On \((M,N)\)-SI (implicative) filters in \(R_0\)-algebras

Jianming Zhan\(^a\); Yang Xu\(^b\), Young Bae Jun \(^c\)

\(^a\) Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province, 445000, PR China
E-mail: zhanjianming@hotmail.com

\(^b\) School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 610031, PR China
E-mail: xuyang@home.swjtu.edu.cn

\(^c\) Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail: skywine@gmail.com

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Abstract

Molodtsov’s soft set theory provides a general mathematical framework for dealing with uncertainty. This paper aims to put forward a new soft set–\((M,N)\)-soft intersection set, which is a generalization of soft intersection sets. We introduce \((M,N)\)-SI (implicative) filters of \(R_0\)-algebras. Some characterizations of these kinds of filters are established. In particular, we discuss the properties of \((M,N)\)-soft congruences in \(R_0\)-algebras. It can lay a foundation for providing a new soft algebraic tool in considering many problems that contain uncertainties.

Keywords: Soft set; \(R_0\)-algebras; filter; implicative filter; \((M,N)\)-soft congruence; \((M,N)\)-SI implicative(Boolean) filter.

1. Introduction

The concept of \(R_0\)-algebras was first introduced by Wang\(^{39}\) by providing an algebraic proof of the completeness theorem of a formal deductive system\(^{37,38,31,45}\). In 2005, Liu and Li\(^{21,22}\) have extended the notions of implicational filters and Boolean filters to \(R_0\)-algebras by considering the fuzzification of such notions. It can be easily observed that \(R_0\)-algebras are different from the \(BL\)-algebras\(^{14,46}\) because the identity \(x \land y = x \circ (x \rightarrow y)\) holds in \(BL\)-algebras, but it does not hold in \(R_0\)-algebras. We note that \(R_0\)-algebras are also different from the lattice implication algebras\(^{41,42}\) because the identity \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\) holds in lattice implication algebras, but it does not hold in \(R_0\)-algebras. Although they are essentially different, they still have some similarities, that is, they all have the implication operator \(\rightarrow\). Therefore, it is meaningful to generalize the lattice implication algebras and \(BL\)-algebras to \(R_0\)-algebras. In\(^7\), Esteva and Godo introduced the \(MTL\)-algebra, which is an algebra in-

\(^*\)Corresponding address: Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province, 445000, China. Tel: 13597795069

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duced by using a continuous $t$-norm and its corresponding residuum. It can be proved that an $R_0$-algebra is, in particular a $MTL$-algebra in which its $t$-norm $\odot$ is a nilpotent minimum $t$-norm. In particular, Ma$^{25,27}$ discussed fuzzy filters of $R_0$-algebras.

It is well known that the complexities of modelling uncertain data in economics, engineering, environmental science, sociology, information sciences and many other fields can not be successfully dealt with by classical methods. Although probability theory, fuzzy set theory and rough set theory are well-known and effective tools for dealing with vagueness and uncertainty, each of them has certain inherent limitations. Based on this reason, Molodtsov$^{30}$ proposed a completely new approach called soft set theory. Since then, especially soft set operations, have undergone tremendous studied, such as$^{2,3,10,11,26,32,33,34}$.

We note that soft set theory emphasizes a balanced coverage of both theory and practice. Nowadays, it has promoted a breath of the discipline of information sciences, intelligent systems, expert and decision support systems, expert and decision support systems, knowledge systems and decision making, and so on. For examples, see$^{5,6,8,12,13,15,16,17,29,40,47}$ At the same time, soft set theory has been found its wide-ranging applications in the algebraic structures, such as$^{1,9,18,20,23,24,26,43,44,35}$ Recently, Çağman and Sezgin$^{4,35}$ made a new approach to soft intersection theory to groups and near-rings. Further, Jun et al.$^{19}$ applied this idea to $R_0$-algebras. They introduced the concept of soft intersection filters of $R_0$-algebras. Some new characterizations were provided.

The present paper is organized as follows. In section 2, we recall some concepts and results of $R_0$-algebras and soft sets. In section 3, we investigate some characterizations of $(M,N)$-$SI$ filters of $R_0$-algebras. In particular, some important properties of $(M,N)$-soft congruences of $R_0$-algebras are discussed in section 4. Finally, we study $(M,N)$-$SI$ implication (Boolean) filter of $R_0$-algebras. It is shown that $(M,N)$-$SI$ Boolean filters and $(M,N)$-$SI$ implication filters of $R_0$-algebras are equivalent in section 5.

2. Preliminaries

By an $R_0$-algebra$^{39}$, we mean a bounded lattice $L = (L, \leq, \land, \lor, ‘, \rightarrow, 0, 1)$, which ‘ is an order-reversing involution and with a binary operation $\rightarrow$ such that the following conditions hold:

$$(R_1) \ x \rightarrow y = y’ \rightarrow x’;$$

$$(R_2) \ 1 \rightarrow x = x;$$

$$(R_3) \ (y \rightarrow z) \land ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y \rightarrow z;$$

$$(R_4) \ x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);$$

$$(R_5) \ (x \rightarrow y) \lor ((x \rightarrow y) \rightarrow (x \lor y)) = 1.$$ In any $R_0$-algebra $L$, the following statements are true (see$^{31}$):

$$(a_1) \ x \leq y \iff x \rightarrow y = 1.$$ 

$$(a_2) \ x \leq y \rightarrow x.$$ 

$$(a_3) \ x’ = x \rightarrow 0.$$ 

$$(a_4) \ (x \rightarrow y) \lor (y \rightarrow x) = 1.$$ 

$$(a_5) \ x \leq y \Rightarrow x \rightarrow z \leq y \rightarrow z.$$ 

$$(a_6) \ x \leq y \Rightarrow z \leq x \rightarrow z \rightarrow y.$$ 

$$(a_7) \ ((x \rightarrow y) \rightarrow y) = x \rightarrow y.$$ 

$$(a_8) \ x \lor y = ((x \rightarrow y) \rightarrow y) \land ((y \rightarrow x) \rightarrow x).$$ 

$$(a_9) \ x \land x’ = 0, x \lor x’ = 1.$$ 

$$(a_{10}) \ x \land y \leq x \land y, x \lor (x \rightarrow y) \leq x \land y.$$ 

$$(a_{11}) \ (x \lor y) \rightarrow z = x \rightarrow (y \rightarrow z).$$ 

$$(a_{12}) \ x \leq y \rightarrow (x \land y).$$ 

$$(a_{13}) \ x \lor y \leq z \iff x \leq y \rightarrow z.$$ 

$$(a_{14}) \ x \leq y \Rightarrow x \land y \leq y \land z.$$ 

$$(a_{15}) \ x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z).$$ 

$$(a_{16}) \ (x \rightarrow y) \lor (y \rightarrow z) \leq x \rightarrow z.$$ 

Let $L$ be an $R_0$-algebra. For any $x, y \in L$, define $x \land y = (x \rightarrow y)’$ and $x \lor y = x’ \rightarrow y$. It is proved that $\land$ and $\lor$ are commutative, associative and $x \land y = (x’ \lor y)’$, and $(L, \land, \lor, ‘, \rightarrow, 0, 1)$ is a residuated lattice.

Now, we recall some basic concepts of filters in $R_0$-algebras.

A non-empty subset $F$ of $L$ is called a filter of $L$ if it satisfies $(F_1)$ $1 \in F$ and $(F_2)$ $x, x \rightarrow y \in F \Rightarrow y \in F$. A filter $F$ of $L$ is called a Boolean filter of $L$ if $(F_3) x \lor x’ \in F$, for all $x \in L$. A non-empty subset $F$ of $L$ is called an implicature filter of $L$ if it satisfies $(F_1)$ and $(F_4) x \rightarrow (y \rightarrow z) \in F, x \rightarrow y \in A \Rightarrow x \rightarrow z \in F$. We know a filter of $L$ is Boolean if and only it is implicature (see$^{31,21,22}$.)
From now on, \( L \) is an \( R_0 \)-algebra, \( U \) is an initial universe, \( E \) is a set of parameters, \( P(U) \) is the power set of \( U \) and \( A,B,C \subseteq E \).

**Definition 1.** A soft set \( f_A \) over \( U \) is a set defined by \( f_A : E \rightarrow P(U) \) such that \( f_A(x) = \emptyset \) if \( x \notin A \). Here \( f_A \) is also called an approximate function. A soft set over \( U \) can be represented by the set of ordered pairs \( f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\} \). It is clear to see that a soft set is a parameterized family of subsets of \( U \). Note that the set of all soft sets over \( U \) will be denoted by \( S(U) \).

**Definition 2.** Let \( f_A, f_B \in S(U) \). Then,

1. \( f_A \) is called soft subset of \( f_B \) and denoted by \( f_A \subseteq f_B \) if \( f_A(x) \subseteq f_B(x) \), for all \( x \in E \). \( f_A \) and \( f_B \) are called soft equal, denoted by \( f_A = f_B \), if \( f_A \subseteq f_B \) and \( f_B \subseteq f_A \).
2. The union of \( f_A \) and \( f_B \), denoted by \( f_A \cup f_B \), is defined as \( f_A \cup f_B = f_{A \cup B} \), where \( f_{A \cup B}(x) = f_A(x) \cup f_B(x) \), for all \( x \in E \).
3. The intersection of \( f_A \) and \( f_B \), denoted by \( f_A \cap f_B \), is defined as \( f_A \cap f_B = f_{A \cap B} \), where \( f_{A \cap B}(x) = f_A(x) \cap f_B(x) \), for all \( x \in E \).

**Definition 3.** A soft set \( f_L \) over \( U \) is called an \( S^1 \)-filter of \( L \) over \( U \) if it satisfies:

\[ (S_1) \quad f_L(x) \subseteq f_L(1) \quad \text{for all} \quad x \in L. \]

\[ (S_2) \quad f_L(x \rightarrow y) \subseteq f_L(y) \quad \text{for all} \quad x,y \in L. \]

A soft set \( f_L \) over \( U \) is called an \( S^1 \)-implicative filter of \( L \) over \( U \) if it satisfies \((S_1)\) and

\[ (S_3) \quad f_L(x \rightarrow (y \rightarrow z)) \subseteq f_L(x \rightarrow y) \subseteq f_L(x \rightarrow z) \quad \text{for all} \quad x,y,z \in L. \]

**Remark 1.** In \( S^1 \), Jun et al. called these two concepts int-soft filters and int-soft implicative filters, respectively. But it was first introduced this concept by Çağman \(^4\). In the following paper, we will use the terminology in \( S^1 \).

### 3. \((M,N)\)-\( S^1 \) filters

In this section, we introduce the concept of \((M,N)\)-\( S^1 \) filters of \( R_0 \)-algebras and investigate some characterizations. From now on, \( 0 \subseteq M \subseteq N \subseteq U \).

**Definition 4.** A soft set \( f_S \) over \( U \) is called an \((M,N)\)-soft intersection filter (briefly, \((M,N)\)-\( S^1 \) filter) of \( L \) over \( U \) if it satisfies:

\[ (S_1) \quad f_L(x) \cap N \subseteq f_L(1) \cup M \quad \text{for all} \quad x \in L; \]

\[ (S_2) \quad f_L(x \rightarrow y) \cap f_L(x) \cap N \subseteq f_L(y) \cup M \quad \text{for all} \quad x,y \in L. \]

**Remark 2.** If \( f_L \) is an \( S^1 \)-filter of \( L \) over \( U \), then \( f_L \) is an \((0,U)\)-\( S^1 \) filter of \( L \) over \( U \). Hence, every \( S^1 \)-filter of \( L \) is an \((M,N)\)-\( S^1 \) filter of \( L \), but the converse is not true.

**Example 1.** Assume that \( U = S_3 \), symmetric group, is the universal set and let \( L = \{0,a,b,c,1\} \), where \( 0 < a < b < c < 1 \). Define \( \rightarrow \) and \( \rightarrow \) as follows:

<table>
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<tr>
<th>( x )</th>
<th>( x' )</th>
<th>( 0 )</th>
<th>( a )</th>
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<tbody>
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<td>0</td>
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<td>a</td>
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</tr>
</tbody>
</table>

Then \((L,\wedge,\vee',\rightarrow)\) is an \( R_0 \)-algebra\(^{21}\), where \( x \wedge y = \min\{x,y\} \) and \( x \vee y = \max\{x,y\} \).

Let \( M = \{(13),(123)\} \) and \( N = \{(1),(12),(13),(123)\} \).

Define a soft set \( f_L \) over \( U \) by

\[ f_L(1) = \{(1),(12),(123)\}, \quad f_L(c) = \{(1),(12),(13),(123)\} \]

and \( f_L(a) = f_L(b) = f_L(0) = \{(1),(12)\} \).

Then one can easily check that \( f_L \) is an \((M,N)\)-\( S^1 \) filter of \( L \) over \( U \), but it is not \( S^1 \)-filter of \( L \) over \( U \) since \( f_L(c) \not= f_L(1) \).

The following proposition is obvious.

**Proposition 1.** If a soft set \( f_L \) over \( U \) is an \((M,N)\)-\( S^1 \) filter of \( L \) over \( U \), then

\[ (f_L(1) \cap N) \cup M \supseteq (f_L(x) \cap N) \cup M \quad \text{for all} \quad x \in S. \]

**Proposition 2.** If \( f_L \) is an \((M,N)\)-\( S^1 \) filter of \( L \) over \( U \), then \( f_L^+ = \{x \in U | (f_L(x) \cap N) \cup M = (f_L(1) \cap N) \cup M\} \) is a filter of \( L \).

**Proof.** Assume that \( f_L \) is an \((M,N)\)-\( S^1 \) filter of \( L \) over \( U \), then it is clear that \( 1 \in f_L^+ \).

For any \( x \rightarrow y \in f_L^+ \), then

\[ (f_L(x) \cap N) \cup M = (f_L(x \rightarrow y) \cap N) \cup M = (f_L(1) \cap N) \cup M. \]

By Proposition 2, we have \((f_L(y) \cap N) \cup M \subseteq (f_L(1) \cap N) \cup M.\)

Since \( f_L \) is an \((M,N)\)-\( S^1 \) filter of \( L \) over \( U \), we have...
(f_l(y) \cap N) \cup M = ((f_l(y) \cup M) \cap N) \cup M
\supseteq (f_l(x) \cap f_l(x \to y) \cap N) \cup M
= ((f_l(y) \cup N) \cap M) \cup ((f_l(x \to y) \cap N) \cup M)
= (f_l(1) \cap N) \cup M.

Hence, (f_l(y) \cap N) \cup M = (f_l(1) \cap N) \cup M, which implies, y \in f_l^x. This implies that f_l^x is a filter of L.

Define an ordered relation “\(\cong_{(M,N)}\)” on \(S(U)\) as follows:

- For any \(f_{l_1}, g_{l_2} \in S(U), 0 \subseteq M \subseteq N \subseteq U\), we define \(f_{l_1} \cong_{(M,N)} g_{l_2} \iff f_{l_1} \cap N \subseteq g_{l_2} \cup M\).

And we define a relation “\(=_{(M,N)}\)” as follows:

\(f_{l_1} =_{(M,N)} g_{l_2} \iff f_{l_1} \cong_{(M,N)} g_{l_2} \land g_{l_2} \cong_{(M,N)} f_{l_1}\).

Then, we can denote Definition 4 as follows:

**Definition 5.** A soft set \(f_l\) over \(U\) is called an \((M,N)\)-soft intersection filter (briefly, \((M,N)\)-SI filter) of \(L\) over \(U\) if it satisfies:

\((S I'_1)\) \(f_l(x) \subseteq \cong_{(M,N)} f_l(y)\) for all \(x \in L\);
\((S I'_2)\) \(f_l(x \to y) \subseteq \cong_{(M,N)} f_l(x) \cap \cong_{(M,N)} f_l(y)\) for all \(x, y \in L\).

**Proposition 3.** If a soft set \(f_l\) over \(U\) is an \((M,N)\)-SI filter of \(L\), then for any \(x, y, z \in L\):

1. \(x \leq y \Rightarrow f_l(x) \subseteq \cong_{(M,N)} f_l(y)\).
2. \(f_l(x \to y) = f_l(1) \Rightarrow f_l(x) = \cong_{(M,N)} f_l(y)\).
3. \(f_l(x \to y) =_{(M,N)} f_l(x) \cap f_l(y) =_{(M,N)} f_l(x \wedge y)\).
4. \(f_l(0) =_{(M,N)} f_l(x) \cap f_l(x')\).
5. \(f_l(x \to y) \cap f_l(y \to z) =_{(M,N)} f_l(x) \cap f_l(y)\).
6. \(x \odot y \leq z \Rightarrow f_l(x) \cap f_l(y) =_{(M,N)} f_l(z)\).
7. \(f_l(x \to (z' \to y)) \cap f_l(y \to z) =_{(M,N)} f_l(x \to (z' \to y))\).
8. \(f_l(x \to (y \to z)) \cap f_l(x \to y) =_{(M,N)} f_l(x \to (x \to z))\).

**Proof.** (1) Let \(x, y \in L\) be such that \(x \leq y\), then \(x \to y = 1\), and so

\[(f_l(x) \cap N) \cup M = (f_l(x) \cap N) \cap (f_l(1) \cup M) = (f_l(y) \cap N) \cup ((f_l(x) \to y) \cap N) \cup M \subseteq (f_l(x) \cap f_l(x \to y) \cap N) \cup M \subseteq f_l(y) \cup M,
\]

which implies, \(f_l(x) \cong_{(M,N)} f_l(y)\).

(2) Let \(x, y \in L\) be such that \(f_l(x \to y) = f_l(1)\), then

\[f_l(x) \cap N = (f_l(x) \cap N) \cap (f_l(1) \cup M) = (f_l(x) \cap N) \cap (f_l(x \to y) \cup M) \subseteq (f_l(x) \cap f_l(x \to y) \cap N) \cup M \subseteq f_l(y) \cup M,
\]

that is, \(f_l(x) \cong_{(M,N)} f_l(y)\).

(3) By \((a_2)\), \(x \circ y \leq x \wedge y\) for all \(x, y \in L\), then by \((1)\), \(f_l(x \circ y) \cong_{(M,N)} f_l(x) \cap f_l(y)\). On the other hand, by \((a_11)\) and \((1)\), \(f_l(x) \cong_{(M,N)} f_l(y \to (x \circ y))\). It follows from \((S I_2)\) that \(f_l(x) \cap f_l(y) \cong_{(M,N)} f_l(y \to (x \circ y)) \cap f_l(y) \subseteq f_l(x \to y)\). Hence \(f_l(x \to y) =_{(M,N)} f_l(x) \cap f_l(y)\).

By \((a_2)\) and \((a_9)\), we have \(y \leq x \leq y \leq x \wedge y\). Then \(f_l(x) \cong_{(M,N)} f_l(x) \cap f_l(y)\), and \(f_l(x \circ (x \to y)) \cong_{(M,N)} f_l(x) \cap f_l(y)\).

Hence, we have \(f_l(x) \cap f_l(y) =_{(M,N)} f_l(x) \cap f_l(y)\).

Thus, \(f_l(x \circ y) =_{(M,N)} f_l(x) \cap f_l(y) =_{(M,N)} f_l(x \wedge y)\).

(4) Since \(x \circ x' = 0\), then it is a consequence of \((3)\).

(5) By \((6)\) and \((a_15)\), we can deduce it.

(7) By \((a_10)\) and \((a_15)\), we have \((x \to (z' \to y)) \cap (y \to z) = ((x \circ z') \to y) \circ (y \to z) \leq (x \circ z') \to z = x \to (z' \to z)\).

By \((1)\) and \((3)\), we deduce that \(f_l(x) \to (z' \to y)) \cap f_l(y \to z) =_{(M,N)} f_l(x \to (z' \to y)) \circ (y \to z) \cong_{(M,N)} f_l(x \to (z' \to z))\).

(8) By \((R_4)\) and \((a_2)\), we have \(x \rightarrow (y \to z)) \cap (x \to y) = (y \to (x \to z)) \circ (x \to y) \leq x \to (x \to z)\).

Hence, by \((1)\) and \((3)\), we can deduce that \(f_l(x) \to (y \to z) \cap f_l(x \to y) =_{(M,N)} f_l(x \to (y \to z)) \circ (x \to y) \cong_{(M,N)} f_l(x \to (x \to z))\).

**Theorem 4.** A soft set \(f_l\) over \(U\) is an \((M,N)\)-SI filter of \(L\) over \(U\) if and only if it satisfies:

\((S I_3)\) \(\forall x, y \in L, x \leq y \Rightarrow f_l(x) \cong_{(M,N)} f_l(y)\).
\((S I_4)\) \(\forall x, y \in L, f_l(x \circ y) =_{(M,N)} f_l(x) \cap f_l(y)\).

**Proof.** \(\Rightarrow\) By Proposition 3(1) and \((3)\).

\(\Leftarrow\) Let \(x, y \in L\). Since \(x \leq 1\), then by \((S I_3)\), we have \(f_l(x) \cong_{(M,N)} f_l(1)\), that is, \(f_l(x) \cap N \subseteq f_l(1) \cup\)
M. This implies that $(S I_1)$ holds.

By $(a_0)$, $x \circ (x \rightarrow y) \leq y$. Hence, by $(S I_3)$ and $(S I_4)$, $f_L(x) \cap f_L(y) =_{(M,N)} f_L(x \circ (x \rightarrow y) \subseteq_{(M,N)} f_L(y)$. that is, $f_L(x) \cap f_L(y \rightarrow x) \cap N \subseteq f_L(y) \cup M$. This implies that $(S I_2)$ holds.

Therefore, $f_L$ is an $(M,N)$-$S_I$ filter of $L$ over $U$.

The following proposition is obvious.

**Proposition 5.** A soft set $f_L$ over $U$ is an $(M,N)$-$S_I$ filter of $L$ over $U$ if and only if satisfies:

$$(S I_5) \quad x \leq y \rightarrow z \Rightarrow f_L(x) \cap f_L(y) \subseteq_{(M,N)} f_L(z).$$

**Theorem 6.** If a soft set $f_L$ over $U$ is an $(M,N)$-$S_I$ filter of $L$ over $U$, then the following are equivalent:

(i) $\forall x, y, z \in L$, $f_L(x \rightarrow (y \rightarrow z)) \cap f_L(x \rightarrow y) \subseteq_{(M,N)} f_L(x \rightarrow z)$.

(ii) $\forall x, y, z \in L$, $f_L(x \rightarrow (x \rightarrow y)) \subseteq_{(M,N)} f_L(x \rightarrow y)$.

(iii) $\forall x, y, z \in L$, $f_L(x \rightarrow (y \rightarrow z)) \subseteq_{(M,N)} f_L(x \rightarrow y)$.

**Proof.** (i) $\Rightarrow$ (ii) Putting $y = z$ and $z = x$ in (1) and using $(S I_1)$, we can deduce that

$$f_L(x \rightarrow y) \cup M = (f_L(x \rightarrow y) \cup M) \cup M \supseteq (f_L(x \rightarrow y) \cap f_L(x \rightarrow t) \cap N) \cup M \supseteq f_L(x \rightarrow (x \rightarrow y)) \subseteq_{(M,N)} f_L(x \rightarrow y) \cap N.$$

that is, $f_L(x \rightarrow (x \rightarrow y) \subseteq_{(M,N)} f_L(x \rightarrow y)$.

(ii) $\Rightarrow$ (iii) By $(a_1)$ and $(a_1 S)$, we have $x \rightarrow (y \rightarrow z) \leq x \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))$, and so $f_L(x \rightarrow (y \rightarrow z)) \cap N \subseteq f_L(x \rightarrow (x \rightarrow y) \rightarrow (x \rightarrow z))) \cup M$.

Then

$$f_L((x \rightarrow y) \rightarrow (x \rightarrow z)) \cup M = f_L(x \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) \cup M = f_L(x \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) \cup M \supseteq f_L((x \rightarrow (x \rightarrow y \rightarrow (x \rightarrow z))) \cap N) \cup M = f_L((x \rightarrow (x \rightarrow y \rightarrow (x \rightarrow z))) \cap N) \uplus M \supseteq f_L((x \rightarrow (x \rightarrow y \rightarrow (x \rightarrow z))) \cap N) \cap M$$

that is, $f_L(x \rightarrow (y \rightarrow z) \subseteq_{(M,N)} f_L((x \rightarrow y) \rightarrow (x \rightarrow z)))$.

(iii) $\Rightarrow$ (i) By $(S I_2)$ and (iii), we have

$$f_L(x \rightarrow z) \cup M = f_L(x \rightarrow z) \cap f_L(x) \cap f_L(y) \cup M \supseteq f_L((x \rightarrow y) \rightarrow (x \rightarrow z)) \cap f_L(x \rightarrow y) \cap N \cup M \supseteq f_L((x \rightarrow y) \rightarrow (x \rightarrow z)) \cup M \cap N \supseteq f_L((x \rightarrow (y \rightarrow z)) \cap f_L((x \rightarrow y) \cap N).$$

that is, $f_L(x \rightarrow (y \rightarrow z) \subseteq_{(M,N)} f_L((x \rightarrow y) \rightarrow (x \rightarrow z))$.

\section{(M,N)-soft congruences}

In this section, we investigate $(M,N)$-soft congruences, $(M,N)$-soft congruences classes and quotient soft $R_0$-algebras.

**Definition 6.** A soft relation $\theta$ from $f_L \times f_L$ to $P(U \times U)$ is called an $(M,N)$-congruence in $L$ over $U \times U$ if it satisfies:

$(C_1)$ $\theta(1,1) =_{(M,N)} \theta(x,x), \forall x \in L.$

$(C_2)$ $\theta(x,y) =_{(M,N)} \theta(y,x), \forall x \in L.$

$(C_3)$ $\theta(x,y) \cap \theta(y,z) \subseteq_{(M,N)} \theta(x,z), \forall x, y, z \in L.$

$(C_4)$ $\theta(x,y) \subseteq_{(M,N)} \theta(x \circ y, y \circ z), \forall x, y, z \in L.$

$(C_5)$ $\theta(x,y) \subseteq_{(M,N)} \theta(x \circ z, y \circ z), \forall x, y, z \in L.$

\section{Lemma 7.} Let $\theta$ be an $(M,N)$-congruence in $B_L$-algebra $L$ over $U \times U$ and $x \in L$. Define $\theta^\circ$ in $L$ as $\theta^\circ(y) = \theta(x,y), \forall y \in L$. The set $\theta^\circ$ is called an $(M,N)$-congruence class of $x$ by $\theta$ in $L$. The set $L/\theta = \{\theta^\circ | x \in L\}$ is called a quotient soft set by $\theta$.

\section{Lemma 8.} If $\theta$ is an $(M,N)$-congruence in $L$ over $U \times U$, then $\theta(x,y) \subseteq_{(M,N)} \theta(1,1)$, $\forall x, y \in L$.

**Proof.** By $(C_1)$ and $(C_3)$, we have $\theta(1,1) = \theta(x,x) \subseteq_{(M,N)} \theta(x,y) \subseteq_{(M,N)} \theta(x,y) = \theta(x,y).$
\( \theta(1, y) \preceq_{(M,N)} \theta(1, x \to y) \cap \theta(x, 1) = \theta(1, x) \cap \theta(1, x \to y) \),

this is, \( \theta^4(y) \preceq_{(M,N)} \theta^4(x) \cap \theta^4(x \to y) \). This proves that \((S_2)\) holds. Thus, \( \theta^4 \) is an \((M,N)\)-soft congruence in \( L \).

**Lemma 9.** Let \( f_L \) be an \((M,N)\)-SI filter of \( L \) over \( U \), then \( \theta(x, y) = f_L(x \to y) \cap f_L(y \to x) \) is an \((M,N)\)-soft congruence in \( L \).

**Proof.** For any \( x, y, z \in L \), we have

1. \( \theta_f(1, 1) = f_L(1 \to 1) = f_M(1 \to 1) = f_L(x \to y) \cap f_L(y \to x) = \theta_f(x, y) \). This proves that \((C_1)\) holds.

2. It is clear that \((C_2)\) holds.

3. By Proposition 3(5), we have

\[
\theta_f(x, y) \cap \theta_f(y, z) = (f_L(x \to y) \cap f_L(y \to z) \cap f_L(z \to y)) = (f_L(x \to y) \cap f_L(y \to z) \cap f_L(z \to y)) = f_L((N_1(A^x, B^y)) \cap f_L((N_1(A^y, B^z))) \preceq_{(M,N)} f_L(x \to z) \cap f_L(z \to y) = \theta_f(x, z).
\]

Thus, \((C_3)\) holds.

4. Since \( x \to y \leq (x \otimes z) \to (y \otimes z) \) and \( y \to x \leq (y \otimes z) \to (x \otimes z) \), we have \( f_L((x \otimes z) \to (y \otimes z)) \) and \( f_L((y \otimes z) \to (x \otimes z)) \), which implies, \( \theta_f(x, y) \preceq_{(M,N)} \theta_f(x \otimes z, y \otimes z) \). This implies that \((C_4)\) holds.

5. \( \theta_f(x \to z, y \to z) \cap \theta_f(z \to x, y \to z) = f_L((x \to z) \to (y \to z) \cap f_L((y \to z) \to (x \to z)) \cap f_L((z \to x) \to (y \to z) \cap f_L((y \to z) \to (x \to z)) \preceq_{(M,N)} f_L((x \to z) \cap f_L((y \to z) \to (x \to z)) = \theta_f(x, y). \)

Thus, \((C_5)\) holds. Therefore, \( \theta_f \) is an \((M,N)\)-soft congruence in \( L \).

Let \( f_L \) be an \((M,N)\)-SI filter of \( L \) over \( U \) and \( x \in L \). In the following, let \( f^x \) denote the \((M,N)\)-congruence class of \( x \) by \( \theta_f \) in \( L \) and \( L/f \) the quotient set by \( \theta_f \).

**Lemma 10.** If \( f_L \) is an \((M,N)\)-SI filter of \( L \) over \( U \), then \( f^x = \preceq_{(M,N)} f^y \) if and only if \( f_L(x \to y) = f_L(y \to x) = f_L(1 \to 1) \) for all \( x, y \in L \).

**Proof.** If \( f_L \) is an \((M,N)\)-SI filter of \( L \) over \( U \), then \( \forall \varphi (\forall \varphi (x) = f^x \mapsto f^y \mapsto \varphi (y) \preceq_{(M,N)} \varphi (x) \) for all \( x, y \in L \).

For any \( x, y \in L \), \( \varphi (x \lor y) = f^x \lor f^y \mapsto \varphi (x) \lor \varphi (y) \preceq_{(M,N)} \varphi (y) \lor \varphi (x) \preceq_{(M,N)} \varphi (x) \land \varphi (y) \land (x \lor y) = f^x \lor f^y = f^x \land \varphi (x) \land \varphi (y) \). Therefore, \( f_L(1 \to 1) = f_L(1 \to 1) = f_L(1 \to 1) \).

Conversely, assume the given condition holds.

By Proposition 3, we have

\[
f_L(x \to z) \preceq_{(M,N)} f_L(y \to z) \cap f_L(z \to y) \cap f_L(1 \to 1).
\]

If \( f_L(x \to z) = f_L(y \to z) = f_L(z \to y) = f_L(1 \to 1) \), then \( f_L(x \to z) \cap f_L(y \to z) \cap f_L(z \to y) = f_L(1 \to 1) \). Similarly, we can prove that \( f_L(1 \to 1) = f_L(1 \to 1) \).

This implies that \( f^x = f^y \mapsto f^x = f^y \mapsto f^x \land f^y = f^x \lor f^y \). Therefore, \( f_L(1 \to 1) = f_L(1 \to 1) \).

Hence \( f^x = f^y \).

Denote \( f^{(1)} = \{ x \in L \mid f(x) = f(1) \} \).

**Corollary 11.** If \( f \) is an \((M,N)\)-SI filter of \( L \) over \( U \), then \( f^x = \preceq_{(M,N)} f^y \) if and only if \( x \sim f(1) \), \( y \) and \( x \to y \in f(1) \).

Let \( f \) be an \((M,N)\)-SI filter of \( L \) over \( U \). For any \( f^x, f^y \in L/f \), we define

\[
F \lor F = \preceq_{(M,N)} F^x \lor f^y, \quad F \land F = \preceq_{(M,N)} F^x \land f^y,
\]

\[
\langle F \rangle Y = \preceq_{(M,N)} F^x \land f^y, \quad \langle F \rangle Y = \preceq_{(M,N)} F^x \lor f^y.
\]

**Theorem 12.** Let \( f \) be an \((M,N)\)-SI filter of \( L \) over \( U \), then \( L/f = (L/f, \land, \lor, \to, \langle F \rangle Y, f^x) \) is an \( R_0 \)-algebra.

**Proof.** We can claim that the above operations on \( L/f \) are well-defined. In fact, if \( f^x = \preceq_{(M,N)} f^y \) and \( f^y = \preceq_{(M,N)} f^z \), then by Corollary 11, we have \( x \sim f(y, b) \) and \( y \sim f(b) \) for all \( x, y \in f(b) \).

Thus, \( f^x = \preceq_{(M,N)} f^y \). Similarly, we can prove \( f^x = \preceq_{(M,N)} f^y \). Then we can easily check that \( L/f \) is an \( R_0 \)-algebra.

**Theorem 13.** Let \( f_L \) be an \((M,N)\)-SI filter of \( L \) over \( U \), then \( L/f \cong L/f^{(1)} \).

**Proof.** Define \( \varphi : L \to L/f \) by \( \varphi(x) = f^x \) for all \( x \in L \).

For any \( x, y \in L \), \( \varphi(x \lor y) = f^{x \lor y} = f^x \lor f^y = \varphi(x) \lor \varphi(y) \), \( \varphi(x \land y) = f^{x \land y} = f^x \land f^y = \varphi(x) \land \varphi(y) \), \( \varphi(x') = f^{x'} = (\varphi(x))' \) and \( \varphi(x) \).
y) = f^{x\to y} \equiv (M,N) f^x \to f^y = \varphi(x) \to \varphi(y)$. Hence $\varphi$ is an epic.

Moreover, $x \in \ker \varphi \iff \varphi(x) = f^1 \iff f^x = (M,N) f^1 \iff x \sim_{f(1)} 1 \iff x \in f(1)$. Hence, $\ker \varphi = f(1)$. Thus, $L/\text{f} \cong L/\text{f}(1)$.

5. $(M,N)$-SI implicative (Boolean) filters

In this section, we introduce the concept of $(M,N)$-SI implicative (Boolean) filters of $R_0$-algebras and investigate some of their properties.

**Definition 8.** A soft set $f_L$ over $U$ is called an $(M,N)$-soft intersection implicative filter (briefly, $(M,N)$-SI implicative filter) of $L$ over $U$ if it satisfies $(S_I)$ and

$$(S_I) f_L(x \to (y \to z)) \subseteq f_L(x \to y) \cap f_L(x \to z),$$

for all $x,y,z \in L$.

**Remark 3.** If $f_S$ is an $(M,N)$-SI implicative filter of $L$ over $U$, then $f_L$ is an $(\emptyset, U)$-SI implicative filter of $L$ over $U$. Hence, every SI-implicative filter of $L$ is an $(M,N)$-SI implicative filter of $L$, but the converse is not true.

**Example 2.** Assume that $U = D_2 = \{<x,y>: |x|^2 = y^2 = e, xy = yx\} = \{e, x, y, xy\}$, Dihedral group, is the universal set.

Let $L = \{0, a, b, c, d, 1\}$, where $0 < a < b < c < d < 1$. Then $(L, \wedge, \vee, \rightarrow, \to)$ is an $R_0$-algebra.

Let $M = \{e, y\}$ and $N = \{e, x, y\}$.

Define a soft set $f_L$ of $L$ over $U$ by $f_L(1) = \{e, x\}$, $f_L(c) = f_L(d) = \{e, x, y\}$ and $f_L(a) = f_L(b) = f_L(0) = \{e, y\}$.

Then one can easily check that $f_L$ is an $(M,N)$-SI implicative filter of $L$ over $U$, but it is not an SI-implicative filter of $L$ over $U$ since $f_L(1) = \{e, x\} \not\subseteq f_L(c)$.

From Definitions 4 and 8, we have

**Proposition 14.** Every $(M,N)$-SI implicative filter of $L$ over $U$ is an $(M,N)$-SI filter, but the converse may not be true as shown in the following example.

**Example 3.** Consider the soft set $f_L$ of $S$ over $U$ as in Example 1. Let $M = \{(13)\}$ and $N = \{(1), (12), (13), (123)\}$. We can easily check that $f_L$ is an $(M,N)$-SI filter of $L$ over $U$, but it is not an $(M,N)$-SI implicative filter of $L$ over $U$ since $f_L(b \to a) \cup M = f_L(b) \cup M = \{(1), (12)\} \cup \{(13)\} = \{(1), (12), (13)\} \not\subseteq \{(1), (12), (13)\} = f_L(1) \cap N = f_L(b \to (b \to a)) \cap f_L(b \to b) \cap N$.

Now, we discuss some properties of $(M,N)$-SI implicative filters in $R_0$-algebras.

**Theorem 15.** Let $f_L$ be an $(M,N)$-SI filter of $L$ over $U$, then $f_L$ is an $(M,N)$-SI implicative filter of $L$ over $U$ if and only if it satisfies:

$$(S_I) f_L(x \to (y \to z)) \subseteq f_L(x \to y) \cap f_L(x \to z),$$

for all $x, y, z \in L$.

**Proof.** Assume that $f_L$ is an $(M,N)$-SI implicative filter of $L$ over $U$. For any $x, y, z \in L$, we have

$$f_L(x \to z) \subseteq \bigcap_{L} f_L(x \to z) \subseteq \bigcup_{L} f_L(x \to z),$$

that is, $f_L(x \to (y \to z)) \subseteq f_L(x \to y) \cap N,$

Conversely, assume that $f_L$ is an $(M,N)$-SI filter of $L$ over $U$ satisfying $(S_I)$. Then

$$f_L(x \to z) \subseteq \bigcap_{L} f_L(x \to z) \subseteq \bigcup_{L} f_L(x \to z),$$

that is, $f_L(x \to (y \to z)) \subseteq f_L(x \to y) \cap N$, $f_L(x \to y) \cap N,$

Thus, $(S_I)$ holds.

**Theorem 16.** Let $f_L$ be an $(M,N)$-SI filter of $L$ over $U$, then the following are equivalent:

(i) $f_L$ is an $(M,N)$-SI implicative filter of $L$.

(ii) $f_L(x \to z) =_{(M,N)} f_L(x \to (z' \to z))$, for all $x, y, z \in L$. 

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Thus, Proposition 3.7 over U satisfying the condition is an $f_L$ for all $x \in L$.

**Proof.**

(i) $\Rightarrow$ (ii) Assume that $f_L$ is an $(M,N)$-SI implicative filter of $L$ over $U$. Putting $y = z$ in $(SI_I)$,

$$f_L(x \to z) \cup M = (f_L(x \to z) \cup M) \cup M \supseteq (f_L(x \to (z' \to z))) \cap f_L(x \to z) \cup (f_L(1) \cap M) \cup M \supseteq (f_L(x \to (z' \to z)) \cup M) \cap (f_L(1) \cup M) \cap N \supseteq f_L(x \to (z' \to y)) \cap f_L(y \to z),$$

Putting $y = z$ in (iii), we have

$$f_L(x \to z) \cup M \supseteq (f_L(x \to z) \cup M) \cup M \supseteq (f_L(x \to (z' \to z))) \cap f_L(x \to z) \cup (f_L(1) \cap M) \cup M \supseteq (f_L(x \to (z' \to z)) \cup M) \cap (f_L(1) \cup M) \cap N \supseteq f_L(x \to (z' \to y)) \cap f_L(y \to z) \cap N,$$

which implies $f_L(x \to z) \subseteq f_L(y \to z)$. Thus, $(SI_I)$ holds. By Theorem 15, $f_L$ is an $(M,N)$-SI implicative filter of $L$ over $U$. $\square$

**Theorem 17.** Let $f_L$ be an $(M,N)$-SI filter of $L$ over $U$, then the following are equivalent:

1. $f_L$ is an $(M,N)$-SI implicative filter of $L$.
2. $f_L(x) \equiv_{(M,N)} f_L(x' \to x)$, for all $x \in L$.
3. $f_L(x) \equiv_{(M,N)} f_L(y \to x)$, for all $x,y \in L$.
4. $f_L(x) \subseteq f_L(y \to x)$, for all $x,y \in L$.

**Proof.** (1) $\Rightarrow$ (2) By Theorem 16(ii), we have

$$f_L(x) = f_L(1 \to x) \equiv_{(M,N)} f_L(1 \to (x' \to x)) = f_L(x' \to x).$$

(2) $\Rightarrow$ (3) Since $x' \leq x \leq y$, then $(x \to y) \to x \leq x' \to y$, and so $f_L(x' \to y) \subseteq f_L((x \to y) \to x)$. Thus, from (2), we can deduce that

$$f_L(x) \equiv_{(M,N)} f_L(x' \to x) \subseteq f_L(((x \to y) \to x) \to x).$$

On the other hand, since $x \leq (x \to y) \to x$, we have $f_L(x) \subseteq f_L((x \to y) \to x)$. Thus, we can get $f_L(x) \equiv_{(M,N)} f_L((x \to y) \to x)$.

(3) $\Rightarrow$ (4) Since $f_L$ is an $(M,N)$-SI filter of $L$, then $f_L((x \to y) \to x) = f_L((x \to y) \to x) \cap f_L(z)$. It follows from (3) that

$$f_L(x) \equiv_{(M,N)} f_L((x \to y) \to x) \cap f_L(z) = f_L(x' \to x) \cap f_L(z).$$

(4) $\Rightarrow$ (1) Since $z \leq x \leq z'$ and $z' \leq x \leq z$ and $z' \leq x \leq z$, we have $f_L(z') \subseteq f_L(z) \subseteq f_L((x \to y) \to x) \cap f_L(z)$. It follows from (2) that

$$f_L(x) \equiv_{(M,N)} f_L(x' \to x) \subseteq f_L((x \to y) \to x) \cap f_L(z) = f_L(x' \to x).$$

Finally, we introduce the concept of $(M,N)$-SI Boolean filters of $R_0$-algebras.

**Definition 9.** Let $f_L$ be an $(M,N)$-SI filter of $L$ over $U$, then $f_L$ is called an $(M,N)$-SI Boolean filter of $L$ if it satisfies

$$(SI_9)\ f_L(x \vee y') \equiv_{(M,N)} f_L(1),\text{ for all } x \in L.$$
then
\[ f_L((x' \rightarrow x) \rightarrow x) = f_L(x' \rightarrow (x' \rightarrow x')) = f_L((x' \rightarrow x) \rightarrow (x' \rightarrow x')) \]
\[ = f_L(1), \]
and so, \( f_L((x' \rightarrow x) \rightarrow x) = f_L(1) \).

Similarly, we can prove \( f_L((x \rightarrow x') \rightarrow x') = f_L(1) \). Hence, we have
\[ f_L(x \lor x') = f_L((x' \rightarrow x) \land ((x \rightarrow x') \rightarrow x')) = f_L(1) = f_L((x \rightarrow x') \rightarrow (x' \rightarrow x')) = f_L(1) = f_L(x \lor x'). \]

This proves that \( f_L \) is an \((M, N)\)-SI Boolean filter of \( L \).

Conversely, assume that \( f_L \) is an \((M, N)\)-SI Boolean filter of \( L \). For any \( x, y \in L \), we have
\[ f_L(x \rightarrow y) \cup M = (f_L(x \rightarrow y) \cup M) \cup M \]
\[ \supseteq (f_L((y \lor y') \rightarrow (x \rightarrow y)) \cap f_L((y \lor y') \cap N) \cup M)
\[ = (f_L((y \lor y') \rightarrow (x \rightarrow y)) \cap f_L((x \rightarrow x') \rightarrow x')) \cup M \cap N
\[ = f_L(1) \cup M \cap (f_L((y' \rightarrow (x \rightarrow y)) \cup M) \cap N
\[ \supseteq f_L(y' \rightarrow (x \rightarrow y)) \cap N, \]

which implies \( f_L(x \rightarrow y) = f_L(y' \rightarrow (x \rightarrow y)) \).

On the other hand, since \( x \rightarrow y \leq x \rightarrow (y' \rightarrow y) \), we have
\[ f_L(x \rightarrow y) = f_L(x \rightarrow (y' \rightarrow y)). \]

Therefore, it follows from Theorem 16 that \( f_L \) is an \((M, N)\)-SI implicative filter of \( L \). \( \square \)

Remark 4. Every \((M, N)\)-SI implicative filters and \((M, N)\)-SI Boolean filters in \( R_0\)-algebras are equivalent.

6. Conclusion

As a generalization of soft intersection filters of \( R_0\)-algebras, we introduce the concepts of \((M, N)\)-SI (implicative) filters of \( R_0\)-algebras. We investigate their characterizations. In particular, we describe \((M, N)\)-soft congruences in \( R_0\)-algebras.

To extend this work, one can further investigate \((M, N)\)-SI prime (semiprime) filters of \( R_0\)-algebras.

Maybe one can apply this idea to decision making, data analysis and knowledge based systems.

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