Extended hesitant fuzzy linguistic term sets and their aggregation in group decision making

Hai Wang

School of Economics and Management, Southeast University
Nanjing, Jiangsu 211189, China

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Abstract

Group decision making problems which organize a group of experts to evaluate a set of alternatives with respect to several criteria are commonly discussed recently. Hesitant fuzzy linguistic term sets, characterized by a set of consecutive linguistic terms, act as a new model for qualitative settings where experts think of several possible linguistic values or richer expressions than a single term. When evaluating an indicator, alternative or variable in group decision making, however, linguistic terms involved in an expression derived by the group may be not always consecutive. Therefore, we generalize hesitant fuzzy linguistic term sets by enabling any non-consecutive linguistic terms in them, and refer to as extended hesitant fuzzy linguistic term sets (EHFLTSs). EHFLTSs can be constructed by the union of hesitant fuzzy linguistic term sets given by individual expert. As owning more desirable mathematical properties, EHFLTSs are flexible for develop complex decision model. Some basic operation and envelop of EHFLTSs are defined and some mathematical properties are discussed as well. For the sake of application in group decision making, we develop two classes of aggregation operators for aggregating a set of EHFLTSs to suit the cases where weighting vectors take the form of real numbers and linguistic terms respectively. Then a new group decision making model is formed and corresponding processes for two distinct scenarios are developed. A practical application clarifies the rationality and advantages of the proposed technique.

Keywords: Group decision making; Hesitant fuzzy set; Hesitant fuzzy linguistic term set; Extended hesitant fuzzy linguistic term set; Aggregation operator.

1. Introduction

Decision making by individual and group of individuals, such as committees, governing bodies, juries, business partners, teams, and families, is referred to as individual decision making and group decision making (GDM), respectively. Group decision making is a type of participatory procedure in which multi decision makers (DMs) acting collectively, consider and evaluate alternative courses of action, and select among the alternatives a solution or solutions. Along with the increasing complexity of real world decision making problems, uncertainties are unavoidable but quite challenging to modeling. To cope with such uncertainties, several tools have been developed, such as evidential reasoning theories in probabilistic setting and fuzzy sets (Z-FSs) theories in imprecise or vague setting. The latter has been successfully applied to handle kinds of fuzzy information. Recently, some extensions and generalizations of fuzzy sets have been proposed to solve complex problems. These extensions can be concluded as the following two classes. Extensions of the first class are suitable for quantitative...
situations, such as type-2 fuzzy sets, type-n fuzzy sets, intuitionistic fuzzy sets or interval-value fuzzy sets, fuzzy multisets, hesitant fuzzy sets (HFSs) and generalized hesitant fuzzy sets. In many situations, however, the information can only be assessed in a qualitative form instead of a quantitative one. For example, when evaluating the degree of comfort of a car, “good”, “poor” may be used. Thus the second class focuses on fuzzy linguistic approaches regarding computing with words (CWW). Since Zadeh had presented the concept of linguistic variables, several linguistic models were extended, such as the linguistic model based on membership functions, the linguistic model based on type-2 fuzzy sets, the linguistic model based on ordinal scales, the linguistic 2-tuple model, the proportional 2-tuple model and so on.

As the base of linguistic computation, linguistic representation models are quite limited. Most of the existing techniques usually take use of single and very simple linguistic term to represent the information presented by experts. In practical, however, an expert may think of more than one term at the same time and look forward to a more complex linguistic term, instead of a single term, to represent his/her evaluations about problems defined under uncertainty. With this view, Rodriguez et al. presented the concept of hesitant fuzzy linguistic term set (HFLTS) by the idea of HFS in 2012. Based on a predefined linguistic term set, HFLTS is several consecutive ordered linguistic terms of the set. For example, the degree of comfort of a car is evaluated by the linguistic term set {“very good”, “good”, “poor”}. One’s evaluation may be “at least good”. The evaluation can be represented by a HFLTS {“very good”, “good”} and can be seen as a generalized linguistic term “very good” or “good”. It is obvious that HFLTS is a powerful tool for one expert to express his/her assessment by either a single linguistic term or complex linguistic terms.

Recently, Rodriguez et al. used HFLTSs to present a GDM approach dealing with comparative linguistic expressions. In this study, experts’ preferences are expressed by HFLTS. The envelope of each HFLTS, i.e. a linguistic interval, is calculated before aggregating experts’ preferences by proper operators. It is rational and accurate if the linguistic terms in HFLTSs are consecutive. However, it may be not suitable for some more complex GDM situations. In individual decision making, an expert’s evaluation, relative measure and absolute measure, is represented by one term or several consecutive terms according to his/her preference. When it comes to GDM, individual evaluations can be represented by HFLTSs, but the group’s evaluations cannot be always represented by them. Suppose a decision organization with three groups of experts is authorized to assess the satisfactory degree of an alternative with respect to a criterion by a linguistic term set (“very good”, “good”, “indifferent”, “bad”, “very bad”). In Group 1, some experts provide “very good” surely, others provide “good” without hesitancy, and thus the assessment can be represented by a HFLTS (“very good”, “good”). While in Group 2, some experts provide “bad” doubtless, others insist on at least “good”. Thus three linguistic terms are considered in this assessment, i.e. “very good”, “good” and “bad”. We can not use a HFLTS to represent it as linguistic terms are not consecutive here. Group 3 provides between “bad” and “good” consistently, which result to a HFLTS (“good”, “indifferent”, “bad”). An alternative resolution of this problem is that aggregating, by proper aggregating operators, as in Xu, the linguistic information within each group at first and then aggregating the resultant information among groups because we can not represent the assessment of Group 2 by any existing linguistic term set. Therefore, there are at least three steps of aggregations at different levels if multi criteria are taken into account in the problem. Elimination of the aggregation within group by considering all the possible linguistic terms is meaningful during the decision making process. Take the assessment of Group 3 for example, original information is “good”, “indifferent” or “bad”. If the linguistic averaging operator is used, the assessment may result to “indifferent”: if some weights are further considered, some virtual linguistic terms may be derived. Comparing to the original assessment, we lose some important information at the beginning of the decision making process. Another alternative resolution is the evidential reasoning algorithm if corresponding probabilities are provided associated with the linguistic terms. For example, if weights of experts in Group 1 are equal, one half of experts support “very good”, others support “good”, the assessment of Group 1 may be expressed as {“(very good”, 0.5), (“good”, 0.5), (“indifferent”, 0), (“bad”, 0), (“very bad”, 0)}. By the
Evidential reasoning algorithm, the overall assessment of the alternative is expressed by a probabilistic distribution as well. But in practical, we may just have hesitancy on some linguistic terms while not be sure about the probabilistic distribution. Further, ordinal terms are usually used as in this problem. But the evidential reasoning algorithm does not consider the ordinal relation of the linguistic terms at all. In conclusion, when evaluating the satisfactory degree by linguistic terms, the difficulty is not because we have some probabilistic distributions on the possible linguistic terms, but because we have a set of consecutive or nonconsecutive possible linguistic terms. It is useful to deal with all the possible linguistic terms rather than considering just an aggregation operator.

Therefore, in this study, we propose a new linguistic term model named EHFLTS motivated by the idea of HFSs for linguistic GDM setting. An EHFLTS is a subset of a pre-defined linguistic term set. The linguistic information involved in an EHFLTS is considered as a generalized linguistic term, referred to as extended hesitant fuzzy linguistic term (EHFLT). We develop some basic operations for EHFLTSs and some arithmetic operations for EHFLTs. Two classes of aggregation operators with distinct forms of weighting vector are also developed for fusing a set of EHFLTs. A linguistic GDM model based on the proposed EHFLTSs, associated with two specified processes, is presented for potential application. The main advantages of the proposed EHFLTSs are as follow. First, theoretically, EHFLTSs can represent linguistic assessments with consecutive and nonconsecutive linguistic terms, thus all the possible linguistic terms are taken into account without a pre-aggregation process in GDM. Comparing to existing linguistic decision making model, we eliminate at least one aggregation procedure. Second, the probabilistic distribution is not necessary when evaluating. Thus experts can express their evaluations with flexible forms while no extra work is needed. At last, as a generalization of Rodriguez’s HFLTSs, the proposed EHFLTSs own better mathematical properties. For example, the union, intersection and complement of EHFLTSs are closed.

To achieve it, the structure of the paper is as follows. Section 2 reviews some related preliminaries, such as some fuzzy linguistic models, HFSs and HFLTSs. In Section 3, EHFLTSs and EHFLTs are defined, some basic operations associated with their relationships are discussed, and comparison laws are developed as well. Section 4 presents the extension principle and some specific aggregation operators. Section 5 develops the EHFLTSs-based linguistic GDM model and specifies two processes in different scenarios. A practical application is presented in Section 6, as well as comparison with an existing method. Then Section 7 concludes the paper.

2. Preliminaries

Due to the proposal of utilizing HFSs to generalize traditional fuzzy linguistic label sets, this section is devoted to recall some preliminaries involved in fuzzy linguistic approach, HFSs and HFLTSs.

2.1. Fuzzy linguistic models

In many real-world situations, the use of linguistic information is very straightforward and suitable to express the satisfaction associated with an outcome and a state of nature. Fuzzy linguistic approaches are used to model the linguistic information and the fuzzy set theory is utilized to manage the uncertainties. For convenience, let $\hat{S} = \{s_0, s_1, \cdots, s_g\}$ be a finite and totally ordered discrete linguistic term set, where $g$ is a positive even integer, $s_i$ represents a possible value for a linguistic variable such that:

1. The set is ordered: $s_i \succeq s_j$ iff $i \geq j$;
2. The negation operator is defined: $\neg(s_i) = s_j$ such that $j = g - i$.

The cardinality of $\hat{S}$ is odd, and should be neither too small nor too rich.

Example 1. A set of seven linguistic terms, could be:

$\hat{S} = \{s_0 = \text{none}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{medium}, s_4 = \text{high}, s_5 = \text{very high}, s_6 = \text{perfect}\}$.

Further, Fig. 1 shows $\hat{S}$ with the syntax and semantics of the seven terms.

In decision making process, linguistic terms are usually used and computed directly. But it is not easy to define some intuitive operation laws for the above kind of linguistic term set. Therefore, Xu redefine the linguistic term set $\hat{S}$ by another form

$$\hat{S} = \{s_i \mid i = -t, \cdots, t\},$$

where $t$ is a positive even integer.
where $t$ is a positive integer, $s_i$ owns the following characteristics:

1. The set is ordered: $s_i > s_j$ iff $i > j$;
2. The negation operator is defined: $\text{neg}(s_i) = s_{-i}$, especially $\text{neg}(s_0) = s_0$.

**Example 2.** In this case, a set of seven linguistic terms, could be:

$$S = \{ s_{-3} = \text{very poor}, s_{-2} = \text{poor}, s_{-1} = \text{slightly poor}, s_0 = \text{fair}, s_1 = \text{slightly good}, s_2 = \text{good}, s_3 = \text{very good} \}.$$  

Obviously, $s_0$ represents an evaluation of “indifference”. Given a discrete term set with the latter form $S$, Xu further extended it to a continuous term set

$$\mathcal{S} = \{ s_u | \alpha \in [-q, q] \}$$

(2) to preserve all given information, where $q (q > t)$ is a sufficiently large positive integer. If $s_u \in \mathcal{S}$, as seen in Fig. 2, $s_u$ is called an original linguistic term; otherwise, $s_u$ is a virtual linguistic term. In general, original linguistic terms are used for experts to conduct evaluations, and virtual linguistic terms only appear in calculations. To accomplish processes of CW with this representation, the following operation laws are introduced.

**Definition 1.** Let $s_u, s_\beta \in \mathcal{S}$ be any two linguistic terms and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, then

1. $s_u \otimes s_\beta = s_{u+\beta}$;
2. $s_u \otimes s_\beta = s_{u\beta}$;
3. $\lambda s_{u\otimes \beta} = s_{\lambda u \lambda \beta}$;
4. $(s_u)^\lambda = s_{\alpha^\lambda}$.

By the operation laws in Definition 1, we can see the representation $S = \{ s_i | i = -t, \cdots, t \}$ is more in accord with actual situations than the former representation $\mathcal{S} = \{ s_i | i = 0, 1, \cdots, g \}$. In fact, let’s consider the two special linguistic term sets hereinafter. We have $s_{-2} \otimes s_1 = s_0$ in $S$ of Example 2, which means “poor” and “good” result to “fair”. While in $\mathcal{S}$ of Example 1, we have $s_1 \otimes s_3 = s_0$, which means “low” and “high” become “perfect”. However, it should be noted that the results of this symbolic computational model are usually virtual linguistic terms and thus may be out of the universe of discourse of the linguistic variable. In addition, this model does not use semantics or syntax which leads to the result not interpretable.

Serving as another accurate linguistic model, the linguistic 2-tuple model keeps a syntax and fuzzy semantics in its representation. Let $\mathcal{S} = \{ s_i | i = -t, \cdots, t \}$ be a linguistic term set and $\alpha \in [-t, t]$ a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to $\alpha$ is obtained by the following function:

$$\Delta: [-t, t] \rightarrow \mathcal{S} \times [-0.5, 0.5]$$

$$\Delta(\alpha) = (s_i, x), \quad \text{with} \quad \begin{cases} s_i = \text{round}(\alpha) \\ x = \alpha - i, x \in [-0.5, 0.5] \end{cases}$$

Let $(s_i, x)$ be a linguistic 2-tuple associated with $\mathcal{S} = \{ s_i | i = -t, \cdots, t \}$. There is always a function $\Delta^3$ such that, from a linguistic 2-tuple, it returns its equivalent numerical value $\alpha \in [-t, t]$. As can be seen in Fig. 2, the linguistic 2-tuple model keeps the fuzzy representation and syntax.
We can also find in Fig. 2 that there is an interesting relationship between the linguistic virtual model and the linguistic 2-tuple model (See Dong et al. 33 for detail). Given $S = \{s_i\}_{i=-t,\cdots,t}$, if $\alpha \in [-t, t]$, then there is a one to one mapping between virtual term $s_\alpha$ and 2-tuple $(s_\alpha, x)$, such that $\alpha = i + x$. As will be illustrated in Section 4.3, lower indices of virtual terms involved in this study are bounded in $[-t, t]$. We use the linguistic virtual model for the procedure of computing thereafter, and then translate the resultant virtual term into 2-tuple to show its linguistic representation.

2.2. HFSs and HFLTSs

Sometimes, it is difficult to determine the membership of an element into a fixed set and which may be caused by a doubt among a set of different values. For the sake of a better description of this situation, Torra introduced the concept of HFSs as a generalization of fuzzy sets as follow.

**Definition 2** 8. Let $X$ be a fixed set, then a hesitant fuzzy set (HFS) on $X$ in terms of a function $h$ is that when applied to $X$ returns a subset of $[0, 1]$.

Furthermore, given a set of fuzzy sets, a HFS could be defined in accordance with the union of their memberships.

**Definition 3** 8. Given a set of $N$ membership functions: $M = \{\gamma_1, \cdots, \gamma_N\}$, the HFS associated with $M$, that is $h_M$, is defined as follow:

$$h_M(x) = \bigcup_{\gamma \in M} \{\gamma(x)\}.$$  

For a given $x$ in $X$, $h_M(x)$ is a set of some values in $[0, 1]$, denoting the possible memberships of $x$ to a set. To be easily understood, Xia and Xu 34 expressed the HFS by a mathematical symbol $\{<x, h_M(x)> | x \in X\}$, called $h = h_M(x)$ a hesitant fuzzy element (HFE) and $H$ the set of all HFESs.

When decision information is represented by a collection of HFSs, it is necessary to introduce a function or mechanism to aggregate them for final decision making. Torra and Narukawa 35 proposed an extension principle which permits us to export operations on Z-FSs to T-HFSs as follow.

**Definition 4** 35. Let $\Theta$ be a function $\Theta:[0, 1]^N \rightarrow [0, 1]$. $H = \{h_1, h_2, \cdots, h_N\}$ be a set of HFSs on the reference set $X$. Then the extension of $\Theta$ on $H$ is defined for each $x \in X$ by:

$$\Theta_H(x) = \bigcup_{\substack{\gamma \in \Theta \{h_1(x) \circ h_2(x) \cdots h_N(x)\}}} \{\Theta(\gamma)\}.$$  

Bearing in mind the idea of fuzzy linguistic approaches and HFSs, Rodriguez et al. 19 presented the following concept of HFLTSs.

**Definition 5** 19. Let $S$ be a linguistic term set, then a HFLTS, $H_s$, is an ordered finite subset of consecutive linguistic terms of $S$.

Three basic operations of HFLTS are defined as follows.

**Definition 6** 19. Let $\tilde{H}_S$, $\hat{H}_S^1$ and $\hat{H}_S^2$ be three HFLTSs, then the following operations are defined:

1. Complement:

$$\tilde{H}_S^c = S - \tilde{H}_S = \{s_j | s_j \in S, s_j \notin \tilde{H}_S\};$$

2. Union:

$$\hat{H}_S^1 \cup \hat{H}_S^2 = \{s_j | s_j \in \hat{H}_S^1 \text{ or } s_j \in \hat{H}_S^2\};$$

3. Intersection:

$$\hat{H}_S^1 \cap \hat{H}_S^2 = \{s_j | s_j \in \hat{H}_S^1 \text{ and } s_j \in \hat{H}_S^2\}.$$ 

Note that the union of two HFLTSs may be not close. If the linguistic terms set $S$ in Example 2 is used, let $\hat{H}_S^1 = \{s_1, s_2, s_3\}$ and $\hat{H}_S^2 = \{s_4, s_5\}$, then the union of these two HFLTSs is $\{s_1, s_2, s_3, s_4, s_5\}$. Thus the result is not consecutive any more.

3. Extended hesitant fuzzy linguistic term sets

We present the concept of EHFLTSs as well as some basic operation in this section. Then some mathematical properties are discussed.

3.1. Definition of EHFLTSs

As discussed in Introduction, we may have a doubt among several possible linguistic terms when considering the degree of an alternative satisfying a certain criterion in GDM problems. In order to handle this kind of assessment in decision making process directly, instead of preparatory aggregation, we extend the concept of HFLTSs. Let’s begin with an example.
Example 3. Two groups of experts are authorized to evaluate a car company in isolation using the linguistic sets $S$ in Example 2. Caused by their different knowledge, experiences and backgrounds, experts of Group 1 provide “at least good”, some experts of Group 2 insist on “at least good” as well, while others argue “slightly poor”, they can’t persuade each other. The assessment of Group 1 can be expressed by a HFLTS as follows,

$$\tilde{H}_1(x) = \{s_2, s_3\}.$$  

But linguistic terms emerged from Group 2 are not consecutive subset of $S$, thus can’t be represented by HFLTS.

To accommodate this kind of uncertain circumstance, we present the following definition.

Definition 7. Let $S$ be a linguistic term set, then an ordered subset of linguistic terms of $S$, that is,

$$H_1(x) = \{s_j \mid s_j \in S \},$$  

is called an extended hesitant fuzzy linguistic term set (EHFLTS).

Given a linguistic term set $S = \{s_0, \ldots, s_p\}$, the empty EHFLTS and the full EHTLS for a linguistic variable, $x$, are as follows:

Empty EHFLTS: $H_0(x) = \{\}$;

Full EHFLTS: $H_p(x) = S$.

Similarly, the evaluation of Group 2 in Example 3 can be represented by an EHFLTS:

$$H_2(x) = \{s_1, s_2, s_3\}$$

It is clear that HFLTS is the special case of EHFLTS. In the process of information computing, as we will see hereinafter, virtual linguistic terms are used. Thus for the convenience information processing, an ordered finite subset of virtual linguistic terms set $\tilde{S}$ is referred to as EHFLTS as well.

We can see that, usually, an EHFLTS represents one complex evaluation with uncertainties. In this paper, the linguistic terms that appear in an EHFLTS are considered as a generalized linguistic term. Formally, let $X$ be a fixed set, for $x \in X$, $H_j(x)$ can be represented as the following 2-tuple:

$${\langle x, h_j(x) \rangle} = \mathbb{X}$$

where $h_j(x)$ is a set of $p$ linguistic terms, in $S$ (or $\tilde{S}$), i.e. $h_j(x) = \{s_1, \ldots, s_p\}$. For a given $x$, $h_j(x)$ is abbreviated as $h_j$, represents all possible linguistic terms, thus is referred to as extended hesitant fuzzy linguistic term (EHFLT). For simplicity, single linguistic term is considered as an EHFLT.

3.2. Basic operations

Similar to Rodriguez et al. 19, we define some basic concept and basic operations of EHFLTSs hereinafter.

For a given EHFLTS $H_j(x)$, its upper and lower bound are denoted by:

(1) upper bound: $H^+_j = \max \{s_i \mid s_i \in H_j \}$ and $s_i \leq s_j$ for any $i$;

(2) lower bound: $H^-_j = \min \{s_i \mid s_i \in H_j \}$ and $s_i \geq s_j$ for any $i$.

Obviously, $H^-_j$ and $H^+_j$ define an uncertain linguistic variable, which simultaneously form the envelope of the EHFLTS. We describe it in the following definition.

Definition 8. Given a EHFLTS $H_j(x)$, its envelope, $A_m(H_j)$, is defined by an uncertain linguistic variable represented by $[H^-_j, H^+_j]$.

Three basic operations are defined straightforward.

Definition 9. Given three EHFLTSs $H_j(x)$, $H^+_j(x)$ and $H^-_j(x)$ for given $x$ in $X$, then the following operations are defined:

(1) Complement:

$H^+_j(x) = S - H_j(x) = \{s_i \mid s_i \in S, s_i \notin H_j(x)\}$;

(2) Union:

$(H^+_j \cup H^+_k)(x) = \{s_i \mid s_i \in H^+_j(x) \text{ or } s_i \in H^+_k(x)\}$;

(3) Intersection:

$(H^+_j \cap H^+_k)(x) = \{s_i \mid s_i \in H^+_j(x) \text{ and } s_i \in H^+_k(x)\}$.

We can see that the results of complement, union and intersection are EHFLTSs as well. According to the union of EHFLTSs, we can draw the following conclusion.

Theorem 1. (Construction axiom) The union of HFLTSs results to EHFLTS.
Proof of Theorem 1 is omitted. We shall note that this construction axiom show potential application of the presented EHFLTSs. Individuals provide their evaluation information by HFLTSs. And then the group’s evaluation is formed by the union of these HFLTSs, which results to an EHFLTS. In this procedure, we keep all possible linguistic terms rather than preparatory aggregation. Furthermore, it is very useful when individual priorities are absolutely unknown, for example, in anonymous setting.

For the purpose of aggregation, we further define some arithmetic operations for EHFLTs based on virtual linguistic terms set $S$. Note that $S$ contains as many $s_i$ (original linguistic term or virtual linguistic term) as necessary.

Definition 10. Given three EHFLTs $h_1$, $h_2$, and $h_3$, let $\lambda \geq 0$, then:

1. $h_1 \oplus h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{s_i \oplus s_j\} \cup \{s_i\}$;
2. $h_1 \otimes h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{s_i \otimes s_j\} \cup \{s_i\}$;
3. $\lambda h_1 = \bigcup_{s_i \in h_1} \{\lambda s_i\} \cup \{s_i\}$;
4. $(h_1)_1 = \bigcup_{s_i \in h_1} \{s_i\}$;
5. $h_1 \lor h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{\max\{s_i, s_j\}\};$
6. $h_1 \land h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{\min\{s_i, s_j\}\}.

Example 4. Let $S$ be the linguistic term set in Example 2. $h_1 = \{s_1, s_2\}$, $h_2 = \{s_3, s_4, s_5\}$, then:

$h_1 \oplus h_2 = \{s_1 \oplus s_3, s_1 \oplus s_4, s_1 \oplus s_5, s_2 \oplus s_3, s_2 \oplus s_4, s_2 \oplus s_5\};$

$h_1 \otimes h_2 = \{s_1 \otimes s_3, s_1 \otimes s_4, s_1 \otimes s_5, s_2 \otimes s_3, s_2 \otimes s_4, s_2 \otimes s_5\};$

$\lambda h_1 = \{\lambda s_1, \lambda s_2\};$

$(h_1)_1 = \{s_1, s_2\};$

$h_1 \lor h_2 = \{\max\{s_1, s_3\}, \max\{s_1, s_4\}, \max\{s_1, s_5\}, \max\{s_2, s_3\}, \max\{s_2, s_4\}, \max\{s_2, s_5\}\};$

$h_1 \land h_2 = \{\min\{s_1, s_3\}, \min\{s_1, s_4\}, \min\{s_1, s_5\}, \min\{s_2, s_3\}, \min\{s_2, s_4\}, \min\{s_2, s_5\}\}.$

3.3. Properties

In this section, we will discuss some mathematical properties of the operations defined above.

Theorem 2. Let $H_1(x), H_1'(x), H_2(x)$ and $H_2'(x)$ be four EHFLTSs for given $x \in X$, then

1. Involutive: $(H_1'(x))' = H_2(x);$
2. Commutativity: $(H_1 \cup H_2'(x)) = (H_1' \cup H_2)(x), (H_1 \cap H_2'(x)) = (H_1' \cap H_2)(x);$
3. Associative:

$H_1'(x) \cup (H_2 \cup H_2'(x)) = ((H_1' \cup H_2)(x)) \cup H_2'(x).$

$H_2'(x) \cap (H_2 \cap H_2'(x)) = ((H_1' \cap H_2)(x)) \cap H_2'(x).$

(Distributive: $H_1'(x) \cap (H_2 \cup H_2'(x)) = ((H_1' \cap H_2)(x)) \cup (H_2 \cap H_2'(x)).$

$H_2'(x) \cup (H_2 \cap H_2'(x)) = ((H_1' \cap H_2)(x)) \cap (H_2 \cup H_2'(x)).$

We can prove Theorem 2 using the same method of Rodriguez et al. 19. Thus the procedure of proof is omitted.

Furthermore, we will present some properties of operations on EHFLTs, such as commutativity, associative and distributive. The commutativity is given at first.

Theorem 3. (Commutativity) Let $h_1$ and $h_2$ be two EHFLTs, then

1. $h_1 \oplus h_2 = h_2 \oplus h_1$;
2. $h_1 \otimes h_2 = h_2 \otimes h_1$;
3. $h_1 \lor h_2 = h_2 \lor h_1$;
4. $h_1 \land h_2 = h_2 \land h_1$.

Proof.

(1) $h_1 \oplus h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{s_i \oplus s_j\} \cup \{s_i\};$

(2) $h_1 \otimes h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{s_i \otimes s_j\} \cup \{s_i\};$

(3) $h_1 \wedge h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{\max\{s_i, s_j\}\};$

(4) $h_1 \wedge h_2 = \bigcup_{s_i \in h_1, s_j \in h_2} \{\min\{s_i, s_j\}\}.$

Theorem 4. (Associative) Let $h_1$, $h_2$, and $h_3$ be three EHFLTSs, then

1. $h_1 \oplus (h_2 \oplus h_3) = (h_1 \oplus h_2) \oplus h_3$;
2. $h_1 \otimes (h_2 \otimes h_3) = (h_1 \otimes h_2) \otimes h_3$;
3. $h_1 \lor (h_2 \lor h_3) = (h_1 \lor h_2) \lor h_3$;
4. $h_1 \land (h_2 \land h_3) = (h_1 \land h_2) \land h_3.$

Proof.
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(1) $H^i_1 \oplus (H^i_2 \oplus H^i_3) = H^i_2 \oplus \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \oplus s_j\}\right)$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \oplus s_j\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \oplus s_j\}$

$(H^i_2 \oplus H^i_3) \oplus H^i_3 = \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \oplus s_j\}\right) \oplus h^i_3$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \oplus s_j\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \oplus s_j\}$

(2) $H^i_1 \odot (H^i_2 \odot H^i_3) = H^i_2 \odot \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \odot s_j\}\right)

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \odot s_j\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \odot s_j\}$

$(H^i_2 \odot H^i_3) \odot H^i_3 = \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \odot s_j\}\right) \odot h^i_3$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \odot s_j\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{s_j \odot s_j\}$

(3) $H^i_1 \cup (H^i_2 \cup H^i_3) = H^i_2 \cup \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{\max \{s_j, s_j\}\}\right)

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\max \{s_j, s_j\}\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\max \{s_j, s_j\}\}$

$(H^i_2 \cup H^i_3) \cup H^i_3 = \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{\max \{s_j, s_j\}\}\right) \cup h^i_3$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\max \{s_j, s_j\}\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\max \{s_j, s_j\}\}$

(4) $H^i_1 \cap (H^i_2 \cap H^i_3) = H^i_2 \cap \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{\min \{s_j, s_j\}\}\right)

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\min \{s_j, s_j\}\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\min \{s_j, s_j\}\}$

$(H^i_2 \cap H^i_3) \cap H^i_3 = \left(\bigcup_{s_j \in \lambda, s_j \in \lambda} \{\min \{s_j, s_j\}\}\right) \cap h^i_3$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\min \{s_j, s_j\}\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\min \{s_j, s_j\}\}$

Theorem 5. (Distributive) Let $h^i_1$, $h^i_2$, and $h^i_3$ be three EHFTLs, moreover, $\lambda \geq 0$, then we have:

(1) $\lambda(h^i_1 \oplus h^i_2) = \lambda h^i_1 \oplus \lambda h^i_2$;

(2) $(h^i_1 \odot h^i_2)^4 = (h^i_1)^4 \odot (h^i_2)^4$.

Proof. According to (3) and (4) of Definition 10, we have:

$s_j \in h^i_2 \Leftrightarrow \lambda s_j \in \lambda h^i_2$, $s_j \in h^i_3 \Leftrightarrow (s_j)^4 \in (h^i_3)^4$.

(1) $\lambda(h^i_1 \oplus h^i_2) = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\lambda(s_j \oplus s_j)\}$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\lambda(s_j \oplus s_j)\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\lambda(s_j \oplus s_j)\}$

$\lambda h^i_2 \oplus \lambda h^i_2 = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\lambda(s_j \oplus s_j)\}$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\lambda(s_j \oplus s_j)\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{\lambda(s_j \oplus s_j)\}$

(2) $(h^i_1 \odot h^i_2)^4 = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{(s_j)^4 \odot (s_j)^4\}$

$= \bigcup_{s_j \in \lambda, s_j \in \lambda} \{(s_j)^4 \odot (s_j)^4\} = \bigcup_{s_j \in \lambda, s_j \in \lambda} \{(s_j)^4 \odot (s_j)^4\}$.

3.4 Comparison laws of EHFTLs

Rodriguez et al. utilized envelopes of HFTTs, interval linguistic terms, to compare information included in HFTTs. However, as linguistic terms in EHFTTs are not consecutive, envelopes are not accurate for comparison. In this study, we introduce some concepts to distinguish two EHFTTs.

Definition 11. Given an EHFLT $h_3$,

$$E(h_3) = \frac{1}{\# h_3} \left(\bigoplus_{s_j \in h_3} s_j\right)$$

is called the expected linguistic term of $h_3$, where $\# h_3$ is the number of linguistic terms, $s_j$, in $h_3$.

Definition 12. Given an EHFLT $h_3$, $s_j$ and $s_j$ are the smallest and the biggest linguistic terms of $h_3$, then $D(h_3) = (j - 1)/(2t + 1)$ is called the degree of hesitancy of $h_3$, where $2t + 1$ is the cardinality of linguistic term set $S$.

Expected linguistic term reflects the averaging linguistic term of an EHFLT, and degree of hesitancy represents the degree of uncertainty in evaluation. If the smallest and the biggest linguistic terms of $h_3$ are original linguistic terms in $S$, then $D(h_3) \in [0, 1]$. However, if at least one of the two comes from $\overline{S}$, the boundary of $D(h_3)$ does not hold any more. Using these definitions, we can compare two EHFTLs by the following method.

Definition 13. Given two EHFTLs $h_3$ and $h_3$, then

(1) if $E(h_3) < E(h_3)$, then $h_3$ is smaller than $h_3$, denoted by $h_3 < h_3$.
we present some specific aggregation operators for term sets to EHFLTs in this section. Based on which, we may need to aggregate a set of EHFLTs to obtain the expected linguistic terms from \( S \). Herrera and Herrera-Viedma provided the linguistic weighted disjunction (LWD) operators in the setting values and weights of objects are represented by simple linguistic terms. As the extension of the LWD operator, we define the following operator using Definition 14.

Definition 15. Let \( \{ h'_j \} , j=1,2,\cdots,n \) be a set of EHFLTs, \( \omega=(a_1,\cdots,a_n) \) be the weighting vector of \( \{ h'_j \} , a_j \in S \) for all \( j \). A mapping EHFLWD: \( S^n \rightarrow S \) is called an extended hesitant fuzzy linguistic weighted disjunction (EHFLWD) operator of dimension \( n \) if

\[
EHFLWD(h'_1,h'_2,\cdots,h'_n) = \bigvee_{j=1}^n (a_j \land h'_j)
\]

\[
= \bigvee_{j=1}^n \left( \bigwedge_{s_{a_j} < h'_j} \min \{ a_j, s_{a_j} \} \right)
\]

\[
= \bigvee_{j=1}^n \left( \bigwedge_{s_{a_j} < h'_j} \max \{ \min \{ a_j, s_{a_j} \} \} \right).
\]

If \( \omega=(s_1,\cdots,s_n) \), where \( s_j \) is the largest linguistic term in (1), then \( a_j \land h'_j = h'_j \), therefore,

\[
EHFLWD(h'_1,h'_2,\cdots,h'_n) = \bigvee_{j=1}^n h'_j.
\]

However, \( \bigvee_{j=1}^n h'_j \neq \max \{ h'_j \} \) in general.

The ordered weighted averaging (OWA) operator \(^{37}\) provides an aggregation strategy to lie between the max and min operators because of its re-ordering step. In linguistic setting, Yager \(^{38}\) presented an ordinal form of the OWA operator. Motivated by which, we extend the OWA operator to the extended hesitant fuzzy linguistic setting.

Definition 16. Let \( \{ h'_j \} , j=1,2,\cdots,n \) be a set of EHFLTs. An extended hesitant fuzzy ordinal OWA (EHFOOWA) operator of dimension \( n \) is a mapping EHFOOWA: \( S^n \rightarrow S \), which has associated with a
linguistic weighting vector \( w = (w_j, \cdots, w_n) \) with \( w_j \in S \) for all \( j \) such that
\[
EHFOOWA(h^j_1, h^j_2, \cdots, h^j_n) = \bigvee_{j=1}^n (w_j \land h^j_S(\omega))
\]
\[
= \bigvee_{j=1}^n \left( \bigwedge_{s_{a_j(j)}\in\omega_j} \min\{w_j, s_{a_j(j)}\} \right)
\]
\[
= \bigwedge_{s_{a_j(j)}\in\omega_j} \left\{ \max\{\min\{w_j, s_{a_j(j)}\}\} \right\},
\]
where \( h^j_S(\omega) \) is the \( j \)th largest of the linguistic weighted arguments \( h^j_S = \omega_j \land h^j_\omega \), \( \omega = (\omega_1, \cdots, \omega_n) \) be the weighting vector of \( \{h^j_1, \cdots, h^j_n\} \).

Especially, if \( w = (s_1, s_2, \cdots, s_n) \), then \( w_j \land h^j_S = h^j_S(\omega) \) holds for any \( h^j_S(\omega) \),
\[
EHFOOHA(h^j_1, h^j_2, \cdots, h^j_n) = \bigvee_{j=1}^n (w_j \land h^j_S(\omega))
\]
\[
= \bigvee_{j=1}^n \left( \bigwedge_{s_{a_j(j)}\in\omega_j} \min\{w_j, s_{a_j(j)}\} \right)
\]
\[
= \bigwedge_{s_{a_j(j)}\in\omega_j} \left\{ \max\{\min\{w_j, s_{a_j(j)}\}\} \right\},
\]
thus the EHFOHA operator is reduced to the EHFLWA operator. If \( \omega = (s_1, s_2, \cdots, s_n) \), then for any \( h^j_1, \cdots, h^j_n \), \( \omega \land h^j_1 = h^j_S \) holds, then
\[
EHFOOA(h^j_1, h^j_2, \cdots, h^j_n) = \bigvee_{j=1}^n (w_j \land h^j_S(\omega))
\]
\[
= \bigvee_{j=1}^n \left( \bigwedge_{s_{a_j(j)}\in\omega_j} \min\{w_j, s_{a_j(j)}\} \right)
\]
\[
= \bigwedge_{s_{a_j(j)}\in\omega_j} \left\{ \max\{\min\{w_j, s_{a_j(j)}\}\} \right\},
\]
thus the EHFOOA operator is reduced to the EHFOOWA operator.

4.2. Aggregation operators with numerical weights

Except for linguistic weights, numerical weights are often used in application as well. In this case, several aggregation operators are developed in linguistic setting, such as the linguistic weighted averaging (LWA) operator \(^{40}\), the linguistic OWA operator \(^{40}\), the linguistic hybrid aggregation (LHA) operator \(^{41}\), the induced linguistic OWA operator \(^{40} \) and so on. Based on the extension principle and some existing linguistic aggregation operator, we define some new aggregation operators as follows.

Definition 17. Let \( \{h^j_1\}, j = 1, 2, \cdots, n \) be a set of EHFLTs. An extended hesitant fuzzy ordinal hybrid aggregation (EHFOHA) operator of dimension \( n \) is a mapping \( EHFOHA: S^n \to S \), which has associated with a linguistic weighting vector \( w = (w_1, \cdots, w_n) \) with \( w_j \in S \) for all \( j \) such that
\[
EHFOOHA(h^j_1, h^j_2, \cdots, h^j_n) = \bigvee_{j=1}^n (w_j \land h^j_S(\omega))
\]
\[
= \bigvee_{j=1}^n \left( \bigwedge_{s_{a_j(j)}\in\omega_j} \min\{w_j, s_{a_j(j)}\} \right)
\]
\[
= \bigwedge_{s_{a_j(j)}\in\omega_j} \left\{ \max\{\min\{w_j, s_{a_j(j)}\}\} \right\},
\]
where \( h^j_S(\omega) \) is the \( j \)th largest of the linguistic weighted arguments \( h^j_\omega = \omega_j \land h^j_\omega \), \( \omega = (\omega_1, \cdots, \omega_n) \) be the weighting vector of \( \{h^j_1, \cdots, h^j_n\} \).

Definition 18. Let \( \{h^j_1\}, j = 1, 2, \cdots, n \) be a set of EHFLTs. \( \omega = (\omega_1, \cdots, \omega_n) \) be the weighting vector of \( \{h^j_1\} \) such that \( \sum \omega_j = 1 \), \( \omega_j \in [0, 1] \) for all \( j \). A mapping \( EHFLWA: S^n \to S \) is called an extended hesitant fuzzy linguistic weighted averaging (EHFLWA) operator of dimension \( n \) if
\[
EHFLWA(h^j_1, h^j_2, \cdots, h^j_n) = \bigoplus_{j=1}^n (\omega_j h^j_\omega)
\]
\[
= \bigoplus_{j=1}^n \left( \bigwedge_{s_{a_j(j)}\in\omega_j} \min\{w_j, s_{a_j(j)}\} \right)
\]
\[
= \bigwedge_{s_{a_j(j)}\in\omega_j} \left\{ \max\{\min\{w_j, s_{a_j(j)}\}\} \right\}.\]
The EHFLWA operator extend both the weighted averaging (WA) operator and the LWA operator. Especially, if \( \omega = (1/n, 1/n, \ldots, 1/n) \), the EHFLWA operator is reduced to the extended hesitant fuzzy linguistic averaging (EHFLA) operator:

\[
EHFLWA(h^1_k, h^2_k, \ldots, h^n_k) = \sum_{j=1}^{n} w_j \sigma_j \alpha_j \{s_{\sigma_j}\},
\]

where \( \tilde{\alpha} = \sum_{j=1}^{n} \omega_j \alpha_j \).

The fundamental aspect of the LWA2 operator is that it reorders the input arguments based on their importance. If \( (0, 1, 0, \ldots, 0) \), the EHFLWA operator is reduced to the extended hesitant fuzzy linguistic averaging (EHFLA) operator:

\[
EHFLWA(h^1_k, h^2_k, \ldots, h^n_k) = \sum_{j=1}^{n} w_j \sigma_j \alpha_j \{s_{\sigma_j}\},
\]

where \( \tilde{\sigma}_j = \sum_{j=1}^{n} w_j \sigma_j \).

Especially if \( (1, 0, 1, \ldots, 0) \), then \( \tilde{\sigma}_j = \max\{\sigma_j\} \).

The EHFLWA operator is reduced to the EHFLM1 operator. Similarly, if \( (0, \ldots, 0, 1) \),

\[
EHFLWA(h^1_k, h^2_k, \ldots, h^n_k) = \sum_{j=1}^{n} w_j \sigma_j \alpha_j \{s_{\sigma_j}\}.
\]

The EHFLWA operator is reduced to the EHFLM2 operator.

The fundamental aspect of the EHFLWA operator is that it reorders the input arguments based on their weights. From Definitions 18 and 19, it is clear that the EHFLWA operator weights the input EHFLTs, while the EHFLMA operator weights the ordered position instead. The weights represent distinct aspects of inputs in these two operators. We present the following definition to overcome this drawback.

**Definition 20.** Let \( \{h^j_k\}, \ j = 1, 2, \ldots, n \) be a set of EHFLTs. An extended hesitant fuzzy linguistic hybrid aggregation (EHFLHA) operator of dimension \( n \) is a mapping \( \text{EHFLHA}: \mathbb{S}^n \rightarrow \mathbb{S} \), which has associated with a linguistic weighting vector \( w = (w_1, \ldots, w_n) \) with \( \sum_{j=1}^{n} w_j = 1, \ w_j \in [0, 1] \) for all \( j \), such that

\[
EHFLHA(h^1_k, h^2_k, \ldots, h^n_k) = \sum_{j=1}^{n} w_j \beta_j \{h^j_k\},
\]

where \( \tilde{\beta} = \sum_{j=1}^{n} \beta_j \), \( h^j_k \) is the \( j \)th largest of the \( h^j_k \).

Especially, if \( w = (1/n, 1/n, \ldots, 1/n) \), according to (1) of Theorem 3 and (1) of Theorem 4,

\[
EHFLHA(h^1_k, h^2_k, \ldots, h^n_k) = \sum_{j=1}^{n} \frac{1}{n} h^j_k.
\]

then the EHFLHA operator is reduced to the EHFLM1 operator. If \( \omega = (1/n, 1/n, \ldots, 1/n) \), then \( \tilde{\omega} = 1 \), \( \omega_j \in [0, 1] \) for all \( j \), and \( \alpha_j \) is the balancing coefficient.
4.3. Properties

We will discuss some properties of the presented aggregation operators in this subsection. Because of the operation “\( \bigcup \)”, most of the operators do not possess excellent mathematical properties, such as monotonicity, idempotency, commutativity and boundary. But, luckily, we will see some operators own properties like these four.

Theorem 6. Let \( \{h'_i\} \) and \( \{h''_i\} \), \( j = 1, 2, \ldots, n \), be two sets of EHFLTs. If \( \exists \hat{i} \in \{1, \ldots, n\} \) satisfies \# \( h''_\hat{i} = \# \hat{h}''_\hat{i} = N' \), and \( \forall s_{\hat{i}} \in h''_\hat{i}, \; s_{\hat{i}} \in \hat{h}''_\hat{i}, \; s_{\hat{i}} \leq s_{\hat{i}} \) holds. For \( j = 1, \ldots, n \), \( j \neq \hat{i} \), \( h''_j = \hat{h}''_j \). Then

\[
\text{EHFLWD}(h''_1, h''_2, \ldots, h''_n) \leq \text{EHFLWD}(\hat{h}''_1, \hat{h}''_2, \ldots, \hat{h}''_n).
\]

Proof. Since \( s_{\hat{i}} \leq s_{\hat{i}} \), then \( \alpha \wedge s_{\hat{i}} \leq \alpha \wedge s_{\hat{i}} \) for any \( \alpha \in S \). Then

\[
\bigcup_{\alpha \in S} \{ \alpha \wedge s_{\hat{i}} \} \leq \bigcup_{\alpha \in S} \{ \alpha \wedge s_{\hat{i}} \}
\]

which means \( \alpha \wedge h''_\hat{i} \leq \alpha \wedge h''_\hat{i} \). Thus

\[
\text{EHFLWD}(h''_1, h''_2, \ldots, h''_n) = (\alpha \wedge h''_\hat{i}) \lor \cdots \lor (\alpha \wedge h''_n) \leq (\alpha \wedge \hat{h}''_\hat{i}) \lor \cdots \lor (\alpha \wedge \hat{h}''_n) = \text{EHFLWD}(\hat{h}''_1, \hat{h}''_2, \ldots, \hat{h}''_n).
\]

Theorem 7. (Quasi-Boundary) Let \( \{h'_i\} \), \( j = 1, 2, \ldots, n \), be a set of EHFLTs. Then

\[
s_{\hat{\alpha}} \leq \text{EHFLWD}(h''_1, h''_2, \ldots, h''_n) \leq s_{\hat{\alpha}},
\]

where \( s_{\hat{\alpha}} = \min_j \left\{ \min \{ \alpha_j \wedge s_{\hat{i}} \} \right\} \),

\[
s_{\hat{\alpha}} = \max_j \left\{ \max \{ \alpha_j \wedge s_{\hat{i}} \} \right\}.
\]

Proof. For any \( s_{\hat{i}} \in h''_1, \ldots, s_{\hat{i}} \in h''_n \), we have

\[
\min_j \left\{ \min \{ \alpha_j \wedge s_{\hat{i}} \} \right\} \leq \max_j \left\{ \min \{ \alpha_j \wedge s_{\hat{i}} \} \right\} \leq \max_j \left\{ \max \{ \alpha_j \wedge s_{\hat{i}} \} \right\}.
\]

Then

\[
s_{\hat{\alpha}} \leq \text{EHFLWD}(h''_1, h''_2, \ldots, h''_n) \leq s_{\hat{\alpha}}.
\]

Similarly, using the same approach, we can easily proof the following theorem.

Theorem 8. (Quasi-Boundary) Let \( \{h'_j\} \), \( j = 1, 2, \ldots, n \), be a set of EHFLTs. Then

\[
s_{\hat{\alpha}} \leq \text{EHFOOWA}(h''_1, h''_2, \ldots, h''_n) \leq s_{\hat{\alpha}},
\]

where \( s_{\hat{\alpha}} = \min_j \left\{ \min \{ w_j \wedge \min \{ \alpha_j \wedge s_{\hat{i}} \} \} \right\} \),

\[
s_{\hat{\alpha}} = \max_j \left\{ \max \{ w_j \wedge \max \{ \alpha_j \wedge s_{\hat{i}} \} \} \right\}.
\]

Theorem 9. (Commutativity) Let \( \{h'_j\} \), \( j = 1, 2, \ldots, n \), be a set of EHFLTs. Then

(1) \( \text{EHFOOWA}(h''_1, h''_2, \ldots, h''_n) = \text{EHFOOWA}(h''_2, h''_3, \ldots, h''_n) \),

(2) \( \text{EHFLOWA}(h''_1, h''_2, \ldots, h''_n) = \text{EHFLOWA}(h''_2, h''_3, \ldots, h''_n) \),

where \( h''_1, h''_2, \ldots, h''_n \) is any permutation of \( h''_1, h''_2, \ldots, h''_n \).

Proof. Since \( h''_1, h''_2, \ldots, h''_n \) is any permutation of \( h''_1, h''_2, \ldots, h''_n \), then \( h''_n(\hat{j}) = \hat{h}''_n(\hat{j}) \) for all \( \hat{j} \).

(1) \( \text{EHFOOWA}(h''_1, h''_2, \ldots, h''_n) = \sum_{j=1}^{n} (w_j \wedge h''_n(\hat{j})) = \sum_{j=1}^{n} (w_j \wedge \hat{h}''_n(\hat{j})) = \text{EHFOOWA}(h''_2, h''_3, \ldots, h''_n) \);

(2) \( \text{EHFLOWA}(h''_1, h''_2, \ldots, h''_n) = \sum_{j=1}^{n} (w_j \wedge \hat{h}''_n(\hat{j})) = \text{EHFLOWA}(h''_2, h''_3, \ldots, h''_n) \).

Theorem 10. (Quasi-Idempotency) Let \( h'_\alpha \) be an EHFLT. Then

\[
\text{EHFLWA}(h'_\alpha, h'_\alpha, \ldots, h'_\alpha) = h'_\alpha.
\]

Proof. Suppose \# \( h'_\alpha = N \), \( h'_\alpha = \{ s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_n} \} \), then

\[
E(h'_\alpha) = \frac{1}{N} \sum_{\beta_1}^{n} s_{\beta_1} \leq \frac{1}{N} \sum_{\beta_1}^{n} s_{\beta_1}.
\]

First, we prove

\[
E\left( \text{EHFLWA}(h'_\alpha, h'_\alpha, \ldots, h'_\alpha) \right) = E(h'_\alpha)
\]

by using mathematical induction on \( n \).

(1) When \( n = 2 \),

\[
\text{EHFLWA}(h'_\alpha, h'_\alpha) = \alpha h'_\alpha \oplus \alpha h'_\alpha
\]

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For all the possible values of $\alpha_1$ and $\alpha_2$, the derived indices of virtual linguistic term $s$ are listed as follows. (All the possible values of $\alpha_2$ are listed in the first column, while all the possible values of $\alpha_1$ are listed in the first row.)

$$\begin{align*}
\beta_1 & \quad \beta_2 & \quad \cdots & \quad \beta_n \\
\beta_1 & \quad \alpha_2 \beta_1 & \quad \alpha_2 \beta_2 & \quad \cdots & \quad \alpha_2 \beta_n \\
\vdots & \quad \vdots & \quad \vdots & \quad \ddots & \quad \vdots \\
\beta_n & \quad \alpha_1 \beta_n & \quad \alpha_1 \beta_1 & \quad \cdots & \quad \alpha_1 \beta_n
\end{align*}$$

The sum of the indices in the bracket is:

$$N\alpha_1 \sum_{j=1}^{N} \beta_j + N\alpha_2 \sum_{j=1}^{N} \beta_j = N \sum_{j=1}^{N} \beta_j.$$ 

Thus

$$E(\text{EHFLWA}(h_2, h_3)) = s \frac{1}{N} \sum_{j=1}^{N} \beta_j = s \frac{1}{N} \sum_{j=1}^{N} \beta_j = E(h_3).$$

(2) If $E(\text{EHFLWA}(h_3, h_3, \cdots, h_3)) = E(h_3)$ hold for $n = k$, that is

$$E(\alpha h_k \oplus \cdots \oplus \alpha h_k) = E(h_k),$$

where $\sum_{j=1}^{k} \alpha_j = 1$.

Then when $n = k+1$, according to some trivial computing similar to the case $n = 2$, we can get

$$E(\alpha h_k \oplus \cdots \oplus \alpha h_k \oplus \alpha h_{k+1}) = E(h_{k+1}),$$

where $\sum_{j=1}^{k+1} \alpha_j = 1$.

Second, we prove $D(\text{EHFLWA}(h_3, h_3, \cdots, h_3)) = D(h_3)$ in the following. The max and min linguistic terms in $h_3$ are denoted by $s_\omega$ and $s_\alpha$, then $D(h_3) = (\alpha - \omega)/(2t+1)$. The max virtual linguistic term in $\text{EHFLWA}(h_3, h_3, \cdots, h_3)$ is derived by $s_{\omega} = \cdots = s_{\omega} = s_{\omega}$, the max virtual linguistic term in $\text{EHFLWA}(h_3, h_3, \cdots, h_3)$ is derived by $s_{\omega} = \cdots = s_{\omega} = s_{\omega}$, then $\alpha_1 s_{\omega} \oplus \cdots \oplus \alpha_1 s_{\omega} = s_{\omega}$. Thus

$$D(\text{EHFLWA}(h_3, h_3, \cdots, h_3)) = (\alpha - \omega)/(2t+1) = D(h_3).$$

According to Definition 13,

$$E(\text{EHFLWA}(h_2, h_3, \cdots, h_3)) \equiv h_3.$$ 

**Theorem 11.** (Quasi-Boundary) Let $\{h^j_1\}, j = 1, 2, \cdots, n$, be a set of EHFLTs. Then

$$s_{\omega, \beta} \leq \text{EHFLWA}(h^j_1, h^j_1, h^j_2, \cdots, h^j_n) \leq s_{\omega},$$

where $s_{\omega, \beta} = \min_j \left\{ \min_{\alpha_1, \alpha_2} \left\{ s_{\alpha, \beta} \right\} \right\}$, $s_{\omega} = \max_j \left\{ \max_{\alpha_1, \alpha_2} \left\{ s_{\alpha, \beta} \right\} \right\}$.

**Proof.** Since

$$s_{\omega, \beta} = \min_j \left\{ \min_{\alpha_1, \alpha_2} \left\{ s_{\alpha, \beta} \right\} \right\} \leq s_{\alpha} \leq \max_j \left\{ \max_{\alpha_1, \alpha_2} \left\{ s_{\alpha, \beta} \right\} \right\} = s_{\omega},$$

which holds for $i = 1, 2, \cdots, n$, then

$$s_{\omega, \beta} \leq \alpha_1 s_{\omega} \leq s_{\omega},$$

This leads to

$$s_{\omega, \beta} \leq \text{EHFLWA}(h^j_1, h^j_1, h^j_2, \cdots, h^j_n) \leq s_{\omega}.$$ 

Thus $s_{\omega, \beta} \leq \text{EHFLWA}(h^j_1, h^j_1, h^j_2, \cdots, h^j_n) \leq s_{\omega}$. 

**5. Group decision making with EHFLTs**

In this section, we focus on GDM problems with the application of the proposed EHFLTs and EHFLTs. We describe the problem mathematically at first. Then, as a solution, a new linguistic GDM model is presented. Corresponding procedures are specified for some special scenarios.

**5.1. Problem description**

As can be seen in literatures 43-45, practical decision making problems with large scales of evaluations are usually done by a decision organization with several groups of experts instead of one group. A rational way to form the organization is that experts in a group have similar knowledge and experiences while experts in different group own different professional area. This organization can accumulate the advantages and alleviate the negative effects of a group (See Kang et al. for detail). Therefore, in this section, as the application of the proposed techniques, we focus on this kind of GDM problems, which is described mathematically as follow.

A decision organization is formed by $L$ groups (denoting by $G(l=1, 2, \cdots, L)$) of $K_l$ experts denoted by $E = \{\epsilon_k^l | k = 1, 2, \cdots, K_l, l = 1, 2, \cdots, L\}$. The weights of $L$ groups are $\alpha^{(G)} = (\alpha^{(G)}_1, \alpha^{(G)}_2, \cdots, \alpha^{(G)}_L)^T$. Usually, numbers of experts in each group are between 2 and 7, i.e., $2 \leq K_l \leq 7$. The relative weights of experts within a...
group are indifferent, because of their similar backgrounds, or because of anonymity. The organization is authorized to evaluate a set of alternatives \( A = \{a_1, a_2, \ldots, a_m\} \) in terms of a set of criteria \( C = \{c_1, c_2, \ldots, c_N\} \). The weights of criteria are \( \omega^{(c)} = (\omega_1^{(c)}, \omega_2^{(c)}, \ldots, \omega_N^{(c)})^T \). Group \( G_i \) evaluates a subset of \( C \) in isolation, denoted by \( SC_i \), such that \( SC_i \neq \phi \), \( l = 1, 2, \ldots, L \) and \( \bigcup_{l=1,2,\ldots,L} SC_i = C \). The satisfaction degree of \( a_n \), evaluated by \( e_{nhac} \), with respect to criterion \( c_n \), is represented by a function \( h: AXEC \times C \rightarrow S \) in linguistic setting. In this paper, the function \( h \) can be expressed by HFLTSs and EHFLTSs, thus are generally denoted by EHFLTSs. Practically, experts may prefer to use linguistic term sets with distinct cardinalities. Therefore, we need to unify the multi-granular linguistic information in a unique linguistic term set before starting the process of decision making. An rational alternative is the transformation function proposed in Xu \(^{47}\). Suppose that the linguistic expression representing the satisfaction degree of \( a_n \), evaluated by \( e_{nhac} \), with respect to criterion \( c_n \), is transformed to a HFLTS \( h^k(a_n,c_n) \), where \( k = 1, 2, \ldots, K \), \( l = 1, 2, \ldots, L \), \( m = 1, 2, \ldots, M \), \( n = 1, 2, \ldots, N \). The aim of the problem is to synthesize evaluation functions of each alternative and then result to a final decision.

5.2. Group decision making model

As seen in Fig. 3, the proposed linguistic group decision making model includes three parts.

Part 1 Structure of groups in the organization. Experts in each group are homogeneous while experts in different groups are heterogeneous. Each group deals with a part of evaluations according to its knowledge and specialty. Individuals of a group work in isolation if possible. The whole organization works collectively to complete the entire evaluations.

Part 2 Evaluation and transformation. Given a linguistic term set \( S \), HFLTSs can be directly used by the experts to elicit several linguistic values for a linguistic variable when experts hesitate among several values. However, such elements are not similar to human beings’ way of thinking and reasoning. Therefore, Rodriguez \(^{19}\) defined a context-free grammar to generate linguistic expressions that are more similar to human beings’ expressions. Then the linguistic expressions provided by experts are transformed into HFLTS by using a transformation function (See Rodriguez \(^{19}\) for detail). According to way of individual thinking in fuzzy uncertain circumstance and the proposed construction axiom, in this model, individual evaluations are represented by linguistic expressions similar to human beings’ way of thinking and reasoning and then transformed to HFLTSs.

Part 3 Synthesis for decision making. In this phase, two tools are used for synthesis. Union is used to transform HFLTSs to a generalized case, EHFLTSs. Aggregation operators are used to synthesize opinions represented by EHFLTSs. The most difference between union and aggregation operator is that all the original information are kept when the former is used while averaging value is obtained by some means when an aggregation operator is used. Thus the proposed model can reduce the use of aggregation operators and eliminate loss of information.

5.3. Two group decision making processes

In this section, we specify the proposed GDM model in two different scenarios with different types of weights. For simplicity, we suppose that every group is authorized to evaluate alternatives with respect to the whole set of criteria.

Scenario 1 Weights take the form of linguistic terms.

Suppose \( \omega_j^{(c)} \in S, l = 1, 2, \ldots, L \) and \( \omega_j^{(c)} \in S, j = 1, 2, \ldots, N \).

Step 1: Union within each group. The evaluation information of \( a_n \), with respect to criterion \( c_n \), provided by group \( G_i \), denoted by \( h'_i(a_n, c_n) \), is derived by

\[
\bigcup_m h'_i(a_n, c_n) = \bigcup_m h^k(a_n, c_n),
\]

where \( l = 1, 2, \ldots, L, m = 1, 2, \ldots, M, n = 1, 2, \ldots, N \).

Step 2: The collective overall preference values of \( a_n \) with respect to criterion \( c_n \), denoted by \( h_i(a_n, c_n) \), are obtained by the EHFOHA operator:

\[
h_i(a_n, c_n) = EHFOHA(h'_1(a_n, c_n), h'_2(a_n, c_n), \ldots, h'_L(a_n, c_n))
\]

where \( m = 1, 2, \ldots, M, n = 1, 2, \ldots, N, w = (w_1, \ldots, w_L) \) is the associated weighting vector of the EHFOHA operator with \( w_i \in S, l = 1, 2, \ldots, L \).
Step 3: The overall preference values of \( a_m \), denoted by \( h_s(a_m) \) are obtained utilizing the EHFLWD operator:

\[
h_s(a_m) = EHFLWD(h_s(a_m, c_1), h_s(a_m, c_2), \ldots, h_s(a_m, c_n))
\]

where \( m = 1, 2, \ldots, M \).

Step 4: Utilize the overall preference values \( h_s(a_m) \) to rank the alternatives \( a_m \) \((m=1, 2, \cdots, M)\), and then select the best one(s) for final decision.

**Scenario 2** Weights take the form of real numbers.

Suppose \( \sum_{l=1}^{L} \omega^{(l)} = 1 \), \( \omega^{(l)} \in [0,1] \), \( l = 1, 2, \ldots, L \) and

\[
\sum_{j=1}^{N} \omega^{(j)} = 1, \quad \omega^{(j)} \in [0,1], \quad j = 1, 2, \ldots, N.
\]

Fig. 3. Schema of the linguistic group decision making model.
6. Application and Discussion

In this section, we apply the proposed linguistic GDM model and processes in a practical example. Then, some discussions are given to compare the proposed technique with some existing approaches.

6.1. An example

A practical GDM problem of evaluating university faculty for tenure and promotion is used to illustrate the proposed processes. The criteria used in some universities are $c_1$: teaching, $c_2$: research, and $c_3$: service (whose weighting vector is $\omega^{(c)} = (0.14, 0.26, 0.60)^T$). Five alternatives (faculty candidates) $a_m \ (m = 1, 2, 3, 4, 5)$ are to be evaluated using the linguistic term set $S$ in Example 2 by two groups of experts (whose weighting vector $\omega^{(g)} = (0.60, 0.40)^T$). The first group $G_1$ includes 3 experts $e_{11}, e_{12}, e_{13}$, and the second group $G_2$ is formed by 2 experts $e_{21}, e_{22}$. After elicitation, experts’ evaluation information is listed in Tables 1-5.

As the weighting vectors take the form of real number, we utilize the process of Scenario 2 to meet a decision.

Step 1: The evaluation information of $a_m \ (m = 1, 2, 3, 4, 5)$, with respect to criterion $c_n \ (n = 1, 2, 3)$, provided by group $G_1$ and $G_2$, are derived by the union of HFLTSs provided by experts of each group. The results are listed in Tables 6-7.

Step 3: The overall preference values of $a_m$, denoted by $h_3(a_m)$, are obtained utilizing the EHFLWA operator. As the cardinality of $h_3(a_m)$ is huge, only the expected linguistic terms are given:

Table 1. Decision matrix provided by $e_{11}$.

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<tr>
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Table 2. Decision matrix provided by $e_{12}$.

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Table 3. Decision matrix provided by $e_{13}$.

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Table 4. Decision matrix provided by $e_{21}$.

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EHFLTs and their aggregation

Step 4: According to Definition 13, we can obtain

\[ E(h_5(a_1)) = s_{0.970}, \quad E(h_5(a_2)) = s_{0.586}, \]

\[ E(h_5(a_3)) = s_{0.703}, \quad E(h_5(a_4)) = s_{0.702}, \]

\[ E(h_5(a_5)) = s_{0.356} \]

Thus, the order of these faculty candidates is

\[ a_2 > a_3 > a_4 > a_5, \]

then \( a_2 \) is the best candidate.

6.2. Comparison and Discussion

As an alternative solution of the above problem, the LA operator \(^{40}\), the LWA operator and LHA operator are used for comparison. Without the idea of HFS, we cannot deal with several possible values at the same

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</table>

Table 5. Decision matrix provided by \( e_{22} \).

Table 6. Decision matrix of \( G_1 \).

Table 7. Decision matrix of \( G_2 \).

Table 8. The final decision matrix.
time. Thus if experts have hesitancy among several possible linguistic terms, a pre-aggregation step has to be done, and then an averaging value is computed by some means. Following the advice of Xu\textsuperscript{41}, the problem can be processed by some steps.

Step 1: If an expert has hesitancy among several possible linguistic terms, the LA operator is used to obtain the corresponding averaging values. For example, the resultant decision matrix provided by $e_{11}$ is transformed into Table 9.

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<td>$(s_4, 0)$</td>
<td>$(s_5, -0.5)$</td>
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</table>

Step 2: The evaluations within each group are synthesized by the LA operator as well as the relative weights of experts within a group are indifferent. The resultant decision matrices are shown in Tables 10 and 11.

Step 3: We aggregate Tables 10 and 11 by the LHA operator to obtain the final decision matrix. The weighting vector and associated weighting vector are the same as used in Section 6.1. The result is presented in Table 12.

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Step 4: The overall satisfactory degree of $a_m$, denoted by $z_m$, $m=1,2,3,4,5$, are obtained utilizing the LWA operator as follows:

$$z_1 = (s_1, -0.62), z_2 = (s_1, 0.19), z_3 = (s_1, -0.296),$$

$$z_4 = (s_1, 0.042), z_5 = (s_1, -0.268).$$

Therefore, $z_2 > z_4 > z_1 > z_3 > z_5$.

Step 5: The alternatives can be ranked as:

$$a_4 > a_5 > a_1 > a_2 > a_3.$$

Based on the procedures of the comparable processes, we discuss their differences by the following aspects.

1) Number of times of using aggregation operators. As the idea of HFS is used, the proposed model and process need less aggregation. Comparing to the existing process, the pre-aggregation step in expert level is eliminated. Further, aggregation within each group is conducted by the union operation rather than an aggregation operator.

2) Possible values versus averaging values. Because of less aggregation operator is used, all possible values, rather than only averaging values, are maintained for consideration. As in Table 1 and Table 9, the proposed process use HFLTs to represent experts’ opinion, while existing process has to synthesize all possible linguistic terms to an averaging value. The same phenomenon happens in each level of aggregation. Therefore, the proposed process handle all possible values along with the procedure of aggregation, it is more meaningful than considering just averaging values as there is no loss of information.

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<thead>
<tr>
<th>$c_n$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$(s_1, 0)$</td>
<td>$(s_2, -0.25)$</td>
<td>$(s_3, -0.25)$</td>
<td>$(s_4, 0)$</td>
<td>$(s_5, -0.5)$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$(s_1, 0)$</td>
<td>$(s_2, 0)$</td>
<td>$(s_3, -0.25)$</td>
<td>$(s_4, 0)$</td>
<td>$(s_5, 0)$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$(s_1, -0.5)$</td>
<td>$(s_2, -0.5)$</td>
<td>$(s_3, -0.25)$</td>
<td>$(s_4, 0)$</td>
<td>$(s_5, -0.5)$</td>
</tr>
</tbody>
</table>
3) Final decision. Both methods are different while agree on the first choice \( a_2 \), which validates that the proposed process is reasonable and it is useful to consider all possible values. We can also see that the priorities of five alternatives are distinct. There is a rank reversal between \( a_4 \) and \( a_5 \). The proposed process uses all possible values for synthesis and need less aggregation, as analyzed above, thus the final decision would be more rational.

7. Conclusion remarks

In this paper, we have extended HFLTSs to EHFLTSs for the purpose of application. Some basic operations and two types of aggregation operators with distinct forms of weighting vector have been proposed. EHFLTSs have more desirable mathematical properties than HFLTS. The construction axiom and linguistic GDM model have been presented for the potential application of the proposed EHFLTSs. At last, a practical example has shown that the proposed technique owns several advantages and obtains more rational decision than the existing method.

Using the proposed EHFLTS and the proposed model, we can take all possible values into account during the procedure of information aggregation. At the same time, less aggregation operators are needed, which makes final decision more robust because more aggregation may lead to less robust decision. Furthermore, as linguistic term sets and HFLTSs can be seen as special cases of the proposed set, EHFLTSs can model more complex and complicated decision making problems in linguistic setting.

As future work we will develop some more aggregation operators, such as induced OWA operators and generalized OWA operators to support the proposed model. Solutions of more general cases of the proposed model will be also considered. Some measures of EHFLTSs used for clustering and data mining, such as degree of similarity, distance and so on, are also challenges.

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References


