

Entropy Production Estimates For a Total Variation Image Model

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Abstract. The main aim of this paper is to give some theoretical analysis for a nonlinear higher order total variation image model. Applying the algebraic approach, we transform the analytical problem of proving entropy dissipation inequality into an algebraic problem about the nonnegativity of certain polynomial. As a result, we derive some key entropy production estimates of radial symmetric solutions in higher dimensions, which improve and extend some previous results.

Introduction

In the last decades, nonlinear partial differential equations (PDEs) are commonly used in image processing. In this paper, we continue to investigate the following nonlinear fourth order PDE

$$u_t + \Delta(u^{-1}\Delta u) = 0 \quad (1)$$

Eq.1 is an effective model for noise removal in image processing ^[1] and corresponds to the famous total variation diffusivity ^[2, 3, 4, and 5]. The one dimensional case of the initial boundary value problem of Eq.1 was discussed in ^[1, 6], based on the entropy dissipation method. The term “entropy” is widely defined in various subjects and frequently used for a positive Lyapunov functional in mathematics ^[6-10]:

Suppose that a PDE for the variable $U = U(x, t)$ possesses positive functional $E(U)$, such that

$$E(U(x, t)) + \delta_0 \int_0^t P(U(x, \tau), U_x(x, \tau), \dots) d\tau \leq E(U(x, 0)), \quad (2)$$

For a constant $\delta_0 > 0$, then functional $E(U(x, t))$ is called an entropy with $P > 0$ which is the corresponding entropy production. The inequality (2.1) is called entropy dissipation estimates, which can be used to estimate the large time behavior of solutions.

For a model with practical significance, it is natural to investigate it in higher dimensions. Recently, the author ^[11] proved that there also exist entropies for Eq.1 in higher dimensions. Let $U(x, t) = u(r, t)$ with $r = |x|$ be a smooth radially symmetric solution of the following fourth order nonlinear degenerate problem

$$\begin{cases} U_t + \Delta(U^{-1}\Delta U) = 0, & x \in \Omega = B^N, \quad t > 0, \\ \nabla U \cdot \nu = \nabla(U^{-1}\Delta U) \cdot \nu = 0, & x \in \partial\Omega, \quad t > 0, \\ U(x, 0) = U_0(|x|) & x \in \Omega, \end{cases} \quad (3)$$

where B^N is a unit ball in R^N ($N \geq 1$), ν is the outer normal vector at the boundary of the unit sphere, and let

$$E_{\alpha}(U(x, t)) = \int_0^1 \omega_N e(u(r, t)) r^{N-1} dr, \quad (4)$$

Then ^[6,11] the functional $E(t) := E_{\alpha}(U(x, t))$, defined in (4) is an entropy which is nonincreasing in

time t if $1 \leq \alpha \leq 4$ for $N=1$ and $\frac{3N+4-4\sqrt{N}}{N+2} \leq \alpha \leq 4$ for $N=2,3$. Furthermore, there holds

$$u_t(r, t) = -\nabla_x \cdot \left\{ \nabla_x \left[u^{-n} \left(u_{rr} + \frac{N-1}{r} u_r \right) \right] \right\} - \nabla_x \cdot \left\{ F(r) \cdot \frac{x}{r} \right\}, \quad (5)$$

$$F(r) = \left[u^{-n} \left(u_{rr} + \frac{N-1}{r} u_r \right) \right]_r = u^{-n+1} \left\{ \frac{u_{rrr}}{u} - n \frac{u_r}{u} \frac{u_{rr}}{u} + \frac{N-1}{r} \left[\frac{u_{rr}}{u} - n \left(\frac{u_r}{u} \right)^2 - \frac{u_r}{ru} \right] \right\}$$

where

$$= u^{-n+1} \{ \eta_3 - n \eta_1 \eta_2 + (N-1) \xi \eta_2 - n(N-1) \xi \eta_1^2 - (N-1) \xi^2 \eta_1 \}, \quad (6)$$

here we have identified $\left(\frac{\partial_r^i u}{u} \right)^m = \eta_i^m$ ($i=1,2,3$) and $\xi = \frac{1}{r}$ for simplification.

This paper continues to consider the entropy production estimates to the radial solution of problem (3). The main results are as follows:

Theorem. Let $U(x, t) = u(r, t)$ be a positive radially symmetric smooth solution of problem (4), $N=2,3$, then the entropy dissipation inequality (2) holds with

$$P(U) = \int_{\Omega} (\Delta U^{\frac{\alpha-1}{2}})^2 dx \quad (\alpha \neq 1),$$

and

$$0 < \delta_0 \leq \begin{cases} \frac{4[(\alpha-3)^2 N^2 + 4(\alpha-2)^2 + 4N(\alpha^2 - 5\alpha + 2)]}{(\alpha-1)^3 (N+2)(\alpha N - 5N + 2\alpha - 6)}, & \text{if } \frac{3N+4-4\sqrt{N}}{N+2} < \alpha \leq \frac{13N+20}{3(N+2)}, \\ \frac{32(4-\alpha)}{(\alpha-1)^3}, & \text{if } \frac{13N+20}{3(N+2)} < \alpha < 4. \end{cases} \quad (7)$$

The following two lemmas will be used in the proof of main results.

Lemma 1[12] Let

$$P(\xi, \eta_1, \eta_2) = a_1 \eta_1^4 + a_2 \eta_1^2 \eta_2 + a_3 \eta_2^2 + a_4 \xi \eta_1^3 + a_5 \xi^2 \eta_1^2 + a_6 \xi \eta_1 \eta_2, \quad (8)$$

Be a polynomial with real coefficients. Then the quantified formula

$$\forall (\xi, \eta_1, \eta_2) \in R^3 : P(\xi, \eta_1, \eta_2) \geq 0 \quad (9)$$

is equivalent to either $a_3=0$ and $a_2=a_6=0$ and $\left[(4a_1 a_5 - a_4^2 \geq 0, a_5 > 0) \text{ or } (a_1 \geq 0, a_4 = a_5 = 0) \right]$;

$$\text{or } a_3 > 0 \text{ and } \left[(4a_3a_5 - a_6^2 > 0, \quad 4a_1a_3a_5 - a_3a_4^2 - a_2^2a_5 - a_1a_6^2 + a_2a_4a_6 \geq 0) \text{ or } (4a_1a_3 - a_2^2 \geq 0, \right. \\ \left. 2a_3a_4 - a_2a_6 = 4a_3a_5 - a_6^2 = 0) \right]. \quad (10)$$

Lemma 2^[12] let the polynomial $P(x) = b_2x^2 + b_1x + b_0$ with $b_2 \geq 0$, and real numbers $z_1 < z_2$ be given. Then the quantified formula is equivalent to either

$$b_2 = 0 \text{ and } \left[(b_1 < 0, \quad b_1z_2 + b_0 < 0) \text{ or } (b_1 > 0, \quad b_1z_1 + b_0 < 0) \text{ or } (b_1 = 0, \quad b_0 \leq 0) \right]; \quad (11)$$

$$\text{or } b_2 > 0 \text{ and } \left[b_2z_1^2 + b_1z_1 + b_0 < 0 \text{ or } (b_1 + 2b_2z_1 < 0, \quad b_1^2 - 4b_0b_2 \geq 0) \right] \quad (12)$$

$$\text{and } \left[b_2z_2^2 + b_1z_2 + b_0 < 0 \text{ or } (b_1 + 2b_2z_2 > 0, \quad b_1^2 - 4b_0b_2 \geq 0) \right]. \quad (13)$$

Sketch Proof of Main results

In this section, following the ideals in [6, 12], we prove our main theorem. Firstly [11] for any $c_1, c_2 \in R$,

$$\frac{d}{dt} E_\alpha(U(x, t)) = -\omega_N \int_0^1 u^{\alpha-1} S_{c_1, c_2}(\xi, \eta) r^{N-1} dr, \quad (14)$$

$$\text{here } S_{c_1, c_2}(\xi, \eta) = b_1\eta_1^4 + b_2\eta_1^2\eta_2 + b_3\eta_2^2 + b_4\xi\eta_1^3 + b_5\xi^2\eta_1^2 + b_6\xi\eta_1\eta_2, \text{ for } b_1 = c_1(\alpha - 4),$$

$$b_2 = 3c_1 + \alpha - 2, \quad b_3 = 1, \quad b_4 = c_2(\alpha - 3) + (N - 1)(c_1 + 1), \quad b_5 = c_2(N - 2) + (N - 1), \quad b_6 = 2c_2.$$

On the other hand, entropy production

$$P(U) = \int_\Omega (\Delta U^{\frac{\alpha-1}{2}})^2 dx = \omega_N \int_0^1 u^{\alpha-1} S_p(\xi, \eta) r^{N-1} dr, \quad (15)$$

where

$$S_p(\xi, \eta) = \left(\frac{\alpha-3}{2}\right)^2 \left(\frac{\alpha-1}{2}\right)^2 \eta_1^4 + (\alpha-3) \left(\frac{\alpha-1}{2}\right)^2 \eta_1^2 \eta_2 + \left(\frac{\alpha-1}{2}\right)^2 \eta_2^2 + (\alpha-3) \left(\frac{\alpha-1}{2}\right)^2 (N-1) \xi \eta_1^3 \\ + \left(\frac{\alpha-1}{2}\right)^2 (N-1)^2 \xi^2 \eta_1^2 + 2 \left(\frac{\alpha-1}{2}\right)^2 (N-1) \xi \eta_1 \eta_2. \quad (16)$$

We now quantify the positive constant δ_0 for which there exist c_1, c_2 and for any $(\xi, \eta_1, \eta_2) \in R^3$, making

$$S_{\delta_0}(\xi, \eta) = d_1\eta_1^4 + d_2\eta_1^2\eta_2 + d_3\eta_2^2 + d_4\xi\eta_1^3 + d_5\xi^2\eta_1^2 + d_6\xi\eta_1\eta_2 \geq 0, \quad (17)$$

here

$$\begin{aligned} d_1 &= c_1(\alpha - 4) - \delta_0 \left(\frac{\alpha - 3}{2}\right)^2 \left(\frac{\alpha - 1}{2}\right)^2, & d_2 &= 3c_1 + \alpha - 2 - \delta_0(\alpha - 3) \left(\frac{\alpha - 1}{2}\right)^2, & d_3 &= 1 - \delta_0 \left(\frac{\alpha - 1}{2}\right)^2, \\ d_4 &= (N - 1)(c_1 + 1) + \left[c_2 - \delta_0 \left(\frac{\alpha - 1}{2}\right)^2 (N - 1) \right] (\alpha - 3), & d_5 &= c_2(N - 2) + (N - 1) - \delta_0 \left(\frac{\alpha - 1}{2}\right)^2 (N - 1)^2, \\ d_6 &= 2c_2 - 2\delta_0 \left(\frac{\alpha - 1}{2}\right)^2 (N - 1). \end{aligned} \quad (18)$$

According to Lemma 1, we just consider the case for $d_3 > 0$. Hence, (11) is equivalent to either

$$4d_1d_3 - d_2^2 = -9 \left[c_1 - \frac{\delta_0(\alpha - 1)^3 - 8 - 4\alpha}{36} \right]^2 + \frac{1}{144}(\alpha - 1) \left[\delta_0(\alpha - 1)^2 - 4 \right] \left[\delta_0(\alpha - 1)^3 + 32(\alpha - 4) \right] \geq 0 \quad (19)$$

and $2d_3d_4 - d_2d_6 = 4d_3d_5 - d_6^2 = 0$, (20)

or $0 < 4d_3d_5 - d_6^2 = -(c_2 + 1 - N) \left[4c_2 + 4 - (\alpha - 1)^2 N \delta_0 \right] \square -M_1$, (21)

and $0 \leq 4d_1d_3d_5 - d_3d_4^2 - d_2^2d_5 - d_1d_6^2 + d_2d_4d_6$

$$\begin{aligned} &= \frac{c_1^2}{4 - \delta_0(\alpha - 1)^2} \left\{ 9M_1 - \left[(\delta_0(\alpha - 1)^2 + 2)(N - 1) - 6c_2 \right]^2 \right\} + \left[2(c_2 + 1 - N)(N + 2) + \frac{M_1}{2}(\alpha - 1) \right] c_1 \\ &+ N(c_2 + 1 - N) + \frac{M_1}{4}(\alpha^2 - 4\alpha + 3). \end{aligned} \quad (22)$$

Lemma 2 is used and gives that

$$0 < \delta_0 \leq \begin{cases} \frac{4 \left[(\alpha - 3)^2 N^2 + 4(\alpha - 2)^2 + 4N(\alpha^2 - 5\alpha + 2) \right]}{(\alpha - 1)^3 (N + 2)(\alpha N - 5N + 2\alpha - 6)}, & \text{if } \frac{3N + 4 - 4\sqrt{N}}{N + 2} < \alpha \leq \frac{13N + 20}{3(N + 2)}, \\ \frac{32(4 - \alpha)}{(\alpha - 1)^3}, & \text{if } \frac{13N + 20}{3(N + 2)} < \alpha < 4. \end{cases} \quad (23)$$

We conclude the proof.

Conclusions

This paper gave some theoretical analysis for a nonlinear higher order total variation image model. Applying the algebraic approach, we transform the analytical problem of proving entropy dissipation inequality into an algebraic problem about the nonnegativity of certain polynomial. The results obtained improve and extend some previous conclusions.

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