Specificity for interval-valued fuzzy sets

Ramón González-del-Campo¹, Luis Garmendia², Ronald R. Yager³

¹ DSIC, Universidad Complutense de Madrid, Spain
E-mail: rgonzale@estad.ucm.es

² DISIA, Universidad Complutense de Madrid, Spain
E-mail: lgarmend@fdi.ucm.es

³ Iona College, USA
E-mail: yager@panix.com

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Abstract

In this paper some axiomatic definitions about specificity for interval-valued fuzzy sets are proposed. Some examples of measures of specificity for interval-valued fuzzy sets are showed. It is also defined a extension of the notion of alpha cut for interval-valued fuzzy sets and a generalized similarity for interval-valued fuzzy relations. An axiomatic definition of specificity of interval-valued fuzzy sets under the knowledge of a generalized similarity is given.

Keywords: Specificity measure, Interval-valued fuzzy set, Similarity, T-indistinguishability.

1. Introduction

Interval-valued fuzzy sets (IVFS) were introduced in the 60s by Grattan-Guinness⁷, Jahn⁸, Sambuc⁹ and Zadeh¹⁶. They are extensions of classical fuzzy sets where the membership degree of the elements on the universe of discourse (between 0 and 1) is replaced by an interval in [0,1] x [0,1]. They easily allow to model uncertainty and vagueness generalizing the fuzzy sets. Sometimes it is easier for experts to give a “membership interval” than a membership degree to a characteristic of objects on a universe. IVFS are a special case of type-2 fuzzy sets that simplifies the calculations while preserving their richness as well.

The concept of specificity provides a measure of the amount of information contained in a fuzzy set. It is strongly related to the inverse of the cardinality of a set. Specificity measures were introduced by Yager¹⁰,¹¹ showing its usefulness as a measure of tranquility when making a decision. The output information of expert systems and other knowledge-based system should be both specific and correct to be useful.

Measures of specificity have been widely analyzed³,⁴,⁵, for intuitionistic fuzzy sets¹⁴, for interval-valued fuzzy sets and for type 2 fuzzy sets¹³.

2. Preliminaries

Let X = {e₁, ..., eₙ} be a finite set.

Definition 2.1 A fuzzy set μ on X is normal if there exists an element x ∈ X such that μ(x) = 1.

Definition 2.2 Let a_j be the jᵗʰ greatest membership degree of μ. A measure of specificity is a func-
tion $Sp: \{a_j\} \rightarrow [0, 1]$ such that:

- $Sp(\mu) = 1$ if and only if $\mu$ is a singleton.
- $Sp(\emptyset) = 0$
- $Sp(\mu)$ depends on $a_j$ in that way:

  1. $\frac{\partial Sp(\mu)}{\partial a_j} > 0$
  2. $\frac{\partial Sp(\mu)}{\partial a_j} \leq 0$ for all $j \geq 2$

It is also defined a weaker measure of specificity:

**Definition 2.3** Let $[0, 1]^X$ be the class of fuzzy sets of $X$. A weak measure of specificity is a function $Sp:[0, 1]^X \rightarrow [0, 1]$ such that:

- $Sp(\mu) = 1$ if and only if $\mu$ is a singleton.
- $Sp(\emptyset) = 0$
- If $\mu$ and $\eta$ are normal fuzzy sets in $X$ and $\mu \subset \eta$, then $Sp(\mu) \geq Sp(\eta)$.

**Definition 2.4** Let $Sp$ and $Sp'$ be two measures of specificity. $Sp$ is more strict than $Sp'$, denoted by $Sp \leq Sp'$, if for all sets $\mu$, it verifies: $Sp(\mu) \leq Sp'(\mu)$.

Yager introduced the linear measure of specificity on a finite space $X$ as:

$$Sp_\text{Y}(\mu) = a_1 - \sum_{j=1}^{n} w_j a_j$$

where $a_j$ is the $j^{th}$ greatest membership degree of $\mu$ and $\{w_j\}$ is a set of weights verifying:

- $w_j \in [0, 1]$
- $\sum_{j=2}^{n} w_j = 1$
- $\{w_j\}$ is not increasing.

**Definition 2.5** A fuzzy relation $R: X^2 \rightarrow [0, 1]$ is a similarity relation if it is reflexive, symmetric, and transitive under the $t$-norm minimum $(\text{Min}(R(a,b), R(b,c)) \leq R(a,c)$ for all $a, b, c$ in $X$).

Yager also defines a measure of specificity under the knowledge of a similarity to solve the Yager’s jacket problem.

**Definition 2.6** Let $\mu$ be a fuzzy set on $X$ and let $S$ be a similarity $S : X \times X \rightarrow [0, 1]$. Let $\pi_\alpha$ be the set of classes of equivalence of the $\alpha$-cut of $S$. The set of classes of equivalence under the knowledge of $S$ $\mu_\alpha / S$ is the subset of equivalence classes of the $\alpha$-cut of $S$ defined in that way: a equivalence class of the $\alpha$-cut of $S$ belongs to $\mu_\alpha / S$ if its intersection with the $\alpha$-cut of $\mu_\alpha$ is not empty.

**Definition 2.7** Let $[0, 1]^X$ be the set of fuzzy sets on $X$. Let $\mu$ be a fuzzy set on $X$ and let $S$ be a similarity $S : X \times X \rightarrow [0, 1]$. The specificity of $\mu$ under $S$ is defined as follows:

$$Sp(\mu / S) = \int_{0}^{\text{max}} \frac{1}{\text{card}(\mu_\alpha / S)} d\alpha$$

**Definition 2.8** It is denoted by $L$ and $\leq_L$ the following set and an order relation:

1. $L = \{[x_1, x_2] \in [0, 1]^2$ with $x_1 \leq x_2\}$.
2. $[x_1, x_2] \leq_L [y_1, y_2]$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$

Also by definition:

$$[x_1, x_2] <_L [y_1, y_2] \iff x_1 < y_1, x_2 \leq y_2 \text{ or } x_1 \leq y_1, x_2 < y_2$$

$0_L = [0, 0]$ and $1_L = [1, 1]$ are the smallest and the greatest elements in $L$ respectively.

$L$ is a complete lattice and the supremum and infimum are defined as follows:

**Definition 2.9** Let $\{[v_i, w_i]\}$ be a set of intervals on $L$. Then

1. Meet$\{[v_i, w_i]\} \equiv [\text{infimum}\{v_i\}, \text{infimum}\{w_i\}]$
2. Joint$\{[v_i, w_i]\} \equiv [\text{supremum}\{v_i\}, \text{supremum}\{w_i\}]$

**Definition 2.10** An interval-valued fuzzy set $A$ on a universe $X$ can be represented by the mapping:

$$A : X \rightarrow [0, 1]^2$$

**Definition 2.11** Let $X$ be a universe and $A$ and $B$ two interval-valued fuzzy sets. The equality between $A$ and $B$ is defined as: $A =_L B$ if and only if $A(a) =_L B(a) \forall a \in X$.

**Definition 2.12** Let $X$ be a universe and $A$ and $B$ two interval-valued fuzzy sets. The inclusion of $A$ in $B$ is defined as: $A \subseteq_L B$ if and only if $A(a) \leq_L B(a) \forall a \in X$.

**Definition 2.13** An interval-valued negation $\mathcal{N}$ is a decreasing function, $\mathcal{N} : L \rightarrow L$, that satisfies:

1. $\mathcal{N}(0_L) =_L 1_L$
2. $\mathcal{N}(1_L) =_L 0_L$
If \( \mathcal{N}(\mathcal{N}([x_1,x_2])) = L [x_1,x_2] \) then \( \mathcal{N} \) is called an involutive negation.

**Definition 2.14** A strong interval-valued negation \( \mathcal{N} \) is a strictly decreasing and involutive function, \( \mathcal{N} : L \rightarrow L \), that satisfies:

1. \( \mathcal{N}(0_L) = 1_L \)
2. \( \mathcal{N}(1_L) = 0_L \)

**Example 2.1** Let \( \mathcal{N} \) be the involutive mapping defined by:

\[
\mathcal{N} : L \rightarrow L \\
\mathcal{N}([x_1,x_2]) = [1 - x_2, 1 - x_1]
\]

Then \( \mathcal{N} \) is a negation operator for interval-valued fuzzy sets. It is trivial to prove that: \( \mathcal{N}(0_L) = 1_L \), \( \mathcal{N}(1_L) = 0_L \) and \( \mathcal{N}(\mathcal{N}([x_1,x_2])) = L [x_1,x_2] \).

**Definition 2.15** A generalized t-norm function \( \mathcal{T} \) is a monotone increasing, symmetric and associative operator, \( \mathcal{T} : L^2 \rightarrow L \), that satisfies: \( \mathcal{T}(1_L, [x_1,x_2]) = L [x_1,x_2] \) for all \([x_1,x_2] \in L \).

**Example 2.2** Let \( \text{Inf}_L \) be defined as follows:

\[
\text{Inf}_L([x_1,x_2], [y_1,y_2]) = \text{Meet} \{ [x_1,x_2], [y_1,y_2] \}
\]

Then, if \( \text{Inf}_L \) is a generalized t-norm.

### 3. Specificity for Interval-valued Fuzzy Sets

**Definition 3.1** An operator \( G : [0,1]^n \rightarrow [0,1] \) is an operator of specificity if it is continuous and it is increasing for the first argument and decreasing for the others and satisfies:

- \( G(1,0,0) = 1 \)
- \( G(0,0,0) = 0 \)

**Lemma 3.1** Let \( \mu \) be a fuzzy set on \( X \). Let \( \{ \mu(a_i) \} \) for all \( i = 1..n \) the list of membership degrees of \( \mu \) decreasing order. Let \( G : [0,1]^n \rightarrow [0,1] \) be an operator of specificity. Then \( G(\mu(a_1),...,\mu(a_n)) \) is a measure of specificity for \( \mathcal{F}/\mathcal{F}'s \).

**Proof.** Trivial by definition 2.2.

**Definition 3.2** An operator \( f(x,y) : [0,1]^2 \rightarrow [0,1] \) with \( x \leq y \) is called transformation operator if it is continuous, increasing and verifies:

1. \( f(1,1) = 1 \)
2. \( f(0,0) = 0 \)
3. \( f(0,x) > 0 \) for all \( x \in (0,1] \)
4. \( f(x,1) < 1 \) for all \( x \in [0,1) \)

Some examples of transformation operators are the following:

**Example 3.1**

\[
f(x,y) = \frac{x+y}{2}
\]

**Example 3.2**

\[
f(x,y) = \alpha x + \beta y
\]

with \( \alpha + \beta = 1, \alpha > 0, \beta > 0 \)

**Example 3.3**

\[
f(x,y) = \frac{x^2+y^2}{2}
\]

**Definition 3.3** Let \( \mu \) be an interval-valued fuzzy set on \( X \) and let \( \{[x_1,q,x_2] \} \) for all \( q = 1..n \) be its membership intervals. Let \( f \) be a transformation operator. Then, the \( f \)-list of \( \mu \) is the set of all the membership intervals of elements of \( X \), ordered decreasingly through the operator \( f \), that is, \( [x,y] \leq_f [z,t] \) if and only if \( f(x,y) \leq f(z,t) \).

**Example 3.4** Let \( X \) be the universe with cardinality \( 5 \) and let \( \mu \) be the following interval-valued fuzzy set:

\[
\mu = \{ [0.8,0.9]/e_1, [0.2,0.4]/e_2, [0.8,1.0]/e_3, [0.1,0.2]/e_4, [0.0,0.1]/e_5 \}
\]

Then, if \( f(x,y) = (x+y)/2 \) then:

<table>
<thead>
<tr>
<th>([x,y])</th>
<th>([f(x,y)])</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0,0.9]</td>
<td>0.85</td>
</tr>
<tr>
<td>[0.2,0.4]</td>
<td>0.30</td>
</tr>
<tr>
<td>[0.8,1.0]</td>
<td>0.90</td>
</tr>
<tr>
<td>[0.1,0.2]</td>
<td>0.15</td>
</tr>
<tr>
<td>[0.0,0.1]</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The \( f \)-list of \( \mu \) is:

\[
\{ [0.8,0.9], [0.8,0.9], [0.2,0.4], [0.1,0.2], [0.0,0.1] \}
\]

**Definition 3.4** An interval-valued fuzzy set \( \mu \) on \( X \) is a singleton if there exists an element \( a_i \in X \) such that \( \mu(a_i) = 1_L \) and \( \mu(a_j) = 0_L \) (for all \( j \neq i \)) for the others.
Definition 3.5 Let \( ([0,1]^2)^X \) be the set of interval-valued fuzzy sets on \( X \). Let \( f \) be a transformation operator. Let \( \{[x_{1q},x_{2q}]\} \) for all \( q = 1..n \) be the \( f \)-list of \( \mu \). A \( f \)-measure of specificity for interval-valued fuzzy sets is a function \( Sp_f : ([0,1]^2)^X \to [0,1] \) such that:

- \( Sp_f(\mu) = 1 \) if and only if \( \mu \) is a singleton.
- \( Sp_f(\emptyset) = 0 \).
- If \([x_{1q},x_{2q}] \) increases (according to \( \leq_L \)) then \( Sp_f(\mu) \) increases.
- If \([x_{1q},x_{2q}] \) increases (according to \( \leq_L \)) then \( Sp_f(\mu) \) decreases for all \( q : 2..n \).

Definition 3.6 An interval-valued fuzzy set \( \mu \) on \( X \) is normal if there exists an element \( a \in X \) such that \( \mu(a) = 1 \).

Definition 3.7 Let \( ([0,1]^2)^X \) be the set of membership degrees of interval-valued fuzzy sets on \( X \). A weak measure of specificity for interval-valued fuzzy sets is a function \( Sp : ([0,1]^2)^X \to [0,1] \) such that:

- \( Sp(\mu) = 1 \) if and only if \( \mu \) is a singleton.
- \( Sp(\emptyset) = 0 \).
- If \( \mu \) and \( \eta \) are normal fuzzy sets in \( X \) and \( \mu \subseteq \eta \), then \( Sp(\mu) \geq Sp(\eta) \).

Lemma 3.2 If \( Sp_f \) is an \( f \)-measure of specificity for interval-valued fuzzy sets then \( Sp_f \) is a weak measure of specificity for interval-valued fuzzy sets.

Proof. Let \( \{[x_{1q},x_{2q}]\} \) and \( \{[y_{1q},y_{2q}]\} \) for all \( q = 1..n \) be the \( f \)-list of \( \mu \) and \( \eta \) respectively. If \( \mu \) and \( \eta \) are normal and \( \mu \subseteq \eta \) then \( [x_{1q},x_{2q}] \leq_L [y_{1q},y_{2q}] \) for all \( q : 2..n \). According to the fourth axiom of the definition 3.5 \( Sp_f(\mu) \geq Sp_f(\eta) \).

Example 3.5 In Yager shows a particular case of function of transformation, \( f \), (called \( Q_f \)). Let \( \mu \) be an interval-valued fuzzy set on \( X \) with \( \mu(a_q) = [x_{1q},x_{2q}] \) for all \( q : 1..n \). Let \( Q_f(a_q) = f(x_{1q},x_{2q}) \) such that \( x \leq f(x,y) \leq y \) for all \( x,y \). Let \( a_i \) be the element of \( X \) which maximizes \( Q_f \). Then, the following expression is a measure of specificity for interval-valued fuzzy sets:

\[
Sp = Q_f(a_i) - \frac{1}{n-1} \sum_{k \neq i} Q_f(a_k).
\]

Lemma 3.3 Let \( \mu \) be an interval-valued fuzzy set on \( X \) and let \( Sp_f \) be any \( f \)-measure of specificity over \( \mu \). Let \( \{[x_{1q},x_{2q}]\} \) for all \( q : 1..n \) the \( f \)-list of \( \mu \). Then, there exists an operator of specificity \( G : [0,1]^n \to [0,1] \) such that:

\[
Sp_f(\mu) = G(f(x_{11},x_{21}),...,f(x_{1n},x_{2n}))
\] (1)

Corollary 3.1 Let \( G \) be a measure of specificity for \( F \).s. Let \( f \) a transformation operator. Then \( G(f(x_{11},x_{21}),...,f(x_{1n},x_{2n})) \) is a measure for \( F \).s.

Definition 3.8 Let \( Sp_f \) and \( Sp'_{g} \) be two measures of specificity. \( Sp_f \) is more strict than \( Sp'_{g} \), denoted by \( Sp_f \leq Sp'_{g} \), if for all set, \( \mu \), it verifies: \( Sp_f(\mu) \leq Sp'_{g}(\mu) \).

Theorem 3.1 \( Sp_f \) is more strict than \( Sp'_{g} \) if and only if \( f(x,y) \leq g(x,y) \) for all \( x,y \).

Proof. Trivial

Theorem 3.2 Let \( f \) be a transformation operator and \( \{\alpha_i\} \) a set of weights that satisfies:

- \( \alpha_i \in (0,1] \)
- \( \sum_{j=2}^{n} \alpha_j = 1 \)
- \( \{\alpha_j\} \) is not increasing.

Let \( T, T', S \) and \( N \) be, two \( T \)-norms, a \( T \)-conorm and a negation in \( [0,1], \leq \) respectively. Let \( \{f(x_{1q},x_{2q})\} \) be the \( f \)-list of an interval-valued fuzzy set \( \mu \). Then

\[
Sp_f(\mu) = T(f(x_{11},x_{21}),S(N(S'(\alpha_2,f(x_{12},x_{22}))),...,T'(\alpha_n,f(x_{1n},x_{2n})))))
\]

is a \( f \)-measure of specificity for interval-valued fuzzy set.

This expression is a generalization of the \( T \)-norm based measure of specificity given in but extended for \( F \).s.

Proof.

1. \( Sp_f(\mu) = 1 \) if and only if \( \mu \) is a singleton:

- If \( \mu \) is a singleton then \([x_{11},x_{21}] = [1,1]\) and \([x_{1k},x_{2k}] = [0,0]\) for all \( k > 1 \). Then \( f(x_{11},x_{21}) = 1 \) and \( f(x_{1k},x_{2k}) = 0 \) for all \( k > 1 \).
• If \( Sp_f(\mu) = 1 \), it is necessary that \( f(x_1, x_2) = 1 \) and
  \[ S(T(\alpha_2, f(x_1, x_2)), \ldots, T(\alpha_n, f(x_1, x_2))) = 0 \]
  Then \( T(\alpha_k, f(x_1, x_2)) = 0 \) for all \( k \) and \( f(x_1, x_2) = 0 \) for all \( k \).

2. \( Sp_f(\emptyset) = 0 \): trivial.

3. Trivial due to the fact \( T', T \) and \( S \) are monotonic

Let \( \{\alpha_i\} \) be a set of weights which satisfies the conditions of theorem 3.2.

**Example 3.6** With \( T(a, b) = \max\{0, a + b - 1\} \),
\[ N(a) = 1 - a, \]
\[ S(a, b) = \min\{1, a + b\}, \]
\[ T'(a, b) = a * b \text{ and } f(x, y) = \frac{x + y}{2}, \text{ it is obtained:} \]
\[ Sp_f(\mu) = \frac{1}{2}(x_1 + x_2) - \sum_{j=2}^{n} \alpha_j(x_1 + x_2) \]

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\[ S(a, b) = \min\{1, a + b\}, \]
\[ T'(a, b) = a * b \text{ and } f(x, y) = \alpha * x + \beta * y \text{ with } \alpha + \beta = 1, \alpha > 0, \beta > 0, \text{ it is obtained:} \]
\[ Sp_f(\mu) = \alpha * x_1 + \beta * x_2 - \sum_{j=2}^{n} \alpha_j(\alpha * x_1 + \beta * x_2) \]

**Example 3.8** With \( T(a, b) = \max\{0, a + b - 1\} \),
\[ N(a) = 1 - a, \]
\[ S(a, b) = \min\{1, a + b\}, \]
\[ T'(a, b) = a * b \text{ and } f(x, y) = \frac{x^2 + y^2}{2}, \text{ it is obtained:} \]
\[ Sp_f(\mu) = \frac{1}{2}(x_1^2 + x_2^2) - \sum_{j=2}^{n} \alpha_j(x_1^2 + x_2^2) \]

Examples 3.6 and 3.7 are extensions of \( R \), Yager’s linear measure of specificity \( S^\alpha \) for \( \forall \alpha, \alpha^\prime \).

4. **Alpha cuts for interval-valued fuzzy sets**

**Definition 4.1** Let \( \mu \) be an interval-valued fuzzy set on \( X \). The \( \alpha_1, \alpha_2 \) cuts of \( \mu \) are subsets of \( X \) defined as follows:

\[ \mu_{\alpha_1, \alpha_2} = \{ a_i | \mu(a_i) \geq \alpha_2 \} \]

**Definition 4.2** Let \( R \) be an interval-valued relation \( R: X^2 \rightarrow L \). The \( \alpha_1, \alpha_2 \) cut of \( R \), \( R_{\alpha_1, \alpha_2} \), is a crisp relation defined for all \( \alpha_1, \alpha_2 \) in \( [0, 1] \) as follows:
\[ R_{\alpha_1, \alpha_2}(a_i, a_j) = \begin{cases} 1 & R(a_i, a_j) \geq \alpha_2, \alpha_1; \\ 0, & \text{otherwise.} \end{cases} \]

**Lemma 4.1** Let \( R = [R_{\text{down}}, R_{\text{up}}] \) be an interval-valued fuzzy relation on \( X \) where \( R_{\text{down}} \) and \( R_{\text{up}} \) are fuzzy relations on \( X \), it is, \( R(a_i, a_j) = [R_{\text{down}}(a_i, a_j), R_{\text{up}}(a_i, a_j)] \) for all \( a_i, a_j \) in \( X \). Then, \( R_{\alpha_1, \alpha_2}(a_i, a_j) = 1 \) if and only if \( R_{\text{down}}(a_i, a_j) = 1 \) and \( R_{\text{up}}(a_i, a_j) = 1 \)

**Proof.** Trivial due to definition 4.2

**Lemma 4.2** Let \( R, S \) be two fuzzy relations. If
\[ R(a_i, a_j) = S(\alpha_1, \alpha_2) \text{ for all } a_i, a_j \text{ on } X \text{ and for all } \alpha \text{ in } [0, 1] \text{ then } R(a_i, a_j) = S(a_i, a_j). \]

**Proof.** Let’s suppose that there exist \( r, s \) such that:
\[ R(a_r, a_s) \neq S(a_r, a_s). \]
\[ R(p(a_r, a_s)) = 1 \text{ and } S(p(a_r, a_s)) = 0 \text{ where } p = R(a_r, a_s) \text{ which is a contradiction. If } R(a_r, a_s) < S(a_r, a_s) \text{ a similar contradiction can be found.} \]

**Proposition 4.1** The set of all \( \alpha_1, \alpha_2 \) cuts of an interval-valued fuzzy relation \( R \) determine \( R \).

**Proof.** By lemma 4.1 the \( \alpha_1, \alpha_2 \) cuts of an interval-valued fuzzy relation \( R \) are determined by the \( \alpha \) cuts of \( R_{\text{down}} \) and the \( \alpha \) cuts of \( R_{\text{up}} \) which by lemma 4.2 are determined by the fuzzy relations \( R_{\text{down}} \) and \( R_{\text{up}} \) that define \( R = [R_{\text{down}}, R_{\text{up}}] \), so the \( \alpha_1, \alpha_2 \) cuts of \( R \) determine \( R \).

**Corollary 4.1** Let \( R, S \) be two interval-valued fuzzy relations. If \( R_{\alpha_1, \alpha_2}(a_i, a_j) = S(\alpha_1, \alpha_2) \text{ for all } a_i, a_j \text{ on } X \text{ and for all } \alpha_1, \alpha_2 \text{ in } [0, 1] \text{ then } R(a_i, a_j) = S(a_i, a_j). \)

**Proof.** Trivial due to proposition 4.1

**Definition 4.3** Let \( \mathcal{T} \) be a generalized \( t \)-norm. An interval-valued relation \( R: X^2 \rightarrow L \) is a generalized \( \mathcal{T} \)-indistinguishability if it is reflexive, symmetric and \( \mathcal{T} \)-transitive, it is:

1. \( R(a, a) =_L 1L \text{ for all } a \text{ in } X. \)
2. \( R(a, b) =_L R(b, a) \text{ for all } a, b \text{ in } X. \)
3. $\mathcal{T} (R(a,b), R(b,c)) \leq_L R(a,c)$ for all $a,b,c$ in $X$.  

**Lemma 4.3** Let $R : X^2 \to L$ be a generalized $\text{Inf}_{L}$-indistinguishability. Then, for each $\alpha_{1}, \alpha_{2}$, $R_{\alpha_{1}, \alpha_{2}}$ is an equivalence relation.

**Proof.**

1. $R_{\alpha_{1}, \alpha_{2}}(a_{i}, a_{i}) = 1$ trivially.

2. $R_{\alpha_{1}, \alpha_{2}}(a_{i}, a_{j})$ is $R_{\alpha_{1}, \alpha_{2}}(a_{j}, a_{i})$ trivially.

3. Due to the fact that $R$ is a $\text{Inf}_{L}$-indistinguishability:

$$\text{Inf}_{L}(R(a_{i}, a_{k}), R(a_{k}, a_{j})) \leq_L R(a_{i}, a_{j})$$

for all $a_{i}, a_{j}, a_{k}$. Then, $R$ is transitive.

**Theorem 4.1** Let $R : X^2 \to L$ be an interval-valued relation. If for each $\alpha_{1}, \alpha_{2}$, $R_{\alpha_{1}, \alpha_{2}}$ is an equivalence relation if and only if $R$ is a $\text{Inf}_{L}$-indistinguishability.

**Proof.** Trivial due to the lemmas 4.3 and 4.4

**Corollary 4.2** Let $R : X^2 \to L$ be an interval-valued relation. Then, $R$ is a $\text{Inf}_{L}$-indistinguishability if and only if $R_{\alpha_{1}, \alpha_{2}}$ and $R_{\alpha_{1}, \alpha_{2}}$ are equivalence relations for all $\alpha_{1}, \alpha_{2}$.

**Theorem 4.2** Let $R : X^2 \to L$ be a generalized $\mathcal{T}$-indistinguishability (with $\mathcal{T} \neq \text{Inf}_{L}$). Then, there exists some $\alpha_{1}, \alpha_{2}$, such that $R_{\alpha_{1}, \alpha_{2}}$ is not an equivalence relation.

**Proof.** Let $R : X^2 \to L$ be a generalized $\mathcal{T}$-indistinguishability (with $\mathcal{T} \neq \text{Inf}_{L}$). Let $a_{i}, a_{j}$ be elements of the universe $X$ such that: $\mathcal{T}(R(a_{i}, a_{k}), R(a_{k}, a_{j})) = 1$. Then, $\{\alpha_{1}, \alpha_{2}\}$ be such that: $[\alpha_{1}, \alpha_{2}] = \text{Inf}_{L}(R(a_{i}, a_{k}), R(a_{k}, a_{j}))$. Then, due the fact that $\text{Inf}_{L}$ is the greatest of the generalized t-norms:

$$[\alpha_{1}, \alpha_{2}] \leq L \text{Inf}_{L}(R(a_{i}, a_{k}), R(a_{k}, a_{j})) \leq L R(a_{i}, a_{j})$$

therefore: $[\alpha_{1}, \alpha_{2}] \leq L R(a_{i}, a_{j})$ and so $R_{\alpha_{1}, \alpha_{2}}$ is transitive.

**Lemma 4.4** Let $R : X^2 \to L$ be an interval-valued relation. If for each $\alpha_{1}, \alpha_{2}$, $R_{\alpha_{1}, \alpha_{2}}$ is an equivalence relation, then $R$ is a $\text{Inf}_{L}$-indistinguishability.

**Proof.**

1. $R(a_{i}, a_{j}) = 1$ by contradiction.

2. $R(a_{i}, a_{j}) = R(a_{j}, a_{i})$ by contradiction.

3. It is supposed that $R$ is not a $\text{Inf}_{L}$-indistinguishability:

$$\text{Inf}_{L}(R(a_{i}, a_{k}), R(a_{k}, a_{j})) > L R(a_{i}, a_{j})$$

for some $a_{i}, a_{j}, a_{k}$ in $X$.

Then, it is found a $R_{\alpha_{1}, \alpha_{2}}$ that is not a equivalence relation: $\epsilon$ and $\delta$ be two real number arbitrarily small such that $\alpha_{1} = R(a_{i}, a_{j}) - \epsilon$ and $\alpha_{2} = R(a_{i}, a_{j}) - \delta$. Then $R_{\alpha_{1}, \alpha_{2}}(a_{i}, a_{k}) = 1$ and $R_{\alpha_{1}, \alpha_{2}}(a_{k}, a_{j}) = 1$ but $R_{\alpha_{1}, \alpha_{2}}(a_{i}, a_{j}) = 0$, i.e $R_{\alpha_{1}, \alpha_{2}}$ is not transitive.

**5. Specificity for Interval-valued Fuzzy Sets under generalized similarities**

**Proposition 5.1** Let $\mu$ be an interval-valued fuzzy set on $X$. Let $[\alpha_{1}, \alpha_{2}] = \text{Joint} \{\mu(a_{i})\}$ for all $i : 1..n$. Then:

$$Sp(\mu) = 2 \times \int_{0}^{\alpha_{1}} \int_{0}^{\alpha_{2}} \frac{1}{\text{card}(\mu_{\alpha_{1}, \alpha_{2}})} d\alpha_{1} d\alpha_{2}$$

$I$ is a measure of specificity for $\mathcal{T}/\mathcal{T}/\mathcal{T}_s$.

Note that the integration area guarantees that $\text{card}(\mu_{\alpha_{1}, \alpha_{2}}) = 0$.

**Proof.**

- Axiom 1:

1. If $\mu$ is a singleton then $Sp(\mu/S) = 1$: 

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• Let \( a_k \) be the only element on \( X \) such that \( \mu(a_k) = 1_L \).

• Then \( \mu_{\alpha_1, \alpha_2} = a_k \) for all \( \alpha_1, \alpha_2 \) and 
\[ \text{card} (\mu_{\alpha_1, \alpha_2}) = 1 \]
for all \( \alpha_1, \alpha_2 \) and 
\[ [\alpha_1, \alpha_2] = [1, 1] \].

• Then 
\[ 2 \int_0^1 \int_0^1 1d\alpha_1 \, d\alpha_2 = 1 \]

2. So that \( Sp(\mu) = 1 \) it is necessary that 
\[ [\alpha_1, \alpha_2] = [1, 1] \] and \( \text{card} (\mu_{\alpha_1, \alpha_2}) = 1 \). Otherwise \( Sp(\mu) < 1 \). Hence \( \mu \) is a singleton.

- Axiom 2:
  Trivial.

- Axiom 3: Let \( \{x_1, x_2, \ldots, x_n\} \) for all \( q = 1 \ldots n \) be the f-list of \( \mu \).

  1. If \( [x_1, x_2] \) increases then \( [\hat{\alpha}_1, \hat{\alpha}_2] \) increases 
  and \( \text{card} (\mu_{\alpha_1, \alpha_2}) \) does not change.

  2. If \( [x_1, x_2] \) for all \( q : 2 \ldots n \) increases then 
  \( 1/\text{card} (\mu_{\alpha_1, \alpha_2}) \) decreases.

\( \square \)

In a set of axioms that generalize the specificity of a fuzzy set under T-indistinguishabilities is given.

**Definition 5.1** \( ^4 \) Let \( Sp \) a measure of specificity for \( \mathcal{F} X \). \( Sp(\mu/S) \) is a measure of specificity under a generalized similarity \( S \) if it verifies:

1. \( Sp(\mu/S) = 1 \) if and only if \( \mu \) is a singleton.

2. \( Sp(\emptyset/S) = 0 \).

3. \( Sp(\mu/1d) = Sp(\mu) \).

4. \( Sp(\mu/S) \geq Sp(\mu) \).

**Definition 5.2** An interval-valued relation \( R : X \times X \rightarrow L \) is a generalized similarity if it is reflexive, symmetric and \( In_{L} \)-transitive where \( In_{L}([x_1, x_2], [y_1, y_2]) = \min (x_1, y_1, \min (x_2, y_2)) \), it is, \( R \) is an \( In_{L} \)-indistinguishability.

**Definition 5.3** Let \( \mu \) be a fuzzy set on \( X \) and let \( S \) be a similarity \( S : X \times X \rightarrow [0, 1] \). Let \( \pi_{\alpha_1, \alpha_2} \) be the set of classes of equivalence of the \( \alpha_1, \alpha_2 \) cut of \( S \). The set of classes of equivalence under the knowledge of \( S \mu_{\alpha_1, \alpha_2}/S \) is the subset of equivalence classes of the \( \alpha_1, \alpha_2 \) cut of \( S \) defined in that way: a equivalence class of the \( \alpha_1, \alpha_2 \) cut of \( S \) belongs to \( \mu_{\alpha_1, \alpha_2}/S \) if its intersection with \( \mu_{\alpha_1, \alpha_2} \) is not empty.

**Example 5.1** Let \( E = \{e_1, e_2, e_3, e_4\} \). Let \( \mu = \{[0.6, 0.8]/e_1 + [0.7, 0.8]/e_2 + [0.8, 0.8]/e_3 + [0.9, 1.0]/e_4\} \) and
\[ S = \begin{pmatrix}
1 & 0.1 & 0.1 & 0.1 \\
0.1 & 1 & 0.8 & 0.6 \\
0.1 & 0.8 & 1 & 0.6 \\
0.1 & 0.6 & 0.6 & 1
\end{pmatrix} \]

Then, \( \pi_{0.7, 0.8} = \{\{e_1\}, \{e_2, e_3\}, \{e_4\}\} \) \( \mu_{0.7, 0.8} = \{e_2, e_3, e_4\} \) and \( \pi_{0.7, 0.8}/S = \{\{e_2, e_3\}, \{e_4\}\} \)

**Proposition 5.2** Let \( \mu \) be an interval-valued fuzzy set on \( X \) and let \( S \) be a similarity \( S : X \times X \rightarrow [0, 1] \). Then:
\[ Sp(\mu/S) = 2 + \int_0^1 \int_0^1 1d\alpha_1 \, d\alpha_2 \]

It is a measure of specificity for \( \mathcal{F} X \).

Note that the integration area guarantees that \( \text{card} (\mu_{\alpha_1, \alpha_2}/S) \) is not zero.

**Proof.** Let \( \{\pi_{\alpha_1, \alpha_2}\} \) for all \( \alpha \) be the set of equivalence classes of \( \pi_{\alpha_1, \alpha_2} \).

- Axiom 1:

  1. If \( \mu \) is a singleton then \( Sp(\mu/S) = 1 \):

   - Let \( a_k \) be the only element on \( X \) such that \( \mu(a_k) = 1_L \).

   - Then \( \mu_{\alpha_1, \alpha_2} = a_k \) for all \( \alpha_1, \alpha_2 \).

   - There exists only a \( \pi_{\alpha_1, \alpha_2} \) such that \( a_k \) belongs to it.

   - And \( \text{card} (\mu_{\alpha_1, \alpha_2}/S) = 1 \) for all \( \alpha_1, \alpha_2 \).

   - Then 
\[ 2 + \int_0^1 \int_0^1 1d\alpha_1 \, d\alpha_2 = 1 \]

  2. If \( Sp(\mu/S) = 1 \) then \( \mu \) is a singleton:

   If \( Sp(\mu/S) = 1 \) then \( \text{card} (\mu_{\alpha_1, \alpha_2}/S) = 1 \) for all \( \alpha_1, \alpha_2 \) and \( \mu \) is a singleton.

- Axiom 2:
  Trivial.
• Axiom 3:
Remember that $X = \{a_1, \ldots, a_n\}$, then if $R$ is the
relation identity then $\{\pi_{\alpha_1, \alpha_2}\} = \{a_i\}$ for all $i: 1 \ldots n$
and $\text{card}(\mu_{\alpha_1, \alpha_2}/S) = \text{card}(\mu_{\alpha_1, \alpha_2})$.

• Axiom 4:
For a relation $S$ there will exist $\alpha_1, \alpha_2$ such
that $\text{card}(\pi_{\alpha_1, \alpha_2}) > 1$ and $\text{card}(\mu_{\alpha_1, \alpha_2}/S) < \text{card}(\mu_{\alpha_1, \alpha_2})$.

6. Conclusion

Several expression for t-norm based measure of
specificity for $\mathcal{I}$-$\mathcal{V}$-$\mathcal{F}$-$\mathcal{S}$s have been proposed and
studied.

An generalized expression for measures of speci-
city have been proposed for $\mathcal{I}$-$\mathcal{V}$-$\mathcal{F}$-$\mathcal{S}$s and the
measures of specificity under the knowledge of gen-
eralized similarities have also been defined follow-
ning the Yager’s jacket ideas.

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References

1. C. Cornelis, G. Deschrijver, and E. Kerre. Advances
and challenges in interval-valued fuzzy logic. Fuzzy
2. C. Cornelis, G. Deschrijver, and E. Kerre. Implication
in intuitionistic fuzzy and interval-valued fuzzy set
theory: construction, classification, application. Int.
On t-norms based specificity measures. Fuzzy Sets
General measures of specificity of fuzzy sets under
t-indistinguishabilities. IEEE Transactions on Fuzzy
A t-norm based specificity for fuzzy sets on compact
domains. International Journal of General Systems,
6. R. González-del Campo and L. Garmendia. Specific-
ity, uncertainty and entropy measures of interval-
valued fuzzy sets. Proceedings EUROFUSE Work-
shop Preference Modelling and Decision Analysis,
7. I. Grattan-Guiness. Fuzzy membership mapped onto
interval and many-valued quantities. Math. Logik.
9. E. Sanchez and R. Sambuc. Fuzzy relationships. phi-
fuzzy functions. application to diagnostic aid in thy-
roid pathology. Proceedings of an International Sym-
posium on Medical Data Processing, pages 513–524,
1976.
10. R.R. Yager. Measuring tranquility and anxiety in
decision-making - an application of fuzzy-sets. In-
ternational Journal of General Systems, 8:139–146,
1982.
11. R.R. Yager. Ordinal measures of specificity. Interna-
fuzzy sets. International Journal Of Fuzzy Systems,
14. R.R. Yager. On the measure of specificity of intuition-
istic fuzzy sets. NAFIPS 2008 · Annual Meeting of the
North American Fuzzy Information Processing Soci-
ey, pages 677–682, 2008.
15. L.A. Zadeh. Similarity relations and fuzzy orderings.
16. L.A. Zadeh. The concept of a linguistic variable and
its application to approximate reasoning i. Informa-