

# New Periodic Wave Solutions for MNW Hierarchy with the Aid of Computerized Symbolic Computation

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**Abstract.** Computer has received more and more applications in various fields. In this paper, many new periodic wave solutions expressed by Jacobi elliptic functions of the Mikhailov-Novikov-Wang (MNW) hierarchy are obtained by using the extended F-expansion method. In the limit cases, the obtained Jacobi elliptic function solutions degenerate into solitary wave solutions and trigonometric function solutions. It is shown that the extended F-expansion method with the aid of computerized symbolic computation can provide a preferred mathematical tool for finding new periodic wave solutions of nonlinear partial differential equations (PDEs).

## Introduction

In a recent work [1], starting from the stationary zero-curvature equation associated with a  $3 \times 3$  matrix spectral problem Shan and Zhu derived a new hierarchy of nonlinear evolution equations (NLEEs) called the Mikhailov-Novikov-Wang (MNW) hierarchy which includes two cases

$$u_t = -(d_{2m+1,x} + h_{2m+1}), \quad v_t = (va_{2m})_x - 2vb_{2m}, \quad m \geq 0, \quad (1)$$

and

$$u_t = -(d_{2m+2,x} + h_{2m+2}), \quad v_t = (va_{2m+1})_x - 2vb_{2m+1}, \quad m \geq 0, \quad (2)$$

by using the Lie algebra approach. In Eqs. (1) and (2), the coefficient functions  $a_{2m}$ ,  $a_{2m+1}$ ,  $b_{2m}$ ,  $b_{2m+1}$ ,  $d_{2m+1}$ ,  $d_{2m+2}$ ,  $h_{2m+1}$ , and  $h_{2m+2}$  are determined by the following recursive relations

$$a_{mx} = -b_m, \quad b_{mx} = e_m - ua_m - d_{m+1} - vd_{m-1}, \quad c_{mx} = uh_m - f_m, \quad d_{mx} = 2h_m, \quad e_{mx} = ub_m - h_{m+1} - vh_{m-1}, \quad (3)$$

$$f_{mx} = -(3uc_m + g_m) + a_{m+1} + va_{m-1}, \quad g_{mx} = -2uf_m + 2b_{m+1} + 2vb_{m-1}, \quad h_{mx} = 3c_m + ud_m, \quad m \geq 1, \quad (4)$$

$$b_{0x} = e_0 - ua_0 - d_1, \quad e_{0x} = ub_0 - h_1, \quad f_{0x} = -(3uc_0 + g_0) + a_1, \quad g_{0x} = -2uf_0 + 2b_1, \quad (5)$$

with the different initial values

$$a_0 = b_0 = c_0 = d_0 = f_0 = h_0 = g_0 = 0, \quad e_0 = \alpha, \quad (\alpha = \text{const.}, \alpha \neq 0), \quad (6)$$

and

$$a_0 = b_0 = c_0 = d_0 = f_0 = h_0 = e_0 = 0, \quad g_0 = \beta, \quad (\alpha = \text{const.}, \beta \neq 0), \quad (7)$$

For the initial values (6), when  $m=1$  and  $\alpha=9$ , Eq. (1) reduces to the MNW equations

$$u_t = -u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} - 20u_x u_{xx} + 9v_x, \quad v_t = 3vu_{xxx} - 4v_x u_x^2 + v_x u_{xx} - 24vu u_x, \quad (8)$$

which can be converted into the well-known Kaup-Kupershmidt equation

$$u_t = -u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} - 20u_x u_{xx}, \quad (9)$$

as long as setting  $v=0$ .

For the initial values (7), when  $m=1$  and  $\beta=27$ , Eq. (2) gives a new system of NLEEs

$$u_t = 18v_{xxx} - 36(vu)_x - u_{7x} + 49u_x u_{xxxx} + 14uu_{5x} + 84u_{xx} u_{xxx} - 70u_x^3 - 252uu_x u_{xx} - 56u^2 u_{xxx} + \frac{224}{3}u^3 u_x, \quad (10)$$

$$v_t = -36vv_x + v_x v_{xxx} + 3vu_{5x} - 12v_x uu_{xx} - 72vu_x u_{xx} - 36vu u_{xxx} - 6v_x u_x^2 + \frac{32}{3}u^3 v_x + 96vu^2 u_x. \quad (11)$$

As far as we know, Eqs. (10) and (11) have not been solved exactly. When  $v = 0$ , Eqs. (10) and (11) are reduced to

$$u_t = -u_{7x} + 49u_x u_{xxx} + 14uu_{5x} + 84u_{xx} u_{xxx} - 70u_x^3 - 252uu_x u_{xx} - 56u^2 u_{xxx} + \frac{224}{3}u^3 u_x. \quad (12)$$

In [2], Zhang and Tian obtained five new exact solutions including two hyperbolic function solutions, two trigonometric function solutions and a rational solution of Eq. (12) by using tanh-function method. The present paper is motivated by the desire to find new solutions of Eq. (12). Searching for exact solutions, especially new ones, of NLEEs is interesting because exact solutions can help us to gain more insight into some physical phenomena in many fields. Since the Korteweg-de Vries (KdV) equation was solved by Gardner, Green, Kruskal and Miura [3], many effective methods have been proposed for nonlinear PDEs. One of the existing methods is the so-called extended F-expansion method [4] proposed by Wang et al. Later, the extended F-expansion method [4] and its different improvements have received a wide range of applications, such as those in [5-12]. In this paper, the extended F-expansion method [4, 13] is employed to solve Eq. (12). As a result, many periodic wave solutions expressed by Jacobi elliptic functions are simultaneously obtained. These obtained periodic wave solutions can degenerate into solitary wave solutions and trigonometric function solutions in the limit cases of the modulus  $m \rightarrow 1$  and  $m \rightarrow 0$ .

## Periodic Wave Solutions

To begin with, we take the travelling wave transformation:

$$u = u(\xi), \quad \xi = kx + \omega t, \quad (13)$$

then Eq. (12) is transformed into an ordinary differential equation (ODE) of the form

$$\begin{aligned} \omega u' = & -k^7 u^{(7)} + 49k^5 u' u^{(4)} + 14k^5 uu^{(5)} + 84k^5 u'' u''' - 70k^3 u'^3 - 252k^3 uu' u'' \\ & - 56k^3 u^2 u''' + \frac{224}{3}ku^3 u'. \end{aligned} \quad (14)$$

We next suppose that the solution of Eq. (14) can be expressed as

$$u(\xi) = \sum_{i=-n}^n a_i F^i(\xi), \quad (15)$$

where  $a_i$  ( $i = 0, \pm 1, \pm 2, \dots, n$ ) are constants to be determined,  $n$  is an undetermined integer, while  $F(\xi)$  satisfies Jacobi elliptic equation

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R, \quad (16)$$

and hence holds for  $F(\xi)$  and  $F'(\xi)$

$$\begin{cases} F''(\xi) = 2PF^3(\xi) + QF(\xi) \\ F'''(\xi) = (6PF^2(\xi) + Q)F'(\xi) \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi) \\ \vdots \end{cases} \quad (17)$$

where  $P$ ,  $Q$  and  $R$  are all parameters, the prime denotes  $d/d\xi$ . For Eq. (16), different Jacobi elliptic function solutions depending on parameters  $P$ ,  $Q$  and  $R$  were listed in [13]. Noting that besides the 10 solutions [8] of Eq. (16) another 12 solutions were listed in [13] so that more solutions can be found for the considered nonlinear PDEs.

Balancing the highest order linear term  $u^{(7)}$  with the highest order nonlinear term  $u^3 u'$  in Eq. (14) yields  $7 + n = 4n + 1$  which gives  $n = 2$ . Thus, Eq. (15) becomes

$$u = a_0 + a_1 F(\xi) + a_2 F^2(\xi) + a_{-1} F^{-1}(\xi) + a_{-2} F^{-2}(\xi), \quad (18)$$

With the help of Mathematica 4.0, we substitute Eq. (18) along with Eq. (17) into Eq. (14) and equate each coefficient of the same order power of  $F'^\rho(\xi)F^i(\xi)$  ( $\rho = 0, 1; i = 0, \pm 1, \pm 2, \dots$ ) to zero, a

system of nonlinear algebraic equations for  $a_0, a_1, a_2, a_{-1}, a_{-2}, P, Q, R, k$  and  $\omega$  are obtained. Solving the system of nonlinear algebraic equations, we have three cases

$$a_0 = \frac{k^2 Q}{2}, \quad a_1 = 0, \quad a_2 = \frac{3k^2 P}{2}, \quad a_{-1} = 0, \quad a_{-2} = \frac{3k^2 R}{2}, \quad w = \frac{4}{3}(k^7 Q^3 - 36k^7 PQR), \quad (17)$$

$$a_0 = \frac{k^2 Q}{2}, \quad a_1 = 0, \quad a_2 = \frac{3k^2 P}{2}, \quad a_{-1} = 0, \quad a_{-2} = 0, \quad w = \frac{2}{3}(2k^7 Q^3 - 9k^7 PQR), \quad (18)$$

$$a_0 = \frac{k^2 Q}{2}, \quad a_1 = 0, \quad a_2 = 0, \quad a_{-1} = 0, \quad a_{-2} = \frac{3k^2 R}{2}, \quad w = \frac{2}{3}(2k^7 Q^3 - 9k^7 PQR), \quad (19)$$

and hence obtain three formal solutions of Eq. (12) as follows

$$u = \frac{k^2 Q}{2} + \frac{3k^2 P}{2} F^2(\xi) + \frac{3k^2 R}{2} F^{-2}(\xi), \quad \xi = kx + \frac{4}{3}(k^7 Q^3 - 36k^7 PQR)t, \quad (20)$$

$$u = \frac{k^2 Q}{2} + \frac{3k^2 P}{2} F^2(\xi), \quad \xi = kx + \frac{2}{3}(2k^7 Q^3 - 9k^7 PQR)t, \quad (21)$$

$$u = \frac{k^2 Q}{2} + \frac{3k^2 R}{2} F^{-2}(\xi), \quad \xi = kx + \frac{2}{3}(2k^7 Q^3 - 9k^7 PQR)t, \quad (22)$$

Selecting appropriate Jacobi elliptic functions  $F(\xi)$  and different values of  $P, Q$  and  $R$  from Table 1 of [13] and using the formal solutions (20)-(22), we can obtain many periodic wave solutions expressed by Jacobi elliptic functions of Eq. (12). For example, if we select  $F(\xi) = \text{dn}\xi$ ,  $P = -1$ ,  $Q = 2 - m^2$  and  $R = m^2 - 1$ , then Eq. (21) gives

$$u = \frac{k^2(2 - m^2)}{2} - \frac{3k^2}{2} \text{dn}^2(\xi), \quad (23)$$

where

$$\xi = kx + \frac{2}{3}[2k^7(2 - m^2)^3 + 9k^7(2 - m^2)(m^2 - 1)]t. \quad (24)$$

In Fig. 1, a spatial structure of solution (23) with  $m = 0.5$  and  $k = 1$  is shown. Fig. 2 describes the corresponding dynamical evolutions. In Figs. 2 and 3, another spatial structure of solution (23) with  $m = 0.01$  and  $k = 1$  and the corresponding dynamical evolutions are described, respectively. It is easy to see from Figs. 1-4 that solution (30) is a double periodic wave solution.

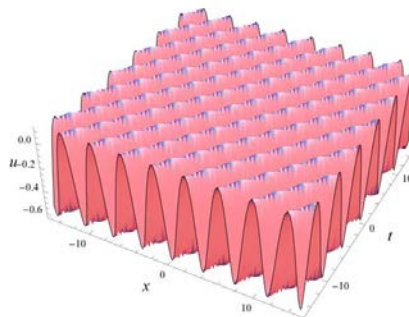


Fig. 1. Spatial structure of double periodic wave solution (23).

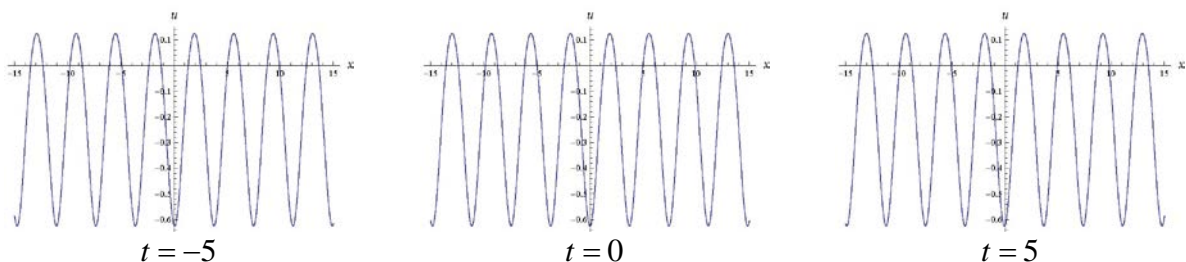


Fig. 2. Dynamical evolutions of double periodic wave solution (23).

In the limit case  $m \rightarrow 1$ , solution (30) degenerate into a solitary wave solution

$$u = \frac{k^2}{2} - \frac{3k^2}{2} \operatorname{sech}^2(\xi), \quad \xi = kx + \frac{4k^7}{3}t, \quad (25)$$

which is displayed in Fig. 5. In Fig. 6, the corresponding dynamical evolutions are shown.

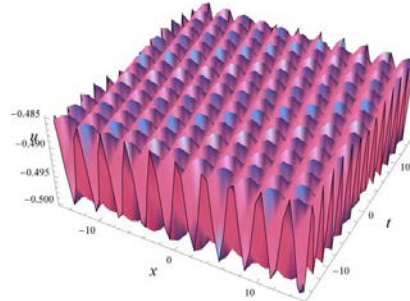


Fig. 3. Spatial structure of double periodic wave solution (25).

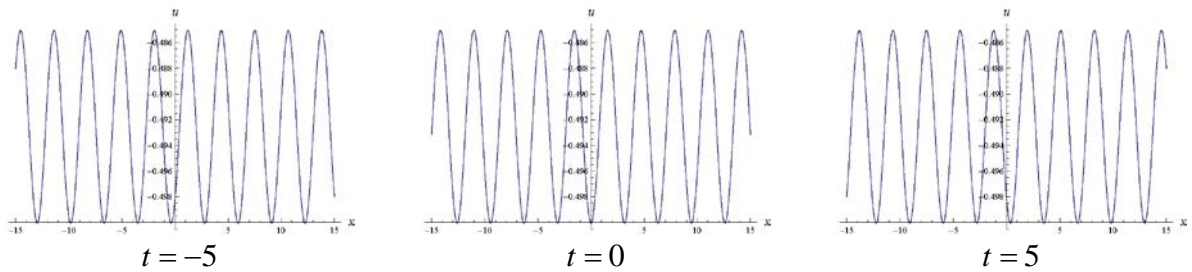


Fig. 4. Dynamical evolutions of double periodic wave solution (25).

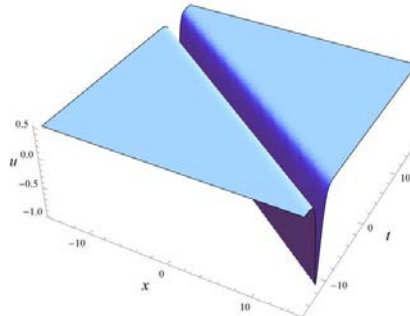


Fig. 5. Spatial structure of solitary wave solution (25).

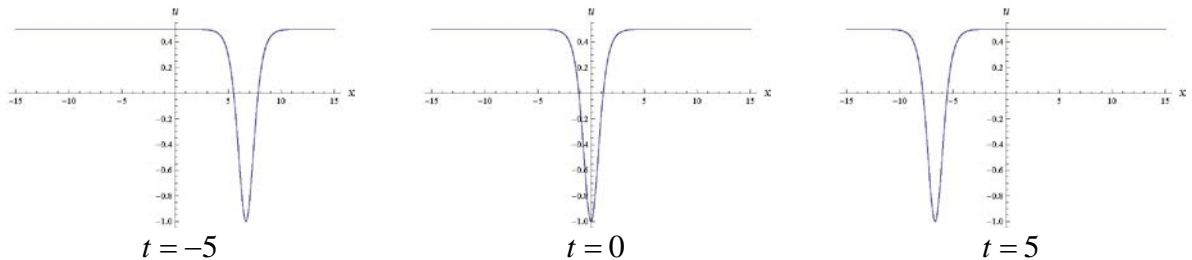


Fig. 6. Dynamical evolutions of solitary wave solution (25).

## Conclusion

In summary, with the aid of computerized symbolic computation we have successfully obtained many periodic wave solutions expressed by Jacobi elliptic functions of one reduced equation (12) of the MNW hierarchy (2) by using the extended F-expansion method [4, 13]. To the best of our knowledge, these obtained periodic wave solutions are new. Three spatial structures with dynamical evolutions of the obtained solution (23) are shown by means of computer simulation. The obtained

periodic wave solutions expressed by Jacobi elliptic function solutions can degenerate into solitary wave solutions and trigonometric function solutions in the limit cases. This paper shows that the extended F-expansion method [4, 13] combined with computerized symbolic computation can provide a preferred mathematical tool for finding new periodic wave solutions of some other new nonlinear PDEs.

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