The Critical Exponent for A Class Non-Autonomous Parabolic Equation in the K-Times Halved Space

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Abstract: This paper investigates the parabolic equation

$$u_t = \Delta u + t^\sigma |x|^m u^p, \quad (x, t) \in D \times (0, T) \quad (P^*)$$

with nonnegative initial data, where $D = R_K^N := \{x \in R^N \mid x_i > 0, i = 1, 2, \dots, K\}$, $0 \leq K \leq N$, $\sigma \geq 0, m > -2, p > 1$ and extend the classical result of Fujita and more recent results of Levine and Meier. We demonstrate that $p_c = 1 + \frac{2+2\sigma+m}{N+K}$ as its critical exponent, which means that problem (P^*) exhibited the following behavior: if $1 < p < p_c$, then every positive solution of the equation blow up in finite time, whereas if $p > p_c$, then there exist both global and nonglobal solutions.

Introduction and the Main Results

In this paper, we will consider existence and nonexistence of the following problem

$$\begin{cases} u_t = \Delta u + t^\sigma |x|^m u^p & (x, t) \in D \times (0, T) \\ u(x, t) = 0 & x \in \partial D \times (0, T) \\ u(x, 0) = u_0(x) \geq 0 & x \in D \end{cases} \quad (1.1)$$

Where $\sigma \geq 0, m > -2, p > 1$, $D = R_K^N := \{x \in R^N \mid x_i > 0, i = 1, 2, \dots, K\}$, $0 \leq K \leq N$, and $u_0(x)$ is a nonnegative, continuous and bounded function.

The study of blow-up for nonlinear parabolic equation originates from Fujita[1]. He considered the following cauchy problem of the semilinear heat equation

$$\begin{cases} u_t = \Delta u + u^p & (x, t) \in R^N \times (0, T) \\ u(x, 0) = u_0(x) \geq 0 & x \in R^N \end{cases} \quad (1.2)$$

He proved the following result:

(a) If $1 < p < 1 + 2/N$, then (1.2) possesses no global nonnegative solutions.

(b) If $p > 1 + 2/N$, both global and nonglobal nonnegative solutions exist.

The case $p = 1 + 2/N$ belongs to case (a), but this was established later (see[5] Weissler give

an elegant proof). We call $p_c = 1 + 2/N$ the critical exponent of (1.2).

Over the past few years, there have been a number of extensions of Fujita's results in various directions (see for example [1]-[13]). Levine [2] treated the following equation

$$\begin{cases} u_t = \Delta u + t^\sigma |x|^m u^p & (x, t) \in \mathbb{R}^N \times (0, T) \\ u(x, 0) = u_0(x) \geq 0 & x \in \mathbb{R}^N \end{cases} \quad (1.3)$$

he showed the critical exponent of (1.3) is $p_c = 1 + \frac{2+2\sigma+m}{N}$.

For the domain $D = \mathbb{R}_K^N$, Meier [6] studied the boundary-value problem

$$\begin{cases} u_t = \Delta u + t^\sigma u^p & (x, t) \in D \times (0, T) \\ u(x, t) = 0 & x \in \partial D \times (0, T) \\ u(x, 0) = u_0(x) \geq 0 & x \in D \end{cases} \quad (1.4)$$

He proved the critical exponent of (1.4) is $p_c = 1 + \frac{2+2\sigma}{N+K}$.

We are concerned problem (1.1) and our Fujita's type results are contained in the following two theorems.

Theorem 1. Suppose that $\sigma \geq 0, m > -2, p > 1, 0 \leq K \leq N$, and $u_0(x)$ is a nonnegative, continuous and bounded function. If $1 < p < 1 + \frac{2+2\sigma+m}{N+K}$, then every nonnegative solution of (1.1) blow up in finite time.

Theorem 2. Suppose that $\sigma \geq 0, m > -2, p > 1, 0 \leq K \leq N$, and $u_0(x)$ is a nonnegative, continuous and bounded function. If $p > 1 + \frac{2+2\sigma+m}{N+K}$, then

- (a) Every nonnegative solution of (1.1) blow up in finite time for $u_0(x)$ large enough.
- (b) (1.1) has global solutions for $u_0(x)$ small enough.

The method that we used is similar to the method used to prove the blow-up result by Pinsky in [8]. So we need the Green's function of the heat equation in the K -times halved space D (see [7]). That is

$$G(x, y, t) = (4\pi t)^{-\frac{N}{2}} \left(\prod_{i=1}^K D(x_i, y_i, t) \right) \left(\prod_{j=k+1}^N E(x_j, y_j, t) \right)$$

where $E(x, y, t) = \exp\left(-\frac{|x-y|^2}{4t}\right)$ and $D(x, y, t) = E(x, y, t) - E(x, -y, t)$.

Throughout the rest of this paper, we will use the notation

$$y = (y_1, \dots, y_K, y_{K+1}, \dots, y_N), \bar{y} = (-y_1, \dots, -y_K, y_{K+1}, \dots, y_N)$$

This paper is organized as follows: In section 2, we will prove some preliminary results. The proofs of the main results are completed in section 3.

Preliminary

In this section, we give some lemmas needed in the proofs of Theorem 1 and Theorem 2.

Lemma 2.1 Assume $G(x, y, t)$ is the Green's function of the heat equation in the K -times halved space D . Then

$$G(x, y, t) \geq 2^{-\frac{N}{2}} t^{-\frac{N-K}{2}} E(0, \bar{y}, \frac{t}{2}) \exp(-\frac{|x|^2}{2t}) (\prod_{i=1}^K x_i y_i) \quad (2.1)$$

Proof. By an elegant calculation, we show $D(x_i, y_i, t) = \exp(-\frac{x_i^2 + y_i^2}{4t}) [\exp(\frac{x_i y_i}{2t}) - \exp(-\frac{x_i y_i}{2t})]$

Applying the Mean Value Theorem, it follows that

$$D(x_i, y_i, t) \geq t^{-1} \exp(-\frac{x_i^2 + y_i^2}{4t}) x_i y_i \quad (2.2)$$

Thus from (2.2), we obtain

$$G(x, y, t) \geq (4\pi t)^{-\frac{N}{2}} t^{-K} \exp(-\frac{(x-\bar{y})^2}{4t}) (\prod_{i=1}^K x_i y_i) \quad (2.3)$$

Using the inequality $|x - y|^2 \leq 2|x|^2 + 2|y|^2$; we complete the proof.

Lemma 2.2 Suppose $u(x, t)$ satisfies (1.1). Then there exist $t_0, c_1 > 0$ such that

$$u(x, t) \geq c_1 t^{-\frac{N}{2}} \exp(-\frac{|x|^2}{2t}) (\prod_{i=1}^K x_i) \quad \text{for any } t > t_0. \quad (2.4)$$

Proof. Using Lemma 1 of [8], we can similarly prove that

$$u(x, t) \geq \int_D G(x, y, t) u_0(y) dy + \int_0^t \int_D G(x, y, t-s) |y|^m s^\sigma u^p(y, s) dy ds \quad (2.5)$$

Since $u_0(x) \geq 0$, it follows from (2.5) that

$$u(x, t) \geq \varepsilon \int_{B_\varepsilon(y_0)} G(x, y, t) dy \quad \text{for some } \varepsilon > 0 \text{ and some } y_0 \in D, \quad (2.6)$$

where $B_\varepsilon(y_0)$ denotes the ball of radius $\varepsilon > 0$ centered at y_0 .

By Lemma (2.1), we complete the proof.

Lemma 2.3 Suppose $u(x, t)$ satisfies (1.1) for all $T > 0$ and that

$$u(x, t) \geq c t^l \exp(-\frac{|x|^2}{t}) (\prod_{i=1}^K x_i) \quad \text{for } t > t_0 \quad (2.8)$$

where $c > 0, t_0 > 0$, and $l \in [-\frac{N}{2} - K, +\infty]$. Then there exist constants $C, t_1 > 0$, such that

$$u(x, t) \geq C t^{l+\sigma+lp+\frac{m+(p-1)K}{2}} \exp(-\frac{|x|^2}{t}) (\prod_{i=1}^K x_i), \quad \text{for } t > t_1 \quad (2.9)$$

Proof. From (2.5) and (2.8), we have

$$u(x, t) \geq c^p \int_{t_0}^t \int_D G(x, y, t-s) |y|^m s^{\sigma+lp} \exp\left(-\frac{p|y|^2}{t}\right) \left(\prod_{i=1}^K y_i^p\right) dy ds \quad (2.10)$$

$$\text{Using the equality } \exp\left(-\frac{(y_i - x_i)^2}{4(t-s)} - \frac{py_i^2}{s}\right) = \exp\left(-\frac{(y_i - r(s, t)x_i)^2}{4r(s, t)(t-s)}\right) \exp\left(-\frac{pr(s, t)x_i^2}{s}\right)$$

$$\text{Where } r(s, t) = \frac{s}{s + 4p(t-s)}.$$

$$\text{We obtain } D(x_i, y_i, t-s) \exp\left(-\frac{py_i^2}{s}\right) = D(r(s, t)x_i, y_i, t-s) \exp\left(-\frac{pr(s, t)x_i^2}{s}\right)$$

$$\begin{aligned} \text{And } G(x, y, t-s) \exp\left(-\frac{py^2}{s}\right) &\geq 2^{-\frac{N}{2}} (t-s)^{-\frac{N}{2}-K} \left(\prod_{i=1}^K x_i y_i\right) E(0, \bar{y}, r(s, t)(t-s)/2) \\ &\times \exp\left(-\frac{r(s, t)|x|^2}{2(t-s)}\right) \exp\left(-\frac{pr(s, t)|x|^2}{s}\right) \end{aligned} \quad (2.11)$$

Clearly $(1 + r(s, t))(t/(t-s)) < 4$, for $s \in [0, t/2]$ So we have

$$\exp\left(-\frac{r(s, t)|x|^2}{2(t-s)}\right) \exp\left(-\frac{pr(s, t)|x|^2}{s}\right) = \exp\left(-\frac{|x|^2}{4t} (1 + r(s, t))(t/(t-s))\right) \geq \exp\left(-\frac{|x|^2}{t}\right) \quad (2.12)$$

Substituting (2.11), (2.12) into (2.10), we conclude

$$u(x, t) = c^p 2^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{t}\right) \left(\prod_{i=1}^K x_i\right) \int_{t_0}^{t/2} \int_D E(0, \bar{y}, r(s, t)(t-s)/2) (t-s)^{-\frac{N}{2}-K} \left(\prod_{i=1}^K y_i^{p+1}\right) |y|^m s^{\sigma+lp} dy ds \quad (2.13)$$

Noticing that there exists constant $c_2 > 0$ such that

$$\int_D (4\pi t)^{-\frac{N}{2}} E(0, \bar{y}, t) \left(\prod_{i=1}^K y_i^{p+1}\right) |y|^m dy \geq c_2 t^{\frac{m+(p+1)K}{2}}, \text{ for } t \geq 1. \quad (2.14)$$

$$\text{We show } \int_D (r(s, t)(t-s))^{-\frac{N}{2}} E(0, \bar{y}, r(s, t)(t-s)/2) \left(\prod_{i=1}^K y_i^{p+1}\right) |y|^m dy \geq c_3 (r(s, t)(t-s))^{\frac{m+(p+1)K}{2}} \quad (2.15)$$

where c_3 is a positive constant, $r(s, t)(t-s)/2 \geq 1$.

An elegant calculation reveals that for sufficiently large $h \geq 1$,

$$r(s, t)(t-s)/2 \geq 1, \text{ if } t \geq 2h \text{ and } h \leq s \leq t-h$$

Without loss of generality, we suppose that $h \geq t_0$. From (2.13) and (2.15), it follows that there

exists constant $c_4 > 0$ such that

$$u(x, t) = c_4 \exp\left(-\frac{|x|^2}{t}\right) \left(\prod_{i=1}^K x_i\right) \int_h^{t/2} \int_D s^{\sigma+lp} (r(s, t))^{\frac{N+m+(p+1)K}{2}} (t-s)^{\frac{m+(p-1)K}{2}} ds, \text{ for } t \geq 2h. \quad (2.16)$$

Making the change of variables $z = s/t$, (2.16) can be rewritten as

$$u(x, t) = c_4 \exp\left(-\frac{|x|^2}{t}\right) \left(\prod_{i=1}^K x_i\right) t^{1+\sigma+lp+\frac{m+(p-1)K}{2}}$$

$$\int_h^{t/2} s^{\sigma+lp} \left(\frac{z}{z+4p(1-z)}\right)^{\frac{N+m+(p+1)K}{2}} (1-z)^{\frac{m+(p-1)K}{2}} dz, \text{ for } t \geq 2h. \quad (2.17)$$

If we now assume that $t \geq 4h$, from (2.17), we assert that there exist constants $C, t_1 \geq 0$ such

$$\text{that } u(x, t) \geq Ct^{l+\sigma+lp+\frac{m+(p-1)K}{2}} \exp\left(-\frac{|x|^2}{t}\right) \left(\prod_{i=1}^K x_i\right), \text{ for } t > t_1.$$

Lemma 2.4 Suppose $u(x, t)$ satisfies (1.1), Ω is bounded subset of D . If $1 < p < 1 + \frac{2+2\sigma+m}{N+K}$,

$$\text{Then } \lim_{t \rightarrow +\infty} u(x, t) = +\infty, \quad x \in \Omega. \quad (2.18)$$

Proof. Define the affine transformation $f(l) = 1 + \sigma + lp + \frac{m+(p-1)K}{2}$, for $l \in \mathbb{R}$.

$$\text{We find that } f^n\left(-\frac{N}{2} - K\right) = \frac{1}{p-1} \left[\frac{2+2\sigma+m-(p-1)(N+K)}{2} p^n - \left(\frac{2+2\sigma+m}{2} + \frac{Kp}{2} - \frac{K}{2}\right) \right].$$

The condition $1 < p < 1 + \frac{2+2\sigma+m}{N+K}$ guarantees that $2+2\sigma+m-(p-1)(N+K) > 0$.

$$\text{then we obtain } \lim_{t \rightarrow +\infty} f^n\left(-\frac{N}{2} - K\right) = +\infty. \quad (2.19)$$

By Lemma (2.2), (2.8) holds with $l = -\frac{N}{2} - K$, after applying n iterations of Lemma (2.3),

$$\text{we have } u(x, t) = Ct^{f^n\left(-\frac{N}{2} - K\right)} \exp\left(-\frac{|x|^2}{t}\right) \left(\prod_{i=1}^K x_i\right), \text{ for } t \geq t_n. \quad (2.20)$$

Now (2.18) follows from (2.19) and (2.20).

The Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1: Suppose that the conclusion of Theorem 1 is false. That is, when $1 < p < 1 + \frac{2+2\sigma+m}{N+K}$, (1.1) possesses global solutions. By the Lemma (2.4), we shift (if

necessary) the origin of time to $\bar{t}$ such that $u(x, \bar{t}) = u_0(x)$ is large enough, where $\bar{t}$ is a nonnegative constant. Let $\Omega \in D$ be a smooth, bounded domain satisfying $\inf_{x \in \Omega} |x|^m \geq \delta$. We consider the following boundary value problem

$$\begin{cases} U_t = \Delta U + \delta \bar{t}^{-\sigma} U^p & (x, t) \in \Omega \times (\bar{t}, T) \\ U(x, t) = 0 & x \in \partial\Omega \times (\bar{t}, T) \\ U(x, 0) = u_0(x) \geq 0 & x \in \Omega \end{cases} \quad (3.1)$$

By The Comparison Theorem, we obtain $u(x, t) \geq U(x, t), (x, t) \in \Omega \times (\bar{t}, T)$. (3.2)

Clearly, when $u_0(x)$ is large enough, $U(x, t)$ will blow up in finite time. It follows from (3.2) that $u(x, t)$ will blow up in finite time. This contradicts to the supposition that (1.1) possesses global solutions.

Proof of Theorem 2: Part (a) is obvious. Since $u_0(x)$ is large enough, the proof is almost the same as the proof of Theorem 1.

Now we prove part (b). We look for a supersolution of (1.1). Take

$$\bar{u}(x, t) = A(T + t)^{-\alpha} e^{-\omega|\eta|^2} \left(\prod_{i=1}^K x_i \right), \text{ where } \eta = \frac{x}{\sqrt{T-t}}. \quad (3.3)$$

By an elegant calculation, $\bar{u}(x, t)$ need satisfy

$$-\alpha + 2\omega N + 4\omega K \geq \omega(4\omega - 1)|\eta|^2 + A^{p-1}(T + t)^{1+\alpha-\alpha p} t^\sigma |x|^m e^{(1-p)\omega|\eta|^2} \left(\prod_{i=1}^K x_i \right)^{p-1} \quad (3.4)$$

Since $p > 1$, there exists a nonnegative constant C such that

$$t^\sigma |x|^m e^{(1-p)\omega|\eta|^2} \left(\prod_{i=1}^K x_i \right) \leq C(T + t)^{\sigma + \frac{m+(p-1)K}{2}} \quad (3.5)$$

$$\text{Choose } \omega = 1/4, (3.4) \text{ is satisfied if } -\alpha + \frac{N}{2} + K \geq CA^{p-1}(T + t)^{1+\alpha-\alpha p+\sigma+\frac{m+(p-1)K}{2}} \quad (3.6)$$

$$\text{Let } 1 + \alpha - \alpha p + \sigma + \frac{m+(p-1)K}{2} = 0. \text{ Then } \alpha = \frac{2+2\sigma+m}{2p-2} + \frac{K}{2}. \text{ Since } p > 1 + \frac{2+2\sigma+m}{N+K}.$$

So (3.6) holds when A is small enough. Thus, we complete the proof of part (b).

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