cooperative games, first we give the definition of interval numbers.

Definition 4.[13] Let R be the real number domain, a bounded closed interval $[\underline{a}, \overline{a}]$ is called an interval number on R, which is expressed as $\overline{A}=[\underline{a}, \overline{a}]$, where $\underline{a}, \overline{a} \in R$, and $\underline{a} \le \overline{a}$. The symbol I(R) denotes the set of all interval numbers.

Definition 5.[2] An interval cooperative game in coalition form is an ordered pair (N,ω), where $N=\{1,2,\cdots,n\}$ is the set of players, and $\omega: 2^N \to I(R)$ is the characteristic function which assigns to each coalition $S \in 2^N$ a closed interval $\omega(S) \in I(R)$, such that $\omega(\varnothing)=[0,0]$. For each $S \in 2^N$, the worth set(or worth interval) $\omega(S)$ is of the form $[\underline{\omega}(S), \overline{\omega}(S)]$, where $\underline{\omega}(S)$ is the lower bound and $\overline{\omega}(S)$ is the upper bound of $\omega(S)$.

In particular, when $\underline{\omega}(S)=\overline{\omega}(S)$, the interval cooperative game is degenerated into classical cooperative game, and $v(S)=\underline{\omega}(S)=\overline{\omega}(S)$. So we can easily get that interval cooperative game is a natural extension of classical cooperative game.

Definition 6.[2] Let (N,ω) be an interval cooperative game, then $v: 2^N \to R$ is called a selection of ω if

$$v(S) \in \omega(S), \forall S \in 2^N \tag{4}$$

the set of all selections is denoted by $Sel(\omega)$.

Definition 7.[2] The imputation set of an interval cooperative game (N,ω) is defined by

$$E(\omega)=\bigcup\{E(v) \mid v \in Sel(\omega)\} \tag{5}$$

In the following we define the core set of (N,ω) which is a frequently used important solution concept.

Definition 8.[2] The core set of an interval cooperative game (N,ω) is defined by

$$C(\omega)=\bigcup\{C(v) \mid v \in Sel(\omega)\} \tag{6}$$

From Definition 7 and 8, we can get that if $\underline{\omega}(S)=\overline{\omega}(S)$, then the imputation set of (N,ω) is degenerated into classical imputation set and core set of (N,ω) is degenerated into classical core set, so the definitions in (5) and (6) are the natural extensions of classical imputation set and core set. For simplification, we call imputation set in (5) the S-imputation, and core set in (6) the S-core, which are denoted by $S\text{-}E(\omega)$ and $S\text{-}C(\omega)$ respectively.

Definition 9.[2] An interval cooperative game (N,ω) is strongly balanced, if for each balanced collection ζ and its balancing vector $\{y_R\}_{R\subseteq N}$, we have

$$\sum_{R\subseteq N\backslash\varnothing} y_R \overline{\omega}(R) \le \underline{\omega}(N) \tag{7}$$

In [2], the following Propositions are given.

Proposition 1. Let (N,ω) be an interval cooperative game, if it is strongly balanced, then $S\text{-}C(\omega) \ne \varnothing$.

Proposition 2. Let (N,ω) be an interval cooperative game, if it is strongly unbalanced, that is, if there exists a balanced collection and its balancing vector which satisfy

$$\sum_{R\subseteq N} y_R \underline{\omega}(R) > \overline{\omega}(N) \tag{8}$$

then $S\text{-}C(\omega)=\varnothing$.

Note that Proposition 1 and 2 are only sufficient but not necessary conditions. By Proposition 1, we can easily get the existence condition of S-core, by Proposition 2, we can also know that if (8) is satisfied, S-core does not exist. However, sometimes we can not discuss S-core only by Proposition 1 and 2. For there may be some special cases that (7) and (8) are not satisfied simultaneously. Next we will give an example to verify the limitations in Proposition 1 and 2.

Example 1. Let (N,ω) be an interval cooperative game, and we know that $N=\{1,2,3\}, \omega(1)=\omega(2)=\omega(3)=[1,3]$, $\omega(1,2)=\omega(1,3)=\omega(2,3)=[5,6], \omega(1,2,3)=[7,10]$. Please discuss the existence of S-core.

Suppose there exists a balanced collection $\zeta=\{\{1,2\},\{3\}\}$ and its balancing vector $y=(1,1)$, for $1\times 6+1\times 3>7$, so (N,ω) is not strongly balanced and we cannot use Proposition 1 to discuss the existence of S-core. At the same time, there is no balanced collection and balancing vector satisfying (8), so Proposition 2 also cannot be used. Thus we cannot discuss the existence of S-core by the existing theorems. However, in fact, S-core in Example 1 exists, we observe that $(3,3,3) \in S\text{-}C(\omega)$. So obtaining a more comprehensive existence theorem to discuss the existence of S-core is the main motivation of the paper. In the following Section, we shall give a theorem which can inevitably be used to discuss the existence of S-core.

3. The Existence Theorem of S-core in Interval Cooperative Game

As an a-priori requirement, following lemmas from [11] are given.

Lemma 1. A balanced collection has a unique balancing vector if and only if it is minimal.

Lemma 2. Any balanced collection is the union of minimal balanced collections.

It follows that Lemma 1 and Lemma 2 hold good for both classical cooperative game as well as interval cooperative games, as they do not require the form of the payoff values.

3.1. *A sufficient condition for nonempty S-core*

The main work to discuss the existence of S-core is that if there exists a $v \in Sel(\omega)$ satisfying $C(v) \neq \varnothing$. So how to get an appropriate characteristic function v is very important. In this Section, we shall firstly try to simplify the inequalities needed to be verified in Proposition 1. Following there is a lemma in [14].

Lemma 3. A necessary and sufficient condition for the game (N,v) to have a nonempty core is that for each minimal balanced collection $\zeta = \{R_1, R_2, \cdots, R_m\}$ and the unique balancing vector $y = (y_1, y_2, \cdots, y_m)$, we have

$$\sum_{j=1}^{m} y_j v(R_j) \leq v(N) \tag{9}$$

To simplify the sufficient condition in Proposition 1, we give the following definition.

Definition 9. An interval cooperative game (N,ω) is minimal strongly balanced, if for each balanced collection and the unique balancing vector $y = (y_1, y_2, \cdots, y_m)$, we have

$$\sum_{j=1}^{m} y_j \overline{\omega}(R_j) \leq \underline{\omega}(N) \tag{10}$$

Theorem 1. If (N,ω) is minimal strongly balanced, then $S\text{-}C(\omega) \neq \varnothing$.

Proof. For $\overline{\omega}(R_j) \geq \underline{\omega}(R_j), \forall j = 1,2,\cdots m$

So

$$\sum_{j=1}^{m} y_j \underline{\omega}(R_j) \leq \sum_{j=1}^{m} y_j \overline{\omega}(R_j) \leq \underline{\omega}(N)$$

it follows from Lemma 3 that the classical cooperative game $(N,\underline{\omega})$ has nonempty core, so $S\text{-}C(\omega) \neq \varnothing$.

Theorem 1 provides a better way of obtaining the non-emptiness of the S-core as it imposes the "minimal" constraint leading to the reduction of a number of redundant inequalities. In [2], similar formulation was made except this constraint, and thereby requiring the inequalities for verification at every point in a convex set. Theorem 1 ensures that we need to check these inequalities only at a finite number of poles.

3.2. *A necessary and sufficient condition for the existence of S-core*

Theorem 1 only gives the sufficient condition for the non-emptiness of S-core. In this Section, we will discuss how to give a necessary and sufficient condition. Prior to giving it, the following definitions and theorems are given firstly.

Definition 10. For a minimal balanced collection ζ, if $\{1,2,\cdots,n\} \in \zeta$, then we call ζ a real minimal balanced collection.

Theorem 2. A necessary and sufficient condition for game (N,v) to have a nonempty core is that for each real minimal balanced collection $\zeta = \{R_1, R_2, \cdots, R_m\}$ and the unique balancing vector $y = (y_1, y_2, \cdots, y_m)$, we have

$$\sum_{j=1}^{m} y_j v(R_j) \leq v(N)$$

Proof. For the classical cooperative game (N,ω), if $\{1,2,\cdots,n\} \in \zeta$, then by the definition of minimal balanced collection, we know $\zeta = \{1,2,\cdots,n\}$ and it has the unique balancing vector 1, so we have

$$\sum_{j=1}^{m} y_j v(R_j) = v(1,2,\cdots,n) = v(N)$$

The result follows from Lemma 3.

Next we define the notions of a weak balanced interval cooperative game and a minimal weak balanced interval cooperative game.

Definition 11. An interval cooperative game (N,ω) is weak balanced, if for each balanced collection ζ and its balancing vector $\{y_R\}_{R\subseteq N}$, we have

$$\sum_{R\subseteq N} y_R \underline{\omega}(R) \le \overline{\omega}(N) \quad (11)$$

Definition 12. An interval cooperative game (N,ω) is minimal weak balanced, if for each minimal balanced collection and the unique balancing vector $y=(y_1,y_2,\cdots,y_m)$, we have

$$\sum_{j=1}^{m} y_j \underline{\omega}(R_j) \le \overline{\omega}(N) \quad (12)$$

Based on Definition 12, we will give the necessary and sufficient condition for the non-emptiness of the S-core.

Theorem 3. $S\text{-}C(\omega)\ne\varnothing$ if and only if (N,ω) is a minimal weak balanced interval cooperative game.

Proof. Let $S\text{-}C(\omega)\ne\varnothing$, thus there exits $v\in Sel(\omega)$, such that $C(v)\ne\varnothing$. In view of Lemma 3, for each minimal balanced collection $\zeta=\{R_1,R_2,\cdots,R_m\}$ and the unique balancing vector $y=(y_1,y_2,\cdots,y_m)$, we have

$$\sum_{j=1}^{m} y_j v(R_j) \le v(N)$$

then it is clear that

$$\sum_{j=1}^{m} y_j \underline{\omega}(R_j) \le \sum_{j=1}^{m} y_j v(R_j) \le v(N) \le \overline{\omega}(N)$$

so (N,ω) is a minimal weak balanced interval cooperative game.

To prove the converse, we construct a classical cooperative game as follows

$$v(S)=\begin{cases}\underline{\omega}(S), S\subset N\\ \overline{\omega}(S), S=N\end{cases}$$

Following Theorem 2, we observe that for each real minimal balanced collection and its unique balancing vector, we only need to verify

$$\sum_{j=1}^{m} y_j v(R_j) \le v(N)$$

For $\{1,2,\cdots,n\}\notin\zeta$, so $v(R_j)=\underline{\omega}(R_j)$, by the definition of minimal weak balanced interval cooperative game, we see that

$$\sum_{j=1}^{m} y_j \underline{\omega}(R_j) \le \overline{\omega}(N)$$

thus we have

$$\sum_{j=1}^{m} y_j v(R_j) = \sum_{j=1}^{m} y_j \underline{\omega}(R_j) \le \overline{\omega}(N) = v(N)$$

and followed by

$$\sum_{j=1}^{m} y_j v(R_j) \le v(N)$$

Consequently in view of Theorem 2, the core of (N,v) is non-empty. This completes the proof.

Example 2. Relative data are the same as Example 1.

In Example 1, we know that $(3,3,3)\in S\text{-}C(\omega)$, but it is hard to achieve. If we use Theorem 1 to discuss the existence, we only need to verify

$$\sum_{j=1}^{m} y_j \underline{\omega}(R_j) \le \overline{\omega}(N)$$

so we can convert it into

$$y_1\underline{\omega}(1)+y_2\underline{\omega}(2)+y_3\underline{\omega}(3)+y_4\underline{\omega}(1,2)+y_5\underline{\omega}(1,3)$$
$$+y_6\underline{\omega}(2,3)\le\overline{\omega}(1,2,3), y_i\ge 0, i=1,2,\cdots,6$$
$$y_1+y_4+y_5=1, y_2+y_4+y_6=1, y_3+y_5+y_6=1.$$

so we need to verify $y_1+y_2+y_3+5(y_4+y_5+y_6)\le 10$. because we have $y_1+y_2+y_3+2(y_4+y_5+y_6)=3$, so $y_4+y_5+y_6\le 1.5$, thus $y_1+y_2+y_3+5(y_4+y_5+y_6)\le$ $3+4.5\le 10$. So Theorem 1 is inevitably can be used to discuss the existence of S-core.

Corollary 1. $S\text{-}C(\omega)\ne\varnothing$ if and only if (N,ω) is a weak balanced interval cooperative game.

Corollary 2. $S\text{-}C(\omega)\ne\varnothing$ if and only if (N,ω) is a real minimal weak balanced interval cooperative game.

Corollary 1 implicates that the weak balanced interval cooperative game is a game with non-empty S-core, Theorem 3 and Corollary 2 have sufficiently reduced the conditions for verifications of this non-emptiness. In order to further reduce the size of the set of inequalities, we next impose the condition of superadditivity on the left endpoints of the interval in the following.

Definition 13. For an interval cooperative game (N,ω), if the following conditions are satisfied

1) $\omega(\varnothing)=[0,0]$;

2) $\forall R,T\subseteq N, R\cap T=\varnothing$, we have

$$\underline{\omega}(R)+\underline{\omega}(T)\le\underline{\omega}(R\cup T) \quad (13)$$

Then (N,ω) is superadditive on the left endpoint of the interval.

Theorem 4. For an interval cooperative game which is superadditive on the left endpoint of the interval, $S\text{-}C(\omega) \neq \varnothing$ if and only if for each proper minimal balanced collection $\zeta = \{R_1, R_2, \cdots, R_m\}$ and its unique balancing vector $y = (y_1, y_2, \cdots, y_m)$, (12) is satisfied.

Proof. By Theorem 3, the necessary condition can be easily proved.

By Theorem 3, we only need to verify that for each minimal balanced collection and its balancing vector, we have

$$\sum_{j=1}^{m} y_j \underline{\omega}(R_j) \leq \bar{\omega}(N)$$

Let us suppose that there is some minimal balanced collection that does not satisfy the above condition. First, along the line of the proof given in [15], we introduce the notation $f(\zeta)$ that represents the total number of disjoint sets of unordered pairs in this minimal balanced collection. We set Ω the smallest $f(\zeta)$ when ζ does not satisfy the above condition in the minimal balanced collection. By hypothesis, Ω is not normal, so $f(\Omega) > 0$, then we choose any two disjoint set P and Q in Ω, and suppose their weights are y_P and y_Q respectively satisfying $y_P \leq y_Q$. Then we set $T = P \bigcup Q$, by the minimality of Ω, $T \notin \Omega$. We construct a new set class

$$\Omega' = \begin{cases} \Omega \bigcup \{T\} \setminus \{P\}, y_P < y_Q \\ \Omega \bigcup \{T\} \setminus \{P, Q\}, y_P = y_Q \end{cases}$$

Then Ω' must satisfy the following:

① Ω' is a balanced collection;
② Ω' is minimal;
③ $f(\Omega') < f(\Omega)$.

①We assign a weight y_p to T; If $Q \in \Omega'$, then assign another weight $y_Q - y_P$ to Q; Assign the other sets in Ω' with some corresponding weights in Ω, so that Ω' is a balanced collection.

② Suppose Ω' is not minimal, then by Corollary 2, we can get a minimal balanced collection $\vartheta \subset \Omega'$, such as $Q \in \vartheta$, and Ω being minimal, T must be an element in ϑ. We then construct a set class

$$\vartheta' = (\vartheta \setminus \{T\}) \bigcup \{P, Q\}$$

It is easy to show that $\vartheta' \subseteq \Omega$ and ϑ' is balanced. Thus we only need to transfer the weight of T in ϑ to P and Q. So by the minimality of Ω, we have $\vartheta' = \Omega$, therefore $\Omega \bigcup \{T\} \setminus \{P\} = \vartheta \bigcup \{Q\}$, and $Q \in \vartheta$, and we have

$$\vartheta = \begin{cases} \Omega \bigcup \{T\} \setminus \{P\}, \vartheta \in \Omega' \\ \Omega \bigcup \{T\} \setminus \{P, Q\}, \vartheta \notin \Omega' \end{cases}$$

It is not difficult to get $\vartheta' = \Omega$, a contradiction to the fact that $\vartheta \subset \Omega'$.

③ For any disjoint sets of unordered pairs in Ω', that is not in Ω, it must have the form $(R, T), R \in \Omega$. Corresponding there is a disjoint set of unordered pairs (R, P), in Ω that is not in Ω'. Furthermore, there is also a disjoint unordered pairs of sets (P, Q) in Ω but not in Ω', so $f(\Omega') < f(\Omega)$.

By ①~③ and the choice of Ω, we see that any balancing vector z of Ω' satisfies

$$\sum_{R \in \Omega'} z_R \underline{\omega}(R) \leq \bar{\omega}(N)$$

and by the definition of Ω', we have

$$\sum_{R \in \Omega'} z_R \underline{\omega}(R) \geq \sum_{R \in \Omega} y_R \underline{\omega}(R) + y_p \underline{\omega}(T) - y_p \underline{\omega}(Q) - y_p \underline{\omega}(P)$$

or interval cooperative games satisfy superadditivity about the interval left endpoint, we must have

$$y_p \underline{\omega}(T) - y_p \underline{\omega}(Q) - y_p \underline{\omega}(P) \geq 0$$

So

$$\sum_{R \in \Omega} y_R \underline{\omega}(R) \leq \sum_{R \in \Omega'} z_R \underline{\omega}(R) \leq \bar{\omega}(N)$$

This contradicts to our assumption that

$$\sum_{R \in \Omega} y_R \underline{\omega}(R) \leq \bar{\omega}(N)$$

So the proof concludes.

4. The Solution Method of S-core

We have already had a further discussion on the existence of S-core in Section 3 and have discussed the necessary and sufficient condition for its non-emptiness. But in reality, in order to solve the profit distribution problems by obtaining the S-core, we often need to know how to get it. The corresponding theorems are as follows.

Theorem 5. Let (N, ω) be an interval cooperative game, and $x \in R^n$. Then $x \in S\text{-}C(\omega)$ if and only if

$$\sum_{i \in R} x_i \geq \underline{\omega}(R), \forall R \subset N \quad (14)$$

$$\underline{\omega}(N) \leq \sum_{i \in N} x_i \leq \bar{\omega}(N) \quad (15)$$

Proof. If $x \in S\text{-}C(\omega)$, then $S\text{-}C(\omega) \neq \varnothing$, so there exists a classical cooperative game (N, v'), such that $x \in C(v')$, $v' \in Sel(\omega)$. It follows that

$$\sum_{i \in R} x_i \geq v'(R) \geq \underline{\omega}(R), \forall R \subset N$$

Then

$$\underline{\omega}(N) \leq \sum_{i \in N} x_i = v'(N) \leq \bar{\omega}(N)$$

The condition in (14) and (15) are obtained.

To prove the converse, suppose there exists $x \in R^n$ satisfying (14) and (15), then we can construct a classical cooperative game (N, v'')

$$v'' = \begin{cases} \underline{\omega}(S), & S \subset N \\ \sum_{i \in N} x_i, & S = N \end{cases}$$

clearly, by (14) and (15) we can show that $v''(S) \in \omega(S), \forall S \subseteq N$, thus $v'' \in Sel(\omega)$. We also have

$$\sum_{i \in R} x_i \geq \underline{\omega}(R) = v''(R), \forall R \subset N$$

$$\sum_{i \in N} x_i = v''(N)$$

by Definition 3, we have $x \in C(v'')$, consequently $x \in S\text{-}C(\omega)$. This completes the proof.

Remark 1. Remark that when fuzzy numbers are degenerated into interval numbers, if the $\alpha\text{-}core$ is also given as follows

$$\alpha\text{-}core(\omega) = \left\{ x \in R^n \mid \sum\nolimits_{i \in R} x_i \geq \underline{\omega}(R)(\forall R \subset N), \right.$$
$$\left. \underline{\omega}(N) \leq \sum_{i \in N} x_i \leq \bar{\omega}(N) \right\}$$

then S-core is same as the α- core given in [3], So Theorem 5 implies $S\text{-}C(\omega) = \alpha\text{-}core(\omega)$.

Observe that S-core is a bounded closed convex set in R^n, so when S-core is nonempty, it can be obtained by a suitably selected linear programming problem. For the properties of S-core, we give the notions of S-dominance and S-dominance core.

Definition 14. In an interval cooperative game (N, ω), suppose $x, y \in S\text{-}E(\omega)$, $\varnothing \neq R \subseteq N$, if the following two conditions are satisfied

$$x_i > y_i, \forall i \in R \tag{16}$$

$$\underline{\omega}(R) \geq \sum_{i \in R} x_i \tag{17}$$

then we call x S-dominance y via R.

If there exists $R \subseteq N$ satisfying x S-dominance y via R, then we call x (S-imputation) S-dominance y (S-imputation) via R.

Definition 15. Let (N, ω) be an interval cooperative game, S-dominance core $S\text{-}DC(\omega)$ consists of all S-imputations which are not S-dominated by any other S-imputations.

Theorem 6. In an interval cooperative game (N, ω), if $\underline{\omega}(S) + \sum_{i \in N \setminus S} \underline{\omega}(i) \leq \bar{\omega}(N), \forall S \subseteq N$, then we have

$$S\text{-}C(\omega) = S\text{-}DC(\omega) = \alpha\text{-}core(\omega)$$

Proof. It follows from Theorem 5 that the S-core is same as the $\alpha\text{-}core$, so we only need to prove that $S\text{-}DC(\omega) = \alpha\text{-}core(\omega)$.

Suppose $x \in R^n$ and $x \in \alpha\text{-}core(\omega)$, then $x \in S\text{-}C(\omega)$, so $x \in S\text{-}E(\omega)$. Suppose $x \notin S\text{-}DC(\omega)$, then there exists $R \subset N, R \neq \varnothing$, and $y \in S\text{-}E(\omega)$, satisfying y S-dominance x via R, that is

$$y_i > x_i, \forall i \in R$$

$$\underline{\omega}(R) \geq \sum_{i \in R} y_i$$

So we get

$$\underline{\omega}(R) \geq \sum_{i \in R} y_i > \sum_{i \in R} x_i$$

This contradicts (14), and therefore we can not have $x \in S\text{-}C(\omega)$, and therefore $x \in S\text{-}DC(\omega)$, we have $\alpha\text{-}core(\omega) \subseteq S\text{-}DC(\omega)$.

Suppose $x \in S\text{-}DC(\omega)$, then $x \in S\text{-}E(\omega)$, so

$$\underline{\omega}(N) \leq \sum_{i \in N} x_i \leq \bar{\omega}(N)$$

that is, (15) holds good. If (14) does not hold, then there exists $R \subseteq N$, such that

$$\sum_{i \in R} x_i < \underline{\omega}(R)$$

So $R \neq \varnothing, R \neq N$ and we set

$$\varepsilon = \frac{1}{|R|}[\underline{\omega}(R) - \sum_{i \in R} x_i] > 0$$

$$\delta = \frac{1}{n - |R|}[\bar{\omega}(N) - \underline{\omega}(R) - \sum_{i \in N \setminus R} \underline{\omega}(i)] \geq 0$$

we suppose a vector $y = (y_1, \cdots, y_n)$

$$y_i = \begin{cases} x_i + \varepsilon, i \in R \\ \underline{\omega}(i) + \delta, i \notin R \end{cases}$$

so $y_i \geq \underline{\omega}(i), \forall i \in N$, and

$$\sum_{i\in N} y_i = \bar{\omega}(N)$$

then we have $y \in S\text{-}E(\omega)$, for $y_i > x_i, \forall i \in R$,

$$\sum_{i\in R} y_i = \underline{\omega}(R)$$

by Definition 15, we know y S-dominance x via R, which contradicts (14), so we get $x \in \alpha\text{-}core(\omega)$, so $S\text{-}DC(\omega) \subseteq \alpha\text{-}core(\omega)$. Theorem 6 is proved.

Remark 2. For an interval cooperative game (N,ω), $S\text{-}DC(\omega) \neq \varnothing$ if and only if it is a weak balanced interval cooperative game, under the condition that

$$\underline{\omega}(S) + \sum_{i\in N\setminus S} \underline{\omega}(i) \le \bar{\omega}(N), \forall S \subseteq N$$

Remark 3. In an interval cooperative game (N,ω), we have $S\text{-}C(\omega) = \alpha\text{-}core(\omega) \subseteq S\text{-}DC(\omega)$.

Theorem 7. Suppose (N,ω) is an interval cooperative game, if $S\text{-}DC(\omega) \neq \varnothing$, the following two conditions are equivalent

1) $\underline{\omega}(S) + \sum_{i\in N\setminus S} \underline{\omega}(i) \le \bar{\omega}(N), \forall S \subseteq N$
2) $S\text{-}C(\omega) = S\text{-}DC(\omega)$

Proof. By Theorem 6, it follows that 1) implies 2).

For $S\text{-}DC(\omega) \neq \varnothing$, so if 2) is true, there exits $x \in S\text{-}C(\omega)$. Suppose 1) is not true, then there exists R, we have

$$\underline{\omega}(R) + \sum_{i\in N\setminus R} \underline{\omega}(i) > \bar{\omega}(N)$$

But

$$\underline{\omega}(N) \le \sum_{i\in N} x_i \le \bar{\omega}(N), \text{ and } \sum_{i\in R} x_i \ge \underline{\omega}(R)$$

So we have the following conclusion

$$\begin{aligned}\underline{\omega}(R) + \sum_{i\in N\setminus R} \underline{\omega}(i) &> \bar{\omega}(N) \ge \sum_{i\in N} x_i \\ &= \sum_{i\in R} x_i + \sum_{i\in N\setminus R} x_i \ge \underline{\omega}(R) + \sum_{i\in N\setminus R} x_i\end{aligned}$$

Thus we obtain that

$$\sum_{i\in N\setminus R} \underline{\omega}(i) > \sum_{i\in N\setminus R} x_i \ge \sum_{i\in N\setminus R} \underline{\omega}(i)$$

which is a contradiction. This completes the proof.

Remark 4. For an interval cooperative game (N,ω), if it is superadditive on the left endpoint of interval, then $S\text{-}DC(\omega) \neq \varnothing$ if and only if for each proper minimal balanced class $\zeta = \{R_1, R_2, \cdots, R_m\}$ and its unique balancing vector $y = (y_1, y_2, \cdots, y_m)$, (12) is satisfied.

Example 3. Let (N,ω) be an interval cooperative game. and N={1, 2, 3}, $\omega(1) = [1,2]$, $\omega(3)$=[3, 4], $\omega(2) = [2,3]$, $\omega(1,2) = [5,6]$, $\omega(1,3) = [6,7]$, $\omega(2,3) = [7,8]$, $\omega(1,2,3) = [8,9]$.

We can verify that the superadditivity on the left endpoint of interval is satisfied and we can have $5+6+7 \le 18$, so $S\text{-}DC(\omega) \neq \varnothing$, we can easily know that $(2,3,4) \in S\text{-}DC(\omega)$.

We also can easily see that the superadditivity on the left endpoint of (N,ω) can be weakened as $\underline{\omega}(R) + \underline{\omega}(T) \le \bar{\omega}(N)$ under the condition that $R \cap T = \varnothing, R \cup T = N$.

5. A Case-Based Example

Suppose there are three computer companies in a science and technology park, represented by the player set $N = \{1,2,3\}$. In order to quickly improve market competitiveness, they decide to join together to produce a new computer M. However, due to their mutual influence and restrictions, they have to cooperate with one another to achieve more profit, and the profit is always uncertain by many factors such as market volatility, incomplete information and so on. For example, company 1 is a new company with many new types of equipments but lack of some experiences, if it produces M alone, it will have no profit in the worst possible case, company 2 is innovative but has some disadvantage in financial support, if it produces M by itself, a worth of 2 million monetary unit will be gained in the best case, company 3 is a leading company with its own market and competitive advantage but is relatively conservative, if it produces M by itself, it will gain about 1million to 2 million monetary unit. After their negotiations and forecast, the estimated uncertain profits are shown in Table 5.1. We discuss on the basis of our model whether three companies are able to cooperate successfully, and how to give an effective and rational profit allocation scheme among three companies.

Table 1. The estimated profit of three companies

Company	1	2	3	1 , 2	1 , 3	2 , 3	1 , 2 , 3
Estimated Profit	[0,1]	[0, 2]	[1, 2]	[6, 7]	[7, 8]	[7, 8]	[8,12]

Thus we define an interval cooperative game as follows. $N=\{1,2,3\}$, $\omega(1)=[0,1]$, $\omega(2)=[0,2]$, $\omega(3)=[1,2]$, $\omega(1,2)=[6,7]$, $\omega(2,3)=[7,8]$, $\omega(1,3)=[7,8]$, $\omega(1,2,3)=[8,12]$. Here we can not apply the model discussed in [2] for (7) and (8) are not satisfied, However our model gives the necessary and sufficient condition for the existence of the S-core. Because (N,ω) satisfies the superadditivity about the interval left endpoint, and when $N=\{1,2,3\}$, it has only one proper minimal balanced collection such as $\{\{1,2\}\{1,3\}\{2,3\}\}$, thus

$$\underline{\omega}(1,2)+\underline{\omega}(1,3)+\underline{\omega}(2,3)=20<2\bar{\omega}(1,2,3)=24$$

So by Theorem 4, ω has a nonempty S-core. Next we will find the S-core.

By Theorem 5, the S-core is the feasible region of the following linear programming problem

$$\begin{cases} x_1\ge\underline{\omega}(1), x_2\ge\underline{\omega}(2), x_3\ge\underline{\omega}(3),\\ x_1+x_2\ge\underline{\omega}(1,2), x_1+x_3\ge\underline{\omega}(1,3), x_2+x_3\ge\underline{\omega}(2,3),\\ \underline{\omega}(1,2,3)\le x_1+x_2+x_3\le\bar{\omega}(1,2,3),\\ x_i\in R, i=1,2,3. \end{cases}$$

Then by Theorem 4, there must be

$$2(x_1+x_2+x_3)\ge\underline{\omega}(1,2)+\underline{\omega}(1,3)+\underline{\omega}(2,3)$$

So S-core is the feasible region of the following linear programming problem

$$\begin{cases} x_1\ge 0, x_2\ge 0, x_3\ge 1,\\ x_1+x_2\ge 6, x_1+x_3\ge 7, x_2+x_3\ge 7,\\ 10\le x_1+x_2+x_3\le 12,\\ x_i\in R, i=1,2,3. \end{cases}$$

Using Mathematica, the software we obtain the solution set which is a tetrahedron shown in Figure 1.

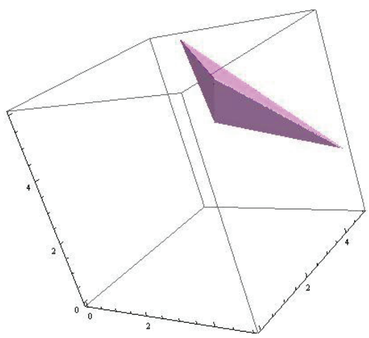

Fig. 1. Solution set space of S-core

The geometric center of the convex set(tetrahedron) is $\omega(x_1,x_2,x_3)=(3.5,3.5,4.5)$.

Remark 5. For a classical cooperative game (N,v), $N=\{1,2,3\}$, $v(R)=\underline{\omega}(R), R\subset N$

$$v(N)=\frac{10+\lambda\bar{\omega}(N)}{1+\lambda}$$

in which $\lambda=3$ (the reason we set 10 here but not $\underline{\omega}(N)$ in $v(N)$ as 10 is the smallest value to ensure S-core nonempty). Then the Shapely value[16] of (N,v) is $(3.5,3.5,4.5)$, it is the same as the geometric center of the above tetrahedron.

Note that we also have the

$$\underline{\omega}(S)+\sum_{i\in N\setminus S}\underline{\omega}(i)\le\bar{\omega}(N), \forall S\subseteq N$$

The S-core we get is also a S-dominance core, so any solution in the S-core can be a distribution scheme among the three companies. In this case study, three companies can cooperate successfully, and a rational way to distribute their shares in the proportion of 7:7:9 of the total profit.

6. Conclusions and Further Study

By studying the S-core of interval cooperative games, in this paper, we further discuss its related properties. Firstly, we propose the concepts of weak balanced and the minimal weak balanced interval cooperative games, prove the necessary and sufficient condition for the non-emptiness of the S-core and simplify the inequalities under the condition that the left endpoints satisfy superadditivity. Secondly, we discuss the whole solution set of the S-core, convert the solution method of S-core into the method of solving a linear programming problem, put forward the concept of a S-dominance core and prove the equivalent condition of S-core and S-dominance core. Finally, by a specific example, the rationality and validity of S-core are verified. Results show that the existence theorem and the solution method of the S-core have strong practicability and reasonableness, which can lay a theoretical foundation to get a reasonable interval distribution solution.

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