

# A New Variable-Coefficient AKNS Hierarchy and its Exact Solutions via Inverse Scattering transform

Sheng Zhang<sup>1, a</sup>, Xudong Gao<sup>2, b</sup>

<sup>1</sup>School of Mathematics and Physics, Bohai University, Jinzhou, 121013, China

<sup>2</sup>School of Mathematics and Statistics, Kashgar University, Kashgar, 844000, China

<sup>a</sup>email: szhangchina@126.com, <sup>b</sup>email: 986242791@163.com

**Keywords:** Variable-coefficient AKNS hierarchy; Exact solution; Inverse scattering transform

**Abstract.** In this paper, a new variable-coefficient AKNS (vcAKNS) hierarchy is derived from the associated with linear spectral problem and its time evolution equation. The derived vcAKNS hierarchy is a solvable mixed hierarchy which contains isospectral nonlinear partial differential equations (PDEs) and nonisospectral nonlinear PDEs. To exactly solve the vcAKNS hierarchy, inverse scattering transform method is employed. Based on a systematic analysis, the formulae of exact solutions of the vcAKNS hierarchy are obtained. In the case of reflectionless potentials, the obtained exact solutions are reduced to  $n$ -soliton solutions.

## Introduction

As we know that nonlinear phenomena involved in many fields including physics, biology, chemistry, mechanics are often related to nonlinear PDEs. Constructing nonlinear PDEs or solving nonlinear PDEs plays an important role in the study of these nonlinear phenomena. Recently, Zhang and Gao [1] derived a mixed AKNS hierarchy from the corresponding linear isospectral problem and its evolution equation. It is shown in [1] that the AKNS spectral problem being nonisospectral is not a necessary condition to construct a nonisospectral AKNS hierarchy. In this paper, starting a linear nonisospectral problem and its evolution equation we construct the following mixed vcAKNS hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \left[ \varsigma(t) \left( \frac{L-2bE}{a} \right)^{m+n} + \tau(t) \left( \frac{L-2bE}{a} \right)^{n+1} x + \frac{a_t}{a} (L-2bE)x + 2b_t x E \right] \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (m=1,2,\dots; n=0,1,2,\dots), \quad (1)$$

which is more general than the one in [1] because when  $\tau(t)=0$  and  $n=0$  Eq. (1) becomes the mixed AKNS hierarchy [1]. The operator  $L$  in Eq. (1) is defined by

$$L = \sigma \partial + 2 \begin{pmatrix} q \\ -r \end{pmatrix} \partial^{-1} (r, q), \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \frac{1}{2} \left( \int_{-\infty}^x - \int_x^{+\infty} \right) dx, \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In nonlinear mathematical physics, the inverse scattering transform (IST) [2] proposed by Gardner, Greene, Kruskal and Miura has achieved considerable development and received a wide range of applications like those in [3]. One of the advantages of the IST is that it can solve a whole hierarchy of nonlinear PDEs associated with a certain spectral problem. The present paper will employ the IST to solve Eq. (1) exactly.

## Derivation of vcAKNS hierarchy

Firstly, let us consider a new and more general AKNS spectral problem [1]

$$\phi_x = M \phi, \quad M = \begin{pmatrix} -a\eta - b & q \\ r & a\eta + b \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (2)$$

and the time evolution

$$\phi_t = N \phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (3)$$

where  $q = q(x, t)$ ,  $r = r(x, t)$  and their derivatives of any order with respect to  $x$  and  $t$  are

smooth potential functions, which vanish as  $x$  tends to infinity; the spectral parameter  $\eta$  is independent with  $x$  and  $\eta_t$  is  $\tau(t)(2\eta)^n/2$ .  $A$ ,  $B$ ,  $C$  are undetermined functions of  $t$ ,  $x$ ,  $q$ ,  $r$  and  $\eta$ , while  $a=a(t)$ ,  $b=b(t)$  are derivable functions of  $t$  and  $a$  is bounded. It should be noted that when  $a=1$ ,  $b=0$ , Eq. (2) becomes the known spectral problem [3].

The compatibility condition of Eqs. (2) and (3) reads

$$M_t - N_x + [M, N] = 0, \quad (4)$$

from which we have

$$\begin{cases} A_x = qC - rB - (a_t\eta + b_t)x + \eta_t \\ q_t = B_x + 2(a\eta + b)B + 2qA \\ r_t = C_x - 2(a\eta + b)C - 2rA \end{cases} \quad (5)$$

If  $N$  in Eq. (3) satisfies the boundary condition:

$$N|_{(q,r)=(0,0)} = \begin{pmatrix} -\frac{1}{2}\varsigma(t)(2\eta)^{m+n} - \frac{1}{2}\tau(t)(2\eta)^n x - (a_t\eta + b_t)x & 0 \\ 0 & \frac{1}{2}\varsigma(t)(2\eta)^{m+n} + \frac{1}{2}\tau(t)(2\eta)^n x + (a_t\eta + b_t)x \end{pmatrix}, \quad (6)$$

here  $\varsigma(t)$  is an arbitrary function of  $t$ , then from Eq. (4) we obtain

$$A = \partial^{-1}(r, q) \begin{pmatrix} -B \\ C \end{pmatrix} - (a_t\eta + b_t)x - \frac{1}{2}\varsigma(t)(2\eta)^{m+n} - \frac{1}{2}\tau(t)(2\eta)^n x, \quad (7)$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L \begin{pmatrix} -B \\ C \end{pmatrix} - 2(a\eta + bE) \begin{pmatrix} -B \\ C \end{pmatrix} + \varsigma(t)(2\eta)^{m+n} \begin{pmatrix} -q \\ r \end{pmatrix} + \tau(t)(2\eta)^n x \begin{pmatrix} -q \\ r \end{pmatrix} + 2(a_t\eta + b_tE)x \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (8)$$

Supposing that

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{i=1}^{m+n} \begin{pmatrix} -b_i \\ c_i \end{pmatrix} (2\eta)^{m+n-i}, \quad (9)$$

we have

$$\begin{pmatrix} -b_{i+1} \\ c_{i+1} \end{pmatrix} = \frac{1}{a}(L - 2bE) \begin{pmatrix} -b_i \\ c_i \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, \dots, m-1, m+1, \dots, m+n-1, \quad (10)$$

$$\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \frac{\varsigma(t)}{a} \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (11)$$

$$\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = \frac{1}{a}(L - 2bE) \begin{pmatrix} -b_{m-1} \\ c_{m-1} \end{pmatrix} + \frac{\tau(t)}{a} x \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (12)$$

$$\begin{pmatrix} -b_{m+n} \\ c_{m+n} \end{pmatrix} = \frac{1}{a}(L - 2bE) \begin{pmatrix} -b_{m+n-1} \\ c_{m+n-1} \end{pmatrix} + \frac{a_t}{a} x \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (13)$$

and hence obtain the vcAKNS hierarchy (1).

## Exact Solutions

Lemma 1 [3]: Suppose that  $\phi(x, k)$  is a solution of linear spectral problem (2), then

$$P(x, k) = \phi_t(x, k) - N\phi(x, k), \quad (14)$$

is a solution of Eq. (2) as well.

Lemma 2 [3]: Suppose that

$$\bar{L}^* = -\sigma\partial + 2 \begin{pmatrix} -r \\ q \end{pmatrix} \partial^{-1}(q, r), \quad \bar{L} = \sigma L \sigma,$$

then  $\bar{L}^*$  is the conjugation operator of  $\bar{L}$ .

Theorem 1: The discrete scattering data

$$\{k(\text{Im } k) = 0, R(k), \kappa_j(\text{Im } \kappa_j > 0), c_j, j = 1, 2, \dots, l\}, \quad (15)$$

$$\{k(\text{Im } k) = 0, \bar{R}(k), \bar{\kappa}_j(\text{Im } \bar{\kappa}_j < 0), \bar{c}_j, j = 1, 2, \dots, l\}, \quad (16)$$

for the spectral problem (2) possess the following time dependence

$$\kappa_j(t) = [\kappa_j^{1-n}(0) + (1-n)(2i)^{n-1} \int_0^t a(s)\tau(s)ds]^{\frac{1}{1-n}}, \quad \bar{\kappa}_j(t) = [\bar{\kappa}_j^{1-n}(0) + (1-n)(2i)^{n-1} \int_0^t a(s)\tau(s)ds]^{\frac{1}{1-n}}, \quad (17)$$

$$c_j^2(t) = c_j^2(0)e^{(2i\kappa_j)^m \int_0^t \zeta(s)ds + 2\ln|a|}, \quad \bar{c}_j^2(t) = \bar{c}_j^2(0)e^{(2i\bar{\kappa}_j)^m \int_0^t \zeta(s)ds + 2\ln|a|}. \quad (18)$$

Proof: Take a solution  $\phi(x, k)$  of Eq. (2), then the solution  $P(x, k) = \phi_t(x, k) - N\phi(x, k)$  can be represented linearly by  $\phi(x, k)$  and  $\tilde{\phi}(x, k)$  which also satisfies Eq. (2) but is independent of  $\phi(x, k)$ , i.e., there exist two functions  $\gamma(t, k)$  and  $\tau(t, k)$  such that

$$\phi_t(x, k) - N\phi(x, k) = \gamma(t, k)\phi(x, k) + \tau(t, k)\tilde{\phi}(x, k). \quad (19)$$

First, we consider  $k = \kappa_j(\text{Im } \kappa_j > 0)$ . It is easy to see that  $\phi(x, \kappa_j)$  decays exponentially while  $\tilde{\phi}(x, \kappa_j)$  must increase exponentially as  $x \rightarrow +\infty$ . Then we have  $\tau(t, k) = 0$ , thus Eq. (19) becomes

$$\phi_t(x, \kappa_j) - N\phi(x, \kappa_j) = \gamma(t, \kappa_j)\phi(x, \kappa_j). \quad (20)$$

Left-multiplying Eq. (20) by the inner product  $(\phi_2(x, \kappa_j), \phi_1(x, \kappa_j))$  yields:

$$\frac{d}{dt} \phi_1(x, \kappa_j) \phi_2(x, \kappa_j) - (C\phi_1^2(x, \kappa_j) + B\phi_2^2(x, \kappa_j)) = 2\gamma(t, \kappa_j)\phi_1(x, \kappa_j)\phi_2(x, \kappa_j). \quad (21)$$

Supposing  $\phi(x, \kappa_j)$  to be the normalization eigenfunction and further integrating Eq. (21) with respect to  $x$  from  $-\infty$  to  $+\infty$ , then noting that  $2 \int_{-\infty}^{\infty} c_j^2 \phi_1(x, \kappa_j) \phi_2(x, \kappa_j) dx = 1$  we have

$$\gamma(t, \kappa_j) = -c_j^2 \int_{-\infty}^{\infty} (C\phi_1^2(x, \kappa_j) + B\phi_2^2(x, \kappa_j)) dx. \quad (22)$$

For convenience, we rewrite Eq. (22)

$$\gamma(t, \kappa_j) = -c_j^2 ((\phi_2^2(x, \kappa_j), \phi_1^2(x, \kappa_j))^T, (B, C)^T), \quad (23)$$

where the following inner product is employed

$$(f(x), g(x)) = \int_{-\infty}^{\infty} (f_1(x)g_1(x) + f_2(x)g_2(x)) dx \quad (24)$$

for arbitrary two vectors  $f(x) = (f_1(x), f_2(x))^T$  and  $g(x) = (g_1(x), g_2(x))^T$ .

Using the spectral problem (2), we have

$$\phi_{1x}(x, \kappa_j) + (ai\kappa_j + b)\phi_1(x, \kappa_j) = q(x)\phi_2(x, \kappa_j), \quad \phi_{2x}(x, \kappa_j) - (ai\kappa_j + b)\phi_2(x, \kappa_j) = r(x)\phi_1(x, \kappa_j), \quad (25)$$

from which we derive

$$(\phi_1(x, \kappa_j)\phi_2(x, \kappa_j))_x = \phi_2(x, \kappa_j)\phi_{1x}(x, \kappa_j) + \phi_{2x}(x, \kappa_j)\phi_1(x, \kappa_j) = q(x)\phi_2^2(x, \kappa_j) + r(x)\phi_1^2(x, \kappa_j), \quad (26)$$

and then obtain

$$\int_{-\infty}^{\infty} (q(x)\phi_2^2(x, \kappa_j) + r(x)\phi_1^2(x, \kappa_j)) dx = \int_{-\infty}^{\infty} (\phi_1(x, \kappa_j)\phi_2(x, \kappa_j))_x dx = 0. \quad (27)$$

In the other hand,

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{i=1}^{m+n} \zeta(t)(2k)^{m+n-i} \frac{(\bar{L} - 2b)^{i-1}}{a^i} \begin{pmatrix} q \\ r \end{pmatrix} + \sum_{j=1}^n \tau(t)(2k)^{n-j} \frac{(\bar{L} - 2b)^{j-1}}{a^{j-1}} \begin{pmatrix} qx \\ rx \end{pmatrix} + \frac{a_t}{a} \begin{pmatrix} qx \\ rx \end{pmatrix}, \quad (28)$$

then we obtain

$$\begin{aligned} \gamma(t, \kappa_j) &= -c_j^2 ((\phi_2^2(x, \kappa_j), \phi_1^2(x, \kappa_j))^T, (B, C)^T) = \left( (\phi_2^2, \phi_1^2)^T, \sum_{i=1}^{m+n} \zeta(t)(2i\kappa_j(t))^{m+n-i} \frac{(\bar{L} - 2bE)^{i-1}}{a^i} \begin{pmatrix} q \\ r \end{pmatrix} \right) \\ &\quad + \left( (\phi_2^2, \phi_1^2)^T, \sum_{j=1}^n \tau(t)(2i\kappa_j(t))^{n-j} \frac{(\bar{L} - 2bE)^{j-1}}{a^{j-1}} \begin{pmatrix} qx \\ rx \end{pmatrix} \right) + \frac{a_t}{a} \left( (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix} \right) \\ &= \sum_{i=1}^{m+n} \zeta(t)(2i\kappa_j(t))^{m+n-i} \frac{1}{a^i} \left( (\phi_2, \phi_1)^T, (\bar{L} - 2b)^{i-1} \begin{pmatrix} q \\ r \end{pmatrix} \right) \\ &\quad + \frac{a_t}{a} \left( (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix} \right) + \sum_{j=1}^n \frac{\tau(t)}{a^{j-1}} (2i\kappa_j(t))^{n-j} \left( (\phi_2^2, \phi_1^2)^T, (\bar{L} - 2b)^{j-1} \begin{pmatrix} qx \\ rx \end{pmatrix} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{m+n} \varsigma(t) (2i\kappa_j(t))^{m+n-i} \frac{1}{a^i} \left( (\bar{L}^* - 2b)^{i-1} (\phi_2, \phi_1)^T, \begin{pmatrix} q \\ r \end{pmatrix} \right) \\
&\quad + \sum_{j=1}^n \frac{\tau(t)}{a^{j-1}} (2i\kappa_j(t))^{n-j} \left( (\bar{L}^* - 2b)^{j-1} (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix} \right) + \frac{a_t}{a} \left( (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix} \right),
\end{aligned} \quad (29)$$

Further using the results

$$\left( (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix} \right) = \int_{-\infty}^{\infty} x(q\phi_2^2 + r\phi_1^2) dx = \int_{-\infty}^{\infty} x(\phi_1\phi_2)_x dx = -\frac{1}{2c_j^2}, \quad (30)$$

$$\left( (\bar{L}^* - 2bE)^{j-1} (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix} \right) = (2ai\kappa_j(t))^{j-1} \int_{-\infty}^{\infty} x(\phi_1\phi_2)_x dx = -\frac{(2ai\kappa_j)^{j-1}}{2c_j^2}, \quad (31)$$

$$\left( (\bar{L}^* - 2bE)^{j-1} (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} q \\ r \end{pmatrix} \right) = (2ai\kappa_j(t))^{j-1} \left( (\phi_2^2, \phi_1^2)^T, \begin{pmatrix} q \\ r \end{pmatrix} \right) = 0, \quad (32)$$

from Eq. (29) we have

$$\gamma = n\tau(t) \frac{(2i\kappa_j(t))^{n-1}}{2} + \frac{a_t}{2a}, \quad (33)$$

where we have used

$$\bar{L}^* \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} = 2(ai\kappa_j(t) + bE) \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}. \quad (34)$$

Thus, Eq. (20) reads

$$\phi_t(x, \kappa_j(t)) - N\phi(x, \kappa_j(t)) = [n\tau(t) \frac{(2i\kappa_j(t))^{n-1}}{2} + \frac{a_t}{2a}] \phi(x, \kappa_j(t)). \quad (35)$$

Noting that

$$N|_{(q,r)=(0,0)} = \begin{pmatrix} -\frac{1}{2}\varsigma(t)(2i\kappa_j(t))^{m+n} - \frac{1}{2}\tau(t)x(2i\kappa_j(t))^n - a_i\kappa_j(t)x - b_t(t)x & 0 \\ 0 & \frac{1}{2}\varsigma(t)(2i\kappa_j(t))^{m+n} + \frac{1}{2}\tau(t)x(2i\kappa_j(t))^n + a_i\kappa_j(t)x + b_t(t)x \end{pmatrix}, \quad (36)$$

$$\phi(x, \kappa_j) \rightarrow c_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{(ai\kappa_j+b)x}, \quad \phi_t(x, \kappa_j) \rightarrow c_{jt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{(ai\kappa_j+b)x} + c_j(a_i\kappa_j + ai\kappa_{jt} + b_t)x \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{(ai\kappa_j+b)x}, \quad (37)$$

as  $x \rightarrow +\infty$ , then we derive that

$$i\kappa_{jt} = \frac{1}{2}\tau(t)(2i\kappa_j)^n, \quad \phi_t \rightarrow c_{jt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{(ai\kappa_j+b)x} + c_j(a_i\kappa_j + b_t)x \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{(ai\kappa_j+b)x} + c_j \frac{a\tau(t)x}{2} (2i\kappa_j)^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{(ai\kappa_j+b)x}, \quad (38)$$

$$c_{jt} - \frac{1}{2}(\varsigma(t)(2i\kappa_j)^{m+n}) = \frac{1}{2}(n\tau(t)(2i\kappa_j)^{n-1} + \frac{a_t}{a})c_j, \quad (39)$$

In a similar way, we have

$$i\bar{\kappa}_{jt} = \frac{1}{2}a\tau(t)(2i\bar{\kappa}_j)^n, \quad \bar{c}_{jt} - \frac{1}{2}(\varsigma(t)(2i\kappa_j)^{m+n})\bar{c}_j = \frac{1}{2}(n\tau(t)(2i\kappa_j)^{n-1} + \frac{a_t}{a})\bar{c}_j. \quad (40)$$

Then the proof is end.

**Theorem 2:** Given the scattering data (15) and (16) for the spectral problem (2), we can obtain exact solutions of the vcAKNS hierarchy (1):

$$q(x, t) = -2K_1(t, x, x), \quad r(x, t) = \frac{K_{2x}(t, x, x)}{K_1(t, x, x)}, \quad (41)$$

where  $K(t, x, y) = (K_1(t, x, y), K_2(t, x, y))^T$  satisfies Gel'fand-Levitan-Marchenko (GLM) integral equation:

$$K(t, x, y) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{F}(t, x+y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_x^\infty F(t, z+x) \bar{F}(t, z+y) dz + \int_x^\infty K(t, x, s) \int_x^\infty F(t, z+s) \bar{F}(t, z+y) dz ds = 0, \quad (42)$$

with

$$F(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(t, k) e^{(aik+b)x} dk + \sum_{j=1}^l c_j^2 e^{(aik_j+b)x}, \quad \bar{F}(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{R}(t, k) e^{(aik+b)x} dk + \sum_{j=1}^{\bar{l}} \bar{c}_j^2 e^{(ai\bar{\kappa}_j+b)x}. \quad (43)$$

In order to give explicit form of solutions (41), we consider the reflectionless potentials  $q(x, t)$  and  $r(x, t)$ , namely  $R(t, k) = \bar{R}(t, k) = 0$ . In this case, the GLM integral equation (42) becomes

$$K(t, x, y) - \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \bar{F}_d(t, x+y) + \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \int_x^\infty F_d(t, z+x) \bar{F}_d(t, z+y) dz + \int_x^\infty K(t, x, s) \int_x^\infty F_d(t, z+s) \bar{F}_d(t, z+y) dz ds = 0, \quad (44)$$

which can be solved exactly. For convenience, we use  $K(t, x, y) = (K_1(t, x, y), K_2(t, x, y))^T$  to rewrite Eq. (44) as:

$$K_1(t, x, y) - \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \bar{F}_d(t, x+y) + \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \int_x^\infty F_d(t, z+x) \bar{F}_d(t, z+y) dz + \int_x^\infty K_1(t, x, s) \int_x^\infty F_d(t, z+s) \bar{F}_d(t, z+y) dz ds = 0, \quad (45)$$

$$K_2(t, x, y) - \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \bar{F}_d(t, x+y) + \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \int_x^\infty F_d(t, z+x) \bar{F}_d(t, z+y) dz + \int_x^\infty K_2(t, x, s) \int_x^\infty F_d(t, z+s) \bar{F}_d(t, z+y) dz ds = 0. \quad (46)$$

Using Eq. (43), we can get

$$\int_x^\infty F_d(t, s+z) \bar{F}_d(t, z+y) dz = \sum_{j=1}^l \sum_{m=1}^{\bar{l}} \frac{ic_j^2(t) \bar{c}_m^2(t)}{a(\kappa_j - \bar{\kappa}_m)} e^{(aik_j+b)(x+s) - (ai\bar{\kappa}_m+b)(x+y)}, \quad (47)$$

and then suppose that

$$K_1(x, y, t) = \sum_{p=1}^{\bar{l}} \bar{c}_p(t) g_p(t, x) e^{-(ai\bar{\kappa}_p+b)y}, \quad K_2(x, y, t) = \sum_{p=1}^{\bar{l}} \bar{c}_p(t) h_p(t, x) e^{-(ai\bar{\kappa}_p+b)y}. \quad (48)$$

Substituting Eqs. (47) and (48) into Eqs. (45) and (46) yields

$$g_m(t, x) + \bar{c}_m(t) e^{-(ai\bar{\kappa}_m+b)x} + \sum_{j=1}^l \sum_{p=1}^{\bar{l}} \frac{c_j^2(t) \bar{c}_m(t) \bar{c}_p(t)}{a^2(\kappa_j - \bar{\kappa}_m)(\kappa_j - \bar{\kappa}_p)} e^{ai(2\kappa_j - \bar{\kappa}_m - \bar{\kappa}_p)x} g_p(t, x) = 0, \quad (49)$$

$$h_m(t, x) + \sum_{j=1}^l \frac{1}{a(\kappa_j - \bar{\kappa}_m)} c_j^2(t) \bar{c}_m(t) e^{ai(2\kappa_j - \bar{\kappa}_m + b)x} + \sum_{j=1}^l \sum_{p=1}^{\bar{l}} \frac{c_j^2(t) \bar{c}_m(t) \bar{c}_p(t)}{a^2(\kappa_j - \bar{\kappa}_m)(\kappa_j - \bar{\kappa}_p)} e^{ai(2\kappa_j - \bar{\kappa}_m - \bar{\kappa}_p)x} h_p(t, x) = 0. \quad (50)$$

Introducing the following vectors

$$g(t, x) = (g_1(t, x), g_2(t, x), \dots, g_{\bar{l}}(t, x))^T, \quad h(t, x) = (h_1(t, x), h_2(t, x), \dots, h_{\bar{l}}(t, x))^T, \quad (51)$$

$$\Lambda = (c_1 e^{-a(i\kappa_1+b)x}, c_2 e^{-a(i\kappa_2+b)x}, \dots, c_{\bar{l}} e^{-a(i\kappa_{\bar{l}}+b)x})^T, \quad \bar{\Lambda} = (\bar{c}_1 e^{-a(i\bar{\kappa}_1+b)x}, \bar{c}_2 e^{-a(i\bar{\kappa}_2+b)x}, \dots, \bar{c}_{\bar{l}} e^{-a(i\bar{\kappa}_{\bar{l}}+b)x})^T, \quad (52)$$

We can write Eqs. (49) and (50) in the matrix form

$$W(t, x) g(t, x) = -\bar{\Lambda}(t, x), \quad W(t, x) h(t, x) = i P(t, x) \Lambda(t, x). \quad (53)$$

Supposing  $W^{-1}(t, x)$  exists, then we have

$$g(t, x) = -W^{-1}(t, x) \bar{\Lambda}(t, x), \quad h(t, x) = i W^{-1}(t, x) P(t, x) \Lambda(t, x), \quad (54)$$

where

$$W(t, x) = E + P(t, x) P^T(t, x), \quad P(t, x) = \left( \frac{c_j(t) \bar{c}_m(t)}{a(\kappa_j - \bar{\kappa}_m)} e^{ai(\kappa_j - \bar{\kappa}_m)x} \right)_{\bar{l} \times l}, \quad (55)$$

and  $E$  is a  $\bar{l} \times \bar{l}$  unit matrix. Substituting Eq. (54) into Eqs. (49) and (50), we have

$$K_1(x, y, t) = -\text{tr}(W^{-1}(t, x) \bar{\Lambda}(t, x) \bar{\Lambda}^T(t, y)), \quad K_2(x, y, t) = -i \text{tr}(W^{-1}(t, x) E(t, x) \Lambda(t, x) \bar{\Lambda}^T(t, y)), \quad (56)$$

where  $\text{tr}(A)$  means the trace of the matrix  $A$ .

Substituting Eq. (56) into Eq. (41), we obtain  $n$ -soliton solutions of the vcAKNS hierarchy (1)

$$q(x, t) = \text{tr}(W^{-1}(t, x) \bar{\Lambda}(t, x) \bar{\Lambda}^T(t, x)), \quad r(x, t) = -\frac{\frac{d}{dx} \text{tr}(W^{-1}(t, x) E(t, x) \frac{d}{dx} \bar{\Lambda}^T(t, x))}{\text{tr}(W^{-1}(t, x) \bar{\Lambda}(t, x) \bar{\Lambda}^T(t, x))}. \quad (57)$$

Particularly, when  $l = \bar{l} = 1$  Eq. (57) gives one-soliton solutions

$$q = \frac{2\bar{c}_1^{-2} e^{-2(ai\bar{\kappa}_1+b)x}}{1 + \left[ \frac{c_1 \bar{c}_1}{a(\kappa_1 - \bar{\kappa}_1)} \right]^2} e^{2ai(\kappa_1 - \bar{\kappa}_1)x}, \quad r = \frac{2c_1^2 e^{2(ai\kappa_1+b)x}}{1 + \left[ \frac{c_1 \bar{c}_1}{a(\kappa_1 - \bar{\kappa}_1)} \right]^2} e^{2ai(\kappa_1 - \bar{\kappa}_1)x},$$

with

$$\kappa_1 = [\kappa_1^{1-n}(0) + (1-n)(2i)^{n-1} \int_0^t a(s)\tau(s)ds]^{\frac{1}{1-n}}, \quad \bar{\kappa}_1 = [\bar{\kappa}_1^{1-n}(0) + (1-n)(2i)^{n-1} \int_0^t a(s)\tau(s)ds]^{\frac{1}{1-n}},$$

$$c_1(t) = c_1(0)e^{\frac{1}{2}(2i\kappa_j)^m \int_0^t \zeta(s)ds + 2\ln|a|}, \quad \bar{c}_1(t) = \bar{c}_1(0)e^{\frac{1}{2}(2i\bar{\kappa}_j)^m \int_0^t \zeta(s)ds + 2\ln|a|}.$$

## Conclusion

We have derived a new vcAKNS hierarchy which includes the known mixed AKNS hierarchy as special case. Exact solutions and  $n$ -soliton solutions of the vcAKNS hierarchy are obtained by the IST method. To the best of our knowledge, the solutions obtained in this paper have not been reported in literature. How to extend the IST method for solving other hierarchies with variable coefficients is worthy of study. This is our task in the future.

## Acknowledgement

This work was supported by the Natural Science Foundation of China (11547005), the PhD Start-up Fund of Liaoning Province of China (20141137) and the Liaoning BaiQianWan Talents Program (2013921055).

## References

- [1] Sheng Zhang, Xudong Gao. Mixed spectral AKNS hierarchy from linear isospectral problem and its exact solutions. *Open Physics*, 13 (2015) 310-322.
- [2] Clifford Garder, John Greene, Martin Kruskal, Robert Miura. Method for solving the Korteweg-de Vries equation. *Physical Review Letters*, 19 (1967) 1095-1097.
- [3] Dengyuan Chen. *Introduction of Soliton Theory*, Beijing: Science Press, 2006.
- [4] Sheng Zhang, Bo Xu, Hongqing Zhang, Exact solutions of a KdV equation hierarchy with variable coefficients. *International Journal of Computer Mathematics*, 91 (2014) 1601-1616.
- [5] Sheng Zhang, Di Wang. Variable-coefficient nonisospectral Toda lattice hierarchy and its exact solutions. *Pramana-Journal of Physics*, 85 (2015) 1143-1156.
- [6] Sheng Zhang, Xu-Dong Gao. Exact solutions and dynamics of a generalized AKNS equations associated with the nonisospectral depending on exponential function. *Journal of Nonlinear Science and Applications*, 19 (2016) 4529-4541.
- [7] Xudong Gao, Sheng Zhang. Time-Dependent-Coefficient AKNS Hierarchy and Its Exact Multi-Soliton Solutions. *International Journal of Applied Science and Mathematics*, 3 (2016) 72-75.