Soft points, $s$-relations and soft rough approximate operations

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Abstract

Soft set theory is a new mathematical tool to deal with uncertain problems. Since soft sets are defined by mappings and they lack “points”, managing them is not convenient. In this paper, the concept of soft points is introduced and the relationship between soft points and soft sets is investigated. We prove that soft sets can be translated into soft point sets and may be expediently handled like ordinary sets. Moreover, we propose $s$-relations on soft sets. By means of soft points and these results, a pair of soft rough approximate operations is defined. Serial, reflexive, symmetric, transitive and Euclidean $s$-relations are characterized by using soft rough approximate operations. In addition, we research soft topologies induced by a reflexive $s$-relation on a special soft set and gives their structure.

Keywords: Soft sets; Soft points; Soft point sets; $s$-relations; Soft rough approximate operations; Soft topologies.

1. Introduction

Most of traditional methods for formal modeling, reasoning and computing are crisp, deterministic and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, fuzzy set theory, interval mathematics, and rough set theory, which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see ). For example, probability theory can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Presently, works on soft set theory are progressing rapidly. Maji et al. further studied soft set theory, used this theory to solve decision making problems and devoted fuzzy soft sets combining soft sets with fuzzy sets. Roy et al. presented a fuzzy soft set theoretic approach towards decision making problems. Li et al. investigated decision making based on intuitionistic fuzzy soft sets. Jiang et al. extended soft sets with description logics. Aktas et al. defined soft groups. Li et al. proposed $L$-fuzzy soft sets based on complete Boolean lattices. Feng et al. investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. discussed relationships between soft sets and topologi-

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cal spaces. Shabir et al. 25 proposed soft topological spaces which are defined on the universe with a fixed set of parameters. Babitha et al. 5 introduced relations on soft sets. Li et al. 13,14 considered roughness of fuzzy soft sets and obtained the relationship among soft sets, soft rough sets and topologies. Li et al. 15 studied parameter reductions of soft coverings.

Rough set theory was proposed by Pawlak 22. It is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions in rough set theory. They can also be seen as a closure operator and an interior operator of the topology induced by an equivalence relation on a universe. We may relax equivalence relations so that rough set theory is able to solve more complicated problems in practice. Pawlak rough set theory has been extended to tolerance relations, similarity relations, binary relations 17,26,30.

Since soft sets are defined by mappings and then lack “points”, managing them is not convenient. Thus, we try to attempt introducing the concept of “soft points” and deal with them as same as ordinary sets.

Feng et al. 7 proposed soft rough approximate operations. But the introduction of these operations seemed suddenly and disposing them is not convenient as soft sets lacks “points” and “soft points” are not proposed. In this paper, we introduce the concept of soft points, prove that soft sets can be translate into soft point sets and then it is convenient to deal with soft sets as same as ordinary sets. We propose s-relations on soft sets. By means of soft points and these results, soft rough approximate operations are defined. And because we do the above work, it is very convenient to deal with the operations introduced by us.

The organization of this paper is as follows: In Section 2, we briefly recall basic concepts about rough sets, soft sets and soft topological spaces. In Section 3, we introduce the concept of soft points and investigate the relationship between soft points and soft sets. In Section 4, we introduce the concepts of serial, reflexive, symmetric, transitive and Euclidean s-relations on soft sets, and investigate the relationships between these s-relations and soft point sets. In Section 5, we propose two soft rough approximate operations. In Section 6, we investigate soft topologies induced by a reflexive s-relation on a special soft set and give their structure. Section 7 concludes this paper and highlights the prospects for potential future development.

2. Overview of rough sets, soft sets and soft topological spaces

In this section, we briefly recall basic concepts about rough sets, soft sets and soft topological spaces.

Throughout this paper, U refers to an initial universe, E refers to the set of parameters and 2^U denotes the power set of U. We only consider the case where both U and E are nonempty finite sets.

2.1. Rough sets

Let R be an equivalence relation on U. The pair (U, R) is called a Pawlak approximation space. Using the equivalence relation R, one can define the following rough approximations:

\[ R_s(X) = \{ x \in U : [x]_R \subseteq X \}, \]
\[ R^s(X) = \{ x \in U : [x]_R \cap X \neq \emptyset \}. \]

Then \( R_s(X) \) and \( R^s(X) \) called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively.

The Pawlak boundary region of X, defined by the difference between these Pawlak rough approximations, that is \( Bnd_R(X) = R^s(X) - R_s(X) \). It can easily be seen that \( R_s(X) \subseteq X \subseteq R^s(X) \).

A set is Pawlak rough if its boundary region is not empty. Otherwise, the set is crisp. Thus X is Pawlak rough if \( R_s(X) \neq R^s(X) \).

We may relax equivalence relations so that rough set theory is able to solve more complicated problems in practice. Pawlak rough set theory has been extended to binary relations 17,26,30.

**Definition 2.1** (30) Let R be a binary relation on U. The pair (U, R) is called a approximation space. Based on the approximation space (U, R), we define a pair of operations \( \overline{R}, \overline{R}^c : 2^U \rightarrow 2^U \) as follows:
\( R(X) = \{ x \in U : R(x) \subseteq X \} \),
\( \overline{R}(X) = \{ x \in U : R(x) \cap X \neq \emptyset \} \),
where \( X \in 2^U \) and \( R(x) = \{ y \in U : xRy \} \) is the successor neighborhood of \( x \). Then \( \overline{R}(X) \) and \( R(X) \) are called the lower approximation and the upper approximation of \( X \), respectively.

\( X \) is called a definable set if \( R(X) = \overline{R}(X) \); \( X \) is called a rough set if \( R(X) \neq \overline{R}(X) \).

### 2.2. Soft sets

**Definition 2.2** \((21)\) Let \( A \subseteq E \). A pair \((f, A)\) is called a soft set over \( U \), if \( f \) is a mapping given by \( f : A \to 2^U \). We denote \((f, A)\) by \( f_A \).

In other words, a soft set over \( U \) is the parameterized family of subsets of the universe \( U \). For \( \varepsilon \in A \), \( f(\varepsilon) \) may be considered as the set of \( \varepsilon \)-approximate elements of the soft set \( f_A \). Obviously, a soft set is not an ordinary set.

Denote \( S(U, E) = \{ f_E : f_E \) is a soft set over \( U \} \).

**Definition 2.3** \((18)\) Let \( A, B \subseteq E \), \( f_A \in S(U, A) \) and \( g_B \in S(U, B) \).

1. \( f_A \) is called a soft subset of \( g_B \), if \( A \subseteq B \) and \( \forall \varepsilon \in A, f(\varepsilon) \subseteq g(\varepsilon) \). We write \( f_A \subseteq g_B \).
2. \( f_A \) is called a soft super set of \( g_B \), if \( g_B \subseteq f_A \).
   We write \( f_A \supseteq g_B \).
3. \( f_A \) and \( g_B \) are called soft equal, if \( A = B \) and \( \forall \varepsilon \in A, f(\varepsilon) = g(\varepsilon) \). We write \( f_A = g_B \).

Obviously, \( f_A = g_B \) if and only if \( f_A \subseteq g_B \) and \( f_A \supseteq g_B \).

**Definition 2.4** \((1, 18)\) Let \( A, B \subseteq E \), \( f_A \in S(U, A) \) and \( g_B \in S(U, B) \).

1. \( h_{A \cup B} \) is called the union of \( f_A \) and \( g_B \), if
   \[
   h(\varepsilon) = \begin{cases} 
   f(\varepsilon), & \text{if } \varepsilon \in A - B, \\
   g(\varepsilon), & \text{if } \varepsilon \in B - A, \\
   f(\varepsilon) \cup g(\varepsilon), & \text{if } \varepsilon \in A \cap B.
   \end{cases}
   \]
   We write \( f_A \cup g_B = h_{A \cup B} \).
2. \( h_{A \cap B} \) is called the soft intersection of \( f_A \) and \( g_B \), if \( \forall \varepsilon \in A \cap B, h(\varepsilon) = f(\varepsilon) \cap g(\varepsilon) \). We write \( f_A \cap g_B = h_{A \cap B} \).

**Remark 2.5** Let \( A, B, C \subseteq E \), \( f_A \in S(U, A) \), \( g_B \in S(U, B) \) and \( h_C \in S(U, C) \). Then
1. \( f_A \cap g_B \subseteq f_A \) (or \( g_B \) \( \subseteq f_A \cup g_B \).
2. \( f_A \cap g_C \subseteq f_A \cap g_C \), hence \( f_A \subseteq f_A \cap g_B \).
3. \( f_A \cap g_C \subseteq f_A \cup g_B \), hence \( f_A \cap g_B \subseteq f_A \cap g_B \).

**Definition 2.6** \((25)\) Let \( A \subseteq E \), \( f_A, g_A, h_A \in S(U, A) \). \( h_A \) is called the difference of \( f_A \) and \( g_A \), if \( \forall \varepsilon \in A, h(\varepsilon) = f(\varepsilon) - g(\varepsilon) \). We write \( h_A = f_A - g_A \).

**Definition 2.7** \((3)\) Let \( A \subseteq E \), \( f_A, g_A \in S(U, A) \). \( g_A \) is called the relative complement of \( f_A \), if \( \forall \varepsilon \in A, g(\varepsilon) = U - f(\varepsilon) \). We write \( g_A = f_A' \) or \((f_A)'\).

**Proposition 2.8** \((3)\) Let \( A \subseteq E \), \( f_A, g_A \in S(U, A) \). Then
1. \((f_A) = f_A \).
2. \((f_A) = (f_A \cap g_A) = (g_A \cup f_A)' = (f_A \cup g_A)' = (g_A \cap f_A)\).

**Remark 2.9** Let \( A \subseteq E \), \( f_A, g_A \in S(U, A) \). Then
1. \((f_A)' = f_A \).
2. \((f_A)' = (f_A \cap g_A)' = (g_A \cup f_A)' = (f_A \cup g_A)' = (g_A \cap f_A)'\).

**Definition 2.10** \((25)\) Let \( X \subseteq 2^U \). The soft set \( X_E \) over \( U \) is defined by \( \forall \varepsilon \in E, X(\varepsilon) = X \).

In this paper, \( U_E \) and \( \emptyset_E \) are also denoted by \( \bar{U} \) and \( \emptyset \), respectively.

**Remark 2.11** Let \( f_A, g_A \in S(U, A) \). Then
1. \( U_A - f_A = f_A' \).
2. \( f_A \cap g_A = \emptyset_A \iff f_A \subseteq g_A' \).
3. \( f_A - g_A = f_A \cap g_A' \).

### 2.3. Soft topological spaces

In what follows we consider problems on the universe \( U \) and the fixed set \( E \) of parameters.

**Definition 2.12** \((25)\) \( \tau \subseteq S(U, E) \) is called a soft topology over \( U \), if (i) \( \emptyset, U \in \tau \); (ii) the union of any number of soft sets in \( \tau \) belongs to \( \tau \); (iii) the intersection of any two soft sets in \( \tau \) belongs to \( \tau \).

The triplet \((U, \tau, E)\) is called a soft topological space over \( U \). Every element of \( \tau \) is called a soft open set in \( U \) and its relative complement is called a soft closed set in \( U \).
In this paper, the family of all soft closed sets is denoted by \( \tau' \).

**Definition 2.13** \((25)\) Let \((U, \tau, E)\) be a soft topological space over \(U\). \(\forall f_E \in S(U, E)\), the soft closure of \(f_E\) is defined by
\[
\text{cl}(f_E) = \bigcap \{g_E : f_E \supseteq g_E \text{ and } g_E \in \tau'\}.
\]

**Definition 2.14** \((25)\) Let \((U, \tau, E)\) be a soft topological space over \(U\). \(\forall f_E \in S(U, E)\), the soft interior of \(f_E\) is defined by
\[
\text{int}(f_E) = \bigcup \{g_E : g_E \subset f_E \text{ and } g_E \in \tau'\}.
\]

**Proposition 2.15** \((25)\) Let \((U, \tau, E)\) be a soft topological space over \(U\). Then \(\forall f_E \in S(U, E)\), \(\text{int}(f_E) = U - \text{cl}(U - f_E)\).

3. Soft points

In this section, we will introduce the concept of soft points and investigate the relationship between soft points and soft sets.

3.1. The concept of soft points

In this subsection we define soft points, which originate from the concept of fuzzy points (see \(^{11,23}\)).

**Definition 3.1** Let \(f_E^* \in S(U, E)\). \(f_E^*\) is called a soft point over \(U\), if there exist \(e \in E\) and \(x \in U\) such that
\[
f_E^*(e) = \begin{cases} 
\{x\}, & \text{if } e = e, \\
\emptyset, & \text{if } e \in E - \{e\}.
\end{cases}
\]

We denote \(f_E^*\) by \((x_e)_E\).

In this case, \(x_e\) is called the support point of \((x_e)_E\), \(\{x\}\) is called the support point set of \((x_e)_E\) and \(e\) is called the expressive parameter of \((x_e)_E\).

**Example 3.2** Let \(U = \{x_1, x_2, x_3, x_4, x_5\}\) and \(E = \{e_1, e_2, e_3, e_4, e_5\}\). We define \(f(e_1) = \{x_1, x_4\}, f(e_2) = U, f(e_3) = \{x_3\}, f(e_4) = \emptyset\). Then \(\mathcal{F}(E) = \{(x_1)_E, (x_4)_E, (x_3)_E, (x_2)_E, (x_1)_E, (x_3)_E, (x_4)_E, (x_5)_E\}\) and \(P(U, E) = \{(x_1)_E \mid 1 \leq i \leq 5\}\). To illustrate the fact that the soft contain relation, the soft intersection operation, the soft union operation and the soft difference operation on two soft sets can be translated into the contain relation, the intersection operation, the union operation and the difference operation on two soft point sets (i.e., two ordinary sets), respectively, we give the following Proposition 3.5.

**Proposition 3.5** Let \(f_E, g_E, h_E \in S(U, E)\).

(1) If \(g_E \subseteq f_E\), then \(\mathcal{F}(E) \subseteq \mathcal{F}(E)\).

(2) If \(f_E = g_E \cap h_E\), then \(\mathcal{F}(E) = \mathcal{F}(E) \cap \mathcal{H}(E)\).

(3) If \(f_E = g_E \cup h_E\), then \(\mathcal{F}(E) = \mathcal{F}(E) \cup \mathcal{H}(E)\).

(4) If \(f_E = g_E \cap h_E\), then \(\mathcal{F}(E) = \mathcal{F}(E) - \mathcal{H}(E)\).

**Proof.** (1) This is obvious.

(2) Let \((x_e)_E \in \mathcal{F}(E)\). Then \(x \in f(e)\). Since \(f_e = g_e \cap h_e\), we have \(x \in g(e)\) and \(x \in h(e)\). Thus \((x_e)_E \in \mathcal{F}(E)\) and \((x_e)_E \in \mathcal{F}(E)\). Hence \((x_e)_E \in \mathcal{F}(E) \cap \mathcal{H}(E)\). Conversely, the proof is similar.

(3) The proof is similar to (2).

(4) The proof is similar to (2). \(\square\)

**Proposition 3.6** (1) If \(f_E = U_E\), then \(P(U, E) = \mathcal{F}(E)\).

(2) \(P(U, E) = \bigcup \{\mathcal{F}(E) : f_E \in S(U, E)\}\).
Proof. (1) This is obvious.
(2) Let \( f_e \in S(U,E) \). Since \( f_e \supseteq U_E \), by Proposition 3.5 and (1), \( \mathcal{F}(E) \subseteq P(U,E) \). Thus \( P(U,E) \supseteq \cup \{ \mathcal{F}(E) : f_e \in S(U,E) \} \).

Conversely, since \( U_E \in S(U,E) \), by (1), we have \( P(U,E) \subseteq \cup \{ \mathcal{F}(E) : f_e \in S(U,E) \} \).

Hence \( \mathcal{F}(E) = \cup \{ \mathcal{F}(E) : f_e \in S(U,E) \} \). \( \square \)

3.2. Soft points and soft sets

In this subsection, we will investigate the relationship between soft points and soft sets.

Definition 3.7 Let \( f_e \in S(U,E) \) and \( (x_e)_E \in P(U,E) \). We define \( (x_e)_E \in f_e \) by \( (x_e)_E \supseteq f_e \).

Note that \( (x_e)_E \notin f_e \), if \( (x_e)_E \not\supseteq f_e \).

Remark 3.8 (1) \( (x_e)_E = (x\prime_e)_E \iff x = x\prime \) and \( e = e\prime \).
(2) \( (x_e)_E \supseteq f_e \iff x \in f(e) \) and \( e \in E \iff (x_e)_E \in \mathcal{F}(E) \).
(3) \( (x_e)_E \in f_e \) and \( f_e \supseteq g_e \Rightarrow (x_e)_E \supseteq g_e \).
(4) \( (x_e)_E \in (x_e\prime)_E \).
(5) \( (x_e)_E \supseteq f_e \iff (x_e)_E \notin f'_e \).

Theorem 3.9 Let \( f_e \in S(U,E) \). Then \( f_e = \bigcup \mathcal{F}(E) \).

Proof. Denote \( h_E = \bigcup \mathcal{F}(E) \). Then \( h_E = \bigcup \{(x_e)_E : x \in f(e) \) and \( e \in E \} \). Thus

\[
h_E = \bigcup_{e \in E} \bigcup_{x \in f(e)} (x_e)_E.
\]

\( \forall e \in E, \)

\[
h(e) = \bigcup_{x \in E \times f(e)} \bigcup_{x \in f(e)} (x_e)_E = \bigcup_{x \in E \times f(e)} \bigcup_{x \in f(e)} (x_e)_E.
\]

This shows \( h_E = f_E \). Hence \( f_E = \bigcup \mathcal{F}(E) \). \( \square \)

Remark 3.10 Theorem 3.9 reveals the fact that a soft set can be translated into a soft point set and vice versa.

Theorem 3.11 Let \( f_E, g_E \in S(U,E) \). Then
(1) \( f_E \supseteq g_E \iff \mathcal{F}(E) \subseteq \mathcal{G}(E) \).
(2) \( f_E = g_E \iff \mathcal{F}(E) = \mathcal{G}(E) \).

Proof. These hold by Proposition 3.5 and Theorem 3.9.

Remark 3.12 Theorem 3.11 illustrates that the soft contain relation and the soft equal relation can be respectively translated into the contain relation and the equal relation on two soft point sets (i.e., two ordinary sets) and vice versa.

When we study some problems of soft sets by using soft points in this paper, we will abide by the following logic thinking: firstly, the soft contain relation, the soft intersection operation, the soft union operation and the soft difference operation on soft sets are translated into the contain relation, the intersection operation, the union operation and the difference operation on soft point sets by Proposition 3.5, respectively; secondly, the relations and operations on ordinary sets (i.e., soft point sets) are realized; thirdly, the results of the relations and operations on ordinary sets are translated into the results on soft sets by Theorem 3.9.

4. \( s \)-relations on soft sets

In this section, we introduce the concepts of serial, reflexive, symmetric, transitive and Euclidean \( s \)-relations on soft sets, and investigate the relationships between these \( s \)-relations and soft point sets.

Definition 4.1 (\( \hat{s} \)) Let \( A, B \subseteq E, f_A \in S(U,A) \) and \( g_B \in S(U,B) \). \( h_{A \times B} \) is called the cartesian product of \( f_A \) and \( g_B \), if \( \forall (a, b) \in A \times B, h(a, b) = f(a) \times g(b) \). We write \( h_{A \times B} = f_A \times g_B \).

Definition 4.2 (\( \hat{s} \)) Let \( A, B \subseteq E, f_A \in S(U,A) \) and \( g_B \in S(U,B) \).
(1) \( R \) is called a relation from \( f_A \) to \( g_B \), if \( R \subseteq f_A \times g_B \).
(2) \( R \) is called a relation on \( f_A \), if \( R \subseteq f_A \times f_A \).

In other words, a relation \( R \) from \( f_A \) to \( g_B \) is of the form \( l_P \), where \( P \subseteq A \times B \) and \( \forall (a, b) \in P \),
Definition 4.3 Let $f_E \in S(U, E)$. $R$ is called a surjective relation (brief: s-relation) on $f_E$, if there exists a soft set $I_{E \times E}$ over $U \times U$ such that $R = I_{E \times E} \subset f_E$. 

Remark 4.4 $R$ is a s-relation on $f_E \Rightarrow R$ is a relation on $f_E$. 

Example 4.5 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $E = \{e_1, e_2\}$. We define $f(e_1) = \{x_1, x_3, x_5\}$, $f(e_2) = \{x_2, x_4\}$. Then $f_E \in S(U, E)$ and $E \times E = \{(e_1, e_1), (e_1, e_2), (e_2, e_1), (e_2, e_2)\}$. 

Let $h_{E \times E} = f_E \times f_E$. Then 

$h(e_1, e_1) = f(e_1) \times f(e_1)$, $h(e_1, e_2) = f(e_1) \times f(e_2)$, 

$h(e_2, e_1) = f(e_2) \times f(e_1)$ and $h(e_2, e_2) = f(e_2) \times f(e_2)$.

(1) Define $l : E \times E \rightarrow 2^{U \times U}$ by

\[ l(e_1, e_1) = \{(x_1, x_1), (x_1, x_3), (x_1, x_5), (x_3, x_3), (x_3, x_5)\}, \]

\[ l(e_1, e_2) = f(e_1) \times f(e_2), \]

\[ l(e_2, e_1) = \{(x_2, x_1), (x_2, x_3), (x_2, x_5), (x_4, x_3), (x_4, x_5)\} \]

and

\[ l(e_2, e_2) = f(e_2) \times f(e_2). \]

Then

\[ l(e_1, e_1) \subseteq h(e_1, e_1), \quad l(e_1, e_2) \subseteq h(e_1, e_2), \]

\[ l(e_2, e_1) \subseteq h(e_2, e_1) \quad \text{and} \quad l(e_2, e_2) \subseteq h(e_2, e_2). \]

So $I_{E \times E} \subset f_E \times f_E$. 

Put $R_1 = I_{E \times E}$. Then $R_1$ is a s-relation on $f_E$.

(2) Put $P = \{(e_1, e_1), (e_1, e_2)\}$. Then $P \subseteq E \times E$. 

Define $k : P \rightarrow 2^{U \times U}$ by 

\[ k(e_1, e_1) = f(e_1) \times f(e_1) \quad \text{and} \quad k(e_1, e_2) = f(e_1) \times f(e_2). \]

Put $R_2 = kP$. Since $R_2 \subset f_E \times f_E$, $R_2$ is a relation on $f_E$. But $R_2$ is not a s-relation on $f_E$. 

Since soft sets can be translated into soft point sets, every relation on a soft set can be translated into a relation on a soft point set. We introduce the following Definition 4.6 for this reason.

Definition 4.6 Let $R$ be a s-relation on $f_E \in S(U, E)$. Define a relation $R^*$ on $\mathcal{F}(E)$ as follows: for any $(x_e) \in \mathcal{F}(E)$, $(x_{e'}) \in \mathcal{F}(E)$, 

\[ (x_e) \in R^*(x_{e'}) \iff (x_e) \in (x_{e'}) \subset R. \]

Then $R^*$ is called the relation induced by $R$. 

Remark 4.7 (1) $(x_e) \times (x_{e'}) \subset f_E \times f_E \iff x \in f(e)$ and $x' \in f(e')$. 

(2) Let $R = I_{E \times E} \subset f_E \times f_E$. Then 

\[ (x_e) \in R^*(x_{e'}) \iff (x, x') \in l(e, e') \]

\[ \quad \quad \quad \quad \quad \Rightarrow \quad x \in f(e), \quad x' \in f(e'). \]

Definition 4.8 Let $R$ be a s-relation on $f_E$. $R$ is called serial (resp. reflexive, symmetric, transitive, Euclidean), if $R^*$ is serial (resp. reflexive, symmetric, transitive, Euclidean).

Let $f_E \in S(U, E)$. Denote $S_f(U, E) = \{g \in S(U, E) : g_e \subset f_E\}$. 

Let $R$ be a s-relation on $f_E$ and $R^*$ the relation induced by $R$. \forall (x_e) \in \mathcal{F}(E), g \in S_f(X, E)$, put 

\[ R^*(x_e) = \{(x_{e'}) \in \mathcal{F}(E) : (x_e) \in R^*(x_{e'})\}, \]

\[ P_{fE}(fE, R) = \{(x_e) \in \mathcal{F}(E) : R^*(x_e) \subseteq \mathcal{F}(E)\}, \]

\[ P_{fE}(fE, R) = \{(x_e) \in \mathcal{F}(E) : R^*(x_e) \cap \mathcal{F}(E) \neq \emptyset\}. \]

Remark 4.9 Let $R$ be a s-relation on $f_E \in S(U, E)$ and $R^*$ the relation induced by $R$. Then 

(1) $R$ is serial $\iff \forall (x_e) \in \mathcal{F}(E), R^*(x_e) \neq \emptyset$. 

(2) $R$ is reflexive $\iff \forall (x_e) \in \mathcal{F}(E), (x_e) \in R^*(x_e)$. 

(3) $R$ is symmetric $\iff \forall (x_e) \in \mathcal{F}(E), (x_{e'}) \in R^*(x_e)$ implies $(x_{e'}) \in R^*(x_e)$. 

(4) $R$ is transitive $\iff \forall (x_e) \in \mathcal{F}(E), (x_{e'}) \in R^*(x_{e'})$ and $(x_{e''}) \in R^*(x_{e''})$ implies $(x_{e''}) \in R^*(x_{e'})$. 

$\iff \forall (x_e) \in \mathcal{F}(E), (x_{e'}) \in R^*(x_{e'})$ and $(x_{e''}) \in R^*(x_{e''})$ implies $R^*(x_{e''}) \subseteq R^*(x_{e'})$. 

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(5) \( R \) is Euclidean \( \iff \forall \, (x_e)_E, (x'_e)_E, (x''_e)_E \in \mathcal{F}(E), \,(x'_e)_E \in R^*( (x_e)_E) \) and \( (x''_e)_E \in R^*( (x_e)_E) \) implies \( R^*( (x''_e)_E) \subseteq R^*( (x'_e)_E) \).

Lemma 4.10 Let \( R \) be a \( s \)-relation on \( f_E \in S(U,E) \). Then \( \forall \, g_E, h_E \in S_f(U,E) \),

(1) \( P_g(f_E, R) = \mathcal{F}(E) \).
(2) \( a \) Let \( (x_e)_E \in P_g(f_E, R) \). Thus \( R^*( (x_e)_E) \subseteq \mathcal{F}(E) \). Since \( R \) is serial, by Remark 4.9, \( R^*( (x_e)_E) \neq \emptyset \). This implies \( R^*( (x_e)_E) \cap \mathcal{F}(E) \neq \emptyset \). So \( (x_e)_E \in P^g(f_E, R) \). Thus \( P_g(f_E, R) \subseteq P^g(f_E, R) \).

\( b \) Let \( (x_e)_E \in P_g(f_E, R) \). Then \( R^*( (x_e)_E) \subseteq \mathcal{F}(E) \). Since \( R \) is reflexive, by Remark 4.9, we have \( (x_e)_E \in R^*( (x_e)_E) \subseteq \mathcal{F}(E) \). Thus \( P_g(f_E, R) \subseteq \mathcal{F}(E) \). Since \( (x_e)_E \in R^*( (x_e)_E) \) and \( (x_e)_E \in \mathcal{F}(E) \), \( R^*( (x_e)_E) \cap \mathcal{F}(E) \neq \emptyset \). Thus \( \mathcal{F}(E) \subseteq P^g(f_E, R) \).

(3) \( a \) Let \( (x_e)_E \in P_h(f_E, R) \). Then \( R^*( (x_e)_E) \subseteq \mathcal{F}(E) \). Since \( g_E \subseteq h_E \), \( \mathcal{F}(E) \subseteq \mathcal{F}(E) \) and \( R^*( (x_e)_E) \subseteq \mathcal{F}(E) \). Thus \( (x_e)_E \in P_h(f_E, R) \). Hence \( P^g(f_E, R) \subseteq P_h(f_E, R) \).

\( b \) The proof is similar to \( a \).

(4) \( a \) Let \( (x_e)_E \in P^h(f_E, R) \). Then \( R^*( (x_e)_E) \cap \mathcal{F}(E) \neq \emptyset \). Since \( l_E = g_E \cup h_E \), by Proposition 3.5, \( R^*( (x_e)_E) \cap \mathcal{F}(E) \neq \emptyset \) and \( R^*( (x_e)_E) \cap \mathcal{F}(E) \neq \emptyset \). Thus \( (x_e)_E \in P^g(f_E, R) \) and \( (x_e)_E \in P^h(f_E, R) \). Hence \( P^h(f_E, R) \subseteq P^g(f_E, R) \cup P^h(f_E, R) \).

Conversely, this is obvious.

\( b \) The proof is similar to \( a \). \( \square \)

5. Soft rough approximate operations

In this section, we propose two soft rough approximate operations. Serial, reflexive, symmetric, transitive and Euclidean \( s \)-relations are characterized by using them.

Definition 5.1 Let \( R \) be a \( s \)-relation on \( f_E \in S(U,E) \). Then the pair \( P = (f_E, R) \) is called a soft approximation space. Based on \( P \), we define the following operations \( \text{apr}_P, \text{appr}_P : S_f(U,E) \to S_f(U,E) \) by

\[ \text{apr}_P(g_E) = \bigcup P_g(f_E, R), \quad \text{appr}_P(g_E) = \bigcap P^g(f_E, R), \]

where \( g_E \in S_f(U,E) \). Then, \( \text{apr}_P \) and \( \text{appr}_P \) are called the soft \( P \)-lower approximation operator and the soft \( P \)-upper approximation operator on \( f_E \), respectively; \( \text{apr}_P(g_E) \) and \( \text{appr}_P(g_E) \) are called the soft \( P \)-lower approximation of \( g_E \) and the soft \( P \)-upper approximation of \( g_E \), respectively.

\( g_E \) is called a soft \( P \)-definable set if \( \text{apr}_P(g_E) \neq \text{appr}_P(g_E) \); \( g_E \) is called a soft \( P \)-rough set if \( \text{apr}_P(g_E) \neq \text{appr}_P(g_E) \).

Remark 5.2 In \( \mathcal{F} \), Feng et al. proposed two operations \( \text{apr}_P, \text{appr}_P : 2^U \to 2^U \) by

\[ \text{apr}_P(X) = \{ u \in U : \exists \ e \in E, \text{ s.t. } u \in f(e) \subseteq X \}, \]

\[ \text{appr}_P(X) = \{ u \in U : \exists \ e \in E, \text{ s.t. } u \in f(e) \text{ and } f(e) \cap X \neq \emptyset \}. \]

where \( X \in 2^U \), \( P = (f_E, E) \) and \( f_E \in S(U,E) \).

Lemma 5.3 Let \( R \) be a \( s \)-relation on \( f_E \in S(U,E) \). Then \( \forall \, g_E, h_E \in S_f(U,E) \), we have

(1) \( h_E = \text{appr}_P(g_E) \iff \mathcal{H}_P = P_h(f_E, R) \).
(2) \( h_E = \text{appr}_P(g_E) \iff \mathcal{H}_P = P^h(f_E, R) \).

Proof. (1) Sufficiency. This holds by Theorem 3.11.

Necessity. Denote \( P_h(f_E, R) = \{ y_a : y \in X \text{ and } a \in A \} \) where \( X \subseteq U \) and \( A \subseteq E \).

Let \( (x_e)_E \in \mathcal{H}_P \). Then \( x \in h(e) = \bigcup \{ y_a : y \in X \text{ and } a \in A \} \).

We claim that \( e \in A \). Otherwise, \( y_a(e) = \emptyset \ \forall \ y \in X \text{ and } a \in A \). Then \( h(e) = \bigcup \{ y_a(e) : y \in X \text{ and } a \in A \} = \emptyset \), a contradiction.
Let $R$ be a $s$-relation on $f_E$. Then
$$\forall (x,y) \in R^*(x,y) \in R \subseteq \mathcal{F}.$$
Suppose \( (x_e)_E \in R^*( (x_e)_E ) \). Then (1) \( ap_{p}(f_e) = ap_{\gamma}(f_e) = f_e \).
(2) \( ap_{p}(0) = ap_{\gamma}(0) = 0 \).

**Proposition 5.9** Let \( R \) be reflexive on \( f_e \in S(U,E) \). Then
(1) \( ap_{p}(f_e) = ap_{\gamma}(f_e) = f_e \).
(2) \( ap_{p}(0) = ap_{\gamma}(0) = 0 \).

**Proof.** (1) By Lemma 4.10, \( ap_{p}(f_e) \subset ap_{\gamma}(f_e) \).

Conversely, since \( P^*(f_e,R) \subset \not F(E) \), then \( ap_{\gamma}(f_e) \subset f_e \). By Lemma 4.10, \( f_e = ap_{\gamma}(f_e) \).
Thus \( ap_{p}(f_e) \subset ap_{\gamma}(f_e) \). Therefore, \( ap_{p}(f_e) = ap_{\gamma}(f_e) \).

(2) This is obvious.

**Proposition 5.10** Let \( R \) be a \( s \)-relation on \( f_e \). Then the following are equivalent:
(1) \( R \) is symmetric.
(2) \( \forall \, ge \in S_f(U,E) \), \( ap_{p}(ap_{p}(ge)) \subset ge \subset ap_{p}(ap_{\gamma}(ge)) \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( ge \in S_f(U,E) \). Denote \( k_e = ap_{p}(ge) \), \( w_e = ap_{\gamma}(k_e) \), \( h_e = ap_{\gamma}(w_e) \).

Suppose \( \not F(E) \subset \not F(E) \neq \emptyset \). Pick \( (x_e)_E \in \not F(E) \) and \( (x_e)_E \in \not F(E) \).
Then \( (x_e)_E \notin \not F(E) \) and \( (x_e)_E \notin \not F(E) \). By Lemma 5.3, \( (x_e)_E \in P^*(f_e,R) \).
This implies \( R^*( (x_e)_E ) \cap \not F(E) \neq \emptyset \). Pick \( (x'_e)_E \in R^*( (x_e)_E ) \cap \not F(E) \). Then \( (x'_e)_E \in \not F(E) \).
By Lemma 5.3, \( (x_e)_E \in P^*(f_e,R) \).
This implies \( R^*( (x'_e)_E ) \subset \not F(E) \). Since \( R \) is symmetric, \( (x_e)_E \in \not F(E) \).
Hence \( (x_e)_E \in \not F(E) \). By Theorem 3.11, \( w_e \subset ge \).

Therefore, \( ap_{p}(ap_{p}(ge)) \subset ge \).

Suppose \( \not F(E) \subset \not F(E) \neq \emptyset \). Pick \( (x_e)_E \in \not F(E) \) and \( (x_e)_E \notin \not F(E) \).
By Lemma 5.3, \( (x_e)_E \notin P^*(f_e,R) \).
This implies \( R^*( (x_e)_E ) \subset \not F(E) \).
Then \( (x_e)_E \notin \not F(E) \) and \( (x_e)_E \notin \not F(E) \). By Lemma 5.3, \( (x_e)_E \notin P^*(f_e,R) \).
This implies \( R^*( (x_e)_E ) \cap \not F(E) = \emptyset \).
Since \( (x_e)_E \in R^*( (x_e)_E ) \), by \( R \) is symmetric, we have \( (x_e)_E \in R^*( (x_e)_E ) \).
Thus \( \not F(E) = \emptyset \), a contradiction. Hence \( \not F(E) \subset \not F(E) \).
By Theorem 3.11, \( w_e \subset l_e \).

Therefore, \( ge \subset ap_{p}(ap_{p}(ge)) \).

(2) \( \Rightarrow \) (1). Let \( (x_e)_E \in \not F(E) \). Then \( (x'_e)_E \in \not F(E) \).
Denote \( g_e = (x_e)_E \), \( h_e = ap_{p}(g_e) \), and \( l_e = ap_{p}(h_e) \).

By Lemma 5.3 and Lemma 5.5, \( l_e = \not F(E) \).
Then \( g_e \subset l_e \subset h_e \). This implies \( (x_e)_E \in \not F(E) \).
(2) \( \Rightarrow \) (1). Let \( ap_{p}(f_e) = ap_{\gamma}(f_e) \).

Then \( \not F(E) \subset \not F(E) \).
By Lemma 5.3, \( (x_e)_E \in \not F(E) \).
Hence \( (x'_e)_E \in \not F(E) \).
By Lemma 5.5, \( (x'_e)_E \in \not F(E) \).
Therefore, \( R \) is symmetric.
Proposition 5.12 Let $R$ be reflexive on $f_{\in E} \in S(U, E)$. Then the following are equivalent.

1. $R$ is transitive;
2. $\forall g_{E} \in S_{f}(U, E)$, 
   \[ \text{apr}_{P}(g_{E}) \supseteq \text{apr}_{P}(\text{apr}_{P}(g_{E})) \supseteq \text{apr}_{P}(\text{apr}_{P}(\text{apr}_{P}(g_{E}))) \]

Proof. (1) \Rightarrow (2). Let $g_{E} \in S_{f}(U, E)$. Denote 

\[ h_{E} = \text{apr}_{P}(g_{E}), \quad k_{E} = \text{apr}_{P}(h_{E}) \quad \text{and} \quad l_{E} = \text{apr}_{P}(k_{E}). \]

We will prove $h_{E} \supseteq \text{apr}_{P}(h_{E})$. We can suppose $P_{E}(f_{E}, R) \neq \emptyset$. Pick $(x_{E})_{E} \in P_{E}(f_{E}, R)$, by Lemma 5.11, $P_{E}(f_{E}, R) \subseteq P_{E}(f_{E}, R)$, then $(x_{E})_{E} \in P_{E}(f_{E}, R)$. Hence $\text{apr}_{P}(g_{E}) \subseteq \text{apr}_{P}(\text{apr}_{P}(g_{E}))$. By Proposition 5.7, $\text{apr}_{P}(\text{apr}_{P}(g_{E})) \subseteq \text{apr}_{P}(\text{apr}_{P}(\text{apr}_{P}(g_{E})))$. Suppose that $\mathcal{L}(E) - \mathcal{K}(E) \neq \emptyset$. Pick $(x_{E})_{E} \in \mathcal{L}(E) - \mathcal{K}(E)$. Then $(x_{E})_{E} \in \mathcal{L}(E)$. By Lemma 5.3, $(x_{E})_{E} \in P^{h}(f_{E}, R)$. This implies $R^{*}((x_{E})_{E}) \cap \mathcal{K}(E) = \emptyset$. Pick $(x_{E})_{E} \in R^{*}((x_{E})_{E}) \cap \mathcal{K}(E)$. Then $(x_{E})_{E} \in \mathcal{K}(E)$. By Lemma 5.3, $(x_{E})_{E} \in P^{h}(f_{E}, R)$. This implies $R^{*}((x_{E})_{E}) \cap \mathcal{K}(E) = \emptyset$. Thus $\mathcal{K}(E) = \emptyset$, a contradiction. Hence $\mathcal{L}(E) \subseteq \mathcal{K}(E)$. By Theorem 3.11, $l_{E} \supseteq k_{E}$.

Therefore, $\text{apr}_{P}(\text{apr}_{P}(g_{E})) \supseteq \text{apr}_{P}(\text{apr}_{P}(g_{E}))$. 

$\Rightarrow (1)$. Let $(x_{E})_{E}, (x_{E})_{E}, (x_{E})_{E} \in \mathcal{F}(E)$ with $(x_{E})_{E} \in R^{*}((x_{E})_{E})$ and $(x_{E})_{E} \in R^{*}((x_{E})_{E})$. Denote 

\[ g_{E} = (x_{E})_{E}, \quad h_{E} = \text{apr}_{P}(g_{E}) \quad \text{and} \quad l_{E} = \text{apr}_{P}(h_{E}). \]

By Lemma 5.3 and Lemma 5.5, $\mathcal{L}_{E} = P^{h}_{E}(f_{E}, R)$ and 

\[ \mathcal{K}(E) = \{(y_{E})_{E} \in \mathcal{F}(E) : (x_{E})_{E} \in R^{*}((y_{E})_{E})\}. \]

$(x_{E})_{E} \in R^{*}((x_{E})_{E})$ implies $(x_{E})_{E} \in \mathcal{K}(E)$. Note that $(x_{E})_{E} \in R^{*}((x_{E})_{E})$. Then $R^{*}((x_{E})_{E}) \cap \mathcal{K}(E) = \emptyset$. So $(x_{E})_{E} \in P^{h}_{E}(f_{E}, R) = \mathcal{L}_{E}$. Since $\text{apr}_{P}(\text{apr}_{P}(g_{E})) \supseteq \text{apr}_{P}(h_{E}), \mathcal{L}_{E} \subseteq \mathcal{K}(E)$. Thus $(x_{E})_{E} \in \mathcal{K}(E)$. By Lemma 5.5, $(x_{E})_{E} \in R^{*}((x_{E})_{E})$.

Therefore, $R$ is transitive. \qed

Proposition 5.13 Let $R$ be reflexive on $f_{E} \in S(U, E)$. Then the following are equivalent.

1. $R$ is Euclidean;
2. $\forall g_{E} \in S_{f}(U, E)$, 
   \[ \text{apr}_{P}(\text{apr}_{P}(g_{E})) \supseteq \text{apr}_{P}(g_{E}) \supseteq \text{apr}_{P}(\text{apr}_{P}(g_{E}))) \]

Proof. (1) \Rightarrow (2). Let $g_{E} \in S_{f}(U, E)$. Denote 

\[ k_{E} = \text{apr}_{P}(g_{E}), \quad l_{E} = \text{apr}_{P}(k_{E}) \quad \text{and} \quad h_{E} = \text{apr}_{P}(l_{E}). \]

Suppose $\mathcal{L}(E) - \mathcal{K}(E) \neq \emptyset$. Pick $(x_{E})_{E} \in \mathcal{L}(E) - \mathcal{K}(E)$. Then $(x_{E})_{E} \not\in \mathcal{K}(E)$ and $(x_{E})_{E} \in \mathcal{L}(E)$. By Lemma 5.3, $(x_{E})_{E} \in P^{h}_{E}(f_{E}, R)$. This implies $R^{*}((x_{E})_{E}) \cap \mathcal{K}(E) = \emptyset$. Pick $(x_{E})_{E} \in R^{*}((x_{E})_{E}) \cap \mathcal{K}(E)$. Then $(x_{E})_{E} \in \mathcal{K}(E)$. By Lemma 5.3, $(x_{E})_{E} \in P^{h}_{E}(f_{E}, R)$. This implies $R^{*}((x_{E})_{E}) \subseteq \mathcal{K}(E)$. Since $(x_{E})_{E} \not\in \mathcal{K}(E)$, by Lemma 5.3, $(x_{E})_{E} \not\in P^{h}_{E}(f_{E}, R)$. So $R^{*}((x_{E})_{E}) \not\subseteq \mathcal{K}(E)$, a contradiction. Hence $\mathcal{L}(E) \subseteq \mathcal{K}(E)$. By Theorem 3.11, $l_{E} \supseteq k_{E}$.

Therefore, $\text{apr}_{P}(\text{apr}_{P}(g_{E})) \supseteq \text{apr}_{P}(g_{E})$.

By Proposition 5.7, $\text{apr}_{P}(\text{apr}_{P}(g_{E})) \supseteq \text{apr}_{P}(g_{E})$.

We will prove $h_{E} \subseteq \text{apr}_{P}(h_{E})$. Suppose $P^{h}_{E}(f_{E}, R) \neq \emptyset$. Pick $(x_{E})_{E} \in P^{h}_{E}(f_{E}, R)$, by Lemma 5.11, $P^{h}_{E}(f_{E}, R) \subseteq P^{h}_{E}(f_{E}, R)$. Then $(x_{E})_{E} \in P^{h}_{E}(f_{E}, R)$. Hence $\text{apr}_{P}(g_{E}) \subseteq \text{apr}_{P}(\text{apr}_{P}(g_{E}))$. 

(2) \Rightarrow (1). Let $(x_{E})_{E}, (x_{E})_{E}, (x_{E})_{E} \in \mathcal{F}(E)$ with $(x_{E})_{E} \in R^{*}((x_{E})_{E})$ and $(x_{E})_{E} \in R^{*}((x_{E})_{E})$. Denote 

\[ g_{E} = (x_{E})_{E}, \quad h_{E} = \text{apr}_{P}(g_{E}) \quad \text{and} \quad l_{E} = \text{apr}_{P}(h_{E}). \]

By Lemma 5.3 and Lemma 5.5, $\mathcal{L}_{E} = P^{h}_{E}(f_{E}, R)$ and 

\[ \mathcal{K}(E) = \{(y_{E})_{E} \in \mathcal{F}(E) : (x_{E})_{E} \in R^{*}((y_{E})_{E})\}. \]

$(x_{E})_{E} \in R^{*}((x_{E})_{E})$ implies $(x_{E})_{E} \in \mathcal{K}(E)$. Note that $(x_{E})_{E} \in R^{*}((x_{E})_{E})$. Then $R^{*}((x_{E})_{E}) \subseteq \mathcal{K}(E)$. Thus $(x_{E})_{E} \in \mathcal{K}(E)$. By Lemma 5.5, $(x_{E})_{E} \in R^{*}((x_{E})_{E})$.

Therefore, $R$ is transitive. \qed

6. Soft topologies induced by s-relations on special soft sets

In this section, we investigate soft topologies induced by a reflexive s-relation on a special soft set and give their structure.
6.1. \(s\)-relations on \(\tilde{U}\)

**Proposition 6.1** Let \(R\) be a \(s\)-relation on \(\tilde{U}\). Then \(\forall \, g_E \in S(U, E), \) we have

1. \(apr_p(g_E) = \tilde{U} - \overline{apr_p}(U - g_E);\)
2. \(\overline{apr_p}(g_E) = \tilde{U} - \overline{apr_p}(U - g_E).\)

**Proof.** (1) Denote \(h_E = \tilde{U} - g_E,\) \(q_E = \overline{apr_p}(h_E),\)

\(l_E = \tilde{U} - q_E\) and \(k_E = \overline{apr_p}(g_E).\)

To prove \(l_E \subset k_E,\) by Theorem 3.11, it suffices to show \(\mathcal{L}(E) \subset \mathcal{H}(E).\)

Suppose \(\mathcal{L}(E) - \mathcal{H}(E) \neq \emptyset.\) Pick \((x_E) \in \mathcal{L}(E) - \mathcal{H}(E).\) By Lemma 5.3, \((x_E) \notin P_g(f_E, R).\) Thus \(R^+(x_E) \notin \mathcal{G}(E).\)

Pick \((x'_E) \in R^+(x_E) \cap \mathcal{G}(E).\) Then \((x'_E) \notin \mathcal{H}(E).\) This implies \(x' \notin g(e').\) Thus \(x' \in U - g(e') = h(e').\)

Since \((x_E) \in \mathcal{L}(E),\) \((x_E) \in \mathcal{W}(E) - \mathcal{D}(E).\) Then \((x_E) \notin \mathcal{D}(E).\) By Lemma 5.3, \((x_E) \notin P^h(f_E, R).\) So \(R^+(x_E) \cap \mathcal{H}(E) = \emptyset.\)

Since \((x'_E) \in R^+(x_E) \cap \mathcal{H}(E),\) \((x'_E) \notin \mathcal{H}(E).\) This implies \(x' \notin h(e'),\) a contradiction. Hence \(\mathcal{L}(E) \subset \mathcal{H}(E).\)

Therefore, \(\tilde{U} - \overline{apr_p}(U - g_E) \subset \overline{apr_p}(g_E).\)

Conversely, to prove \(k_E \subset l_E = \tilde{U} - g_E,\) by Remark 2.11, it suffices to show \(k_E \cap q_E = \emptyset.\)

Suppose \(w_E = k_E \cap q_E \neq \emptyset.\) By Proposition 3.5, \(\mathcal{W}(E) = \mathcal{H}(E) \cap \mathcal{D}(E).\) Pick \((x_E) \in \mathcal{W}(E).\) Then \((x_E) \in \mathcal{H}(E) \cap \mathcal{D}(E).\) By Lemma 5.3, \((x_E) \in P^h(f_E, R).\)

Thus \(R^+(x_E) \subseteq \mathcal{G}(E)\) and \(R^+(x'_E) \cap \mathcal{H}(E) = \emptyset.\)

Pick \((x'_E) \in R^+(x_E) \cap \mathcal{H}(E).\) Then \((x'_E) \in \mathcal{H}(E).\) Thus \(x' \in h(e') = U - g(e')\) and \(x' \notin g(e').\) But \((x'_E) \in \mathcal{H}(E).\) This implies \(x' \notin \mathcal{G}(E).\) Thus \(x' \notin g(e'),\) a contradiction. Hence \(apr_p(g_E) \cap \tilde{U} - \overline{apr_p}(U - g_E).\)

Therefore, \(\overline{apr_p}(g_E) = \tilde{U} - \overline{apr_p}(U - g_E).\)

(2) Denote \(h_E = \tilde{U} - g_E,\) \(q_E = apr_p(h_E),\)

\(l_E = \tilde{U} - q_E\) and \(k_E = apr_p(g_E).\)

To prove \(l_E \subset k_E,\) by Theorem 3.11, it suffices to show \(\mathcal{L}(E) \subset \mathcal{H}(E).\)

Suppose \(\mathcal{L}(E) - \mathcal{H}(E) \neq \emptyset.\) Pick \((x_E) \in \mathcal{L}(E) - \mathcal{H}(E).\) Then \((x_E) \notin \mathcal{H}(E).\) By Lemma 5.3, \((x_E) \notin P^h(f_E, R).\) Thus \(R^+(x_E) \cap \mathcal{H}(E) = \emptyset.\)

Since \((x_E) \in \mathcal{L}(E),\) \((x_E) \in \mathcal{W}(E) - \mathcal{D}(E).\) Then \((x_E) \notin \mathcal{D}(E).\) By Lemma 5.3, \((x_E) \notin \mathcal{H}(E).\)

Thus \(\forall \, g_E \in S(U, E),\) \(apr_p(g_E) \subset \overline{apr_p}(g_E).\)

**Corollary 6.2** Let \(R\) be a \(s\)-relation on \(\tilde{U}.\) Then the following are equivalent.

1. \(R\) is serial;
2. \(\forall \, g_E \in S(U, E),\) \(apr_p(g_E) \subset \overline{apr_p}(g_E);\)
3. \(apr_p(\emptyset) = \emptyset;\)
4. \(\overline{apr_p}(U) = U.\)

**Proof.** This follows from Proposition 5.7 and Proposition 6.1.

**Corollary 6.3** Let \(R\) be a \(s\)-relation on \(\tilde{U}.\) Then the following are equivalent.

1. \(R\) is reflexive;
2. \(\forall \, g_E \in S(U, E),\) \(apr_p(g_E) \subset g_E;\)
3. \(\forall \, g_E \in S(U, E),\) \(g_E \subset \overline{apr_p}(g_E).\)

**Proof.** This follows from Proposition 5.8 and Proposition 6.1.

**Corollary 6.4** Let \(R\) be a \(s\)-relation on \(\tilde{U}.\) Then the following are equivalent.

1. \(R\) is symmetric;
2. \(\forall \, g_E \in S_f(U, E),\) \(g_E \subset apr_p(apr_p(g_E));\)
3. \(\forall \, g_E \in S_f(U, E),\) \(apr_p(apr_p(g_E)) \subset g_E;\)
Proof. This follows from Proposition 5.10 and Proposition 6.1.

Corollary 6.5 Let R be a s-relation on \( \tilde{U} \). Then the following are equivalent.

(1) \( R \) is transitive;
(2) \( \text{aprr}_p(ge) \subseteq \text{aprr}_p(\text{aprr}_p(ge)) \forall ge \in S_f(U,E);
(3) \( \text{aprr}_p(\text{aprr}_p(ge)) \subseteq \text{aprr}_p(ge) \forall ge \in S_f(U,E).

Proof. This follows from Proposition 5.12 and Proposition 6.1.

Corollary 6.6 Let R be a s-relation on \( \tilde{U} \). Then the following are equivalent.

(1) \( R \) is Euclidean;
(2) \( \forall ge \in S_f(U,E), \text{aprr}_p(ge) \subseteq \text{aprr}_p(\text{aprr}_p(ge));
(3) \( \forall ge \in S_f(U,E), \text{aprr}_p(\text{aprr}_p(ge)) \subseteq \text{aprr}_p(ge).

Proof. This follows from Proposition 5.13 and Proposition 6.1.

6.2. Soft topologies induced by relations on \( \tilde{U} \)

Theorem 6.7 Let \( R \) be reflexive on \( \tilde{U} \). Then \( \tau_R = \{ge \in S(U,E) : \text{aprr}_p(ge) = ge\} \) is a soft topology over \( U \).

Proof. (1) By Proposition 5.9, \( \tilde{0}, \tilde{U} \in \tau_R.
(2) Let \( ge, he \in \tau_R \). Since \( ge = \text{aprr}_p(ge) \) and \( he = \text{aprr}_p(h_e) \), by Proposition 5.6, \( ge \cap he = \text{aprr}_p(ge) \cap \text{aprr}_p(h_e) = \text{aprr}_p(ge \cap he) \).
(3) \( \forall \alpha \in \Lambda, \text{aprr}_p(\{ge \in \Lambda \} = \text{aprr}_p(\bigcup \{ge \alpha \in \Lambda \}). \) Since \( R \) is reflexive, by Proposition 5.8, \( \text{aprr}_p(\bigcup \{ge \alpha \in \Lambda \} \subseteq \bigcup \{ge \alpha \in \Lambda \}. \) Conversely, since \( \{ge \alpha \in \Lambda \} \), by Proposition 5.6, we have \( \bigcup \{ge \alpha \in \Lambda \} \subseteq \text{aprr}_p(\bigcup \{ge \alpha \in \Lambda \}). \) Therefore, \( \tau_R = \{ge \in S(U,E) : \text{aprr}_p(ge) = ge\} \) is a soft topology on \( f_E \).

Definition 6.8 Let \( R \) be reflexive on \( \tilde{U} \). Then \( \tau_R \) is called the soft topology induced by \( R \) on \( \tilde{U} \).

The following Theorem 6.9 gives the structure of the soft topology induced by a reflexive s-relation on \( \tilde{U} \).

Theorem 6.9 Let \( R \) be reflexive on \( \tilde{U} \) and \( \tau_R \) the soft topology induced by \( R \) on \( U \). Then

(1) \( \forall ge \in S(U,E) \), \( \text{aprr}_p(ge) : ge \in S(U,E) \}
(2) \( \text{aprr}_p \) is a soft interior operator of \( \tau_R \),
(3) \( \text{aprr}_p \) is a soft closure operator of \( \tau_R \).

Proof. (1) \( a \) Let \( ge \in S(U,E) \). By Corollary 6.5, \( \text{aprr}_p(\text{aprr}_p(ge)) = \text{aprr}_p(ge) \). This implies \( \text{aprr}_p(ge) \subseteq \tau_R \). Hence \( \tau_R = \{\text{aprr}_p(ge) : ge \in S(U,E)\} \).

By Corollary 6.6, \( \{\text{aprr}_p(ge) : ge \in S(U,E)\} \subseteq \tau_R \).

(2) It suffices to show \( \text{aprr}_p(ge) = \text{int}(ge) \) for any \( ge \in S(U,E) \).

By (1), \( \text{aprr}_p(ge) \in \tau_R \). By Corollary 6.3, \( \text{aprr}_p(ge) \subseteq ge \). Thus \( \text{aprr}_p(ge) \subseteq \text{int}(ge) \).

Conversely, suppose \( h_e \in \tau_R \) and \( h_e \subseteq ge \), by Proposition 5.6, \( h_e = \text{aprr}_p(h_e) \subseteq \text{aprr}_p(ge) \). By Remark 2.5,

\[ \text{int}(ge) = \bigcup \{h_e : h_e \in \tau_R \} \subseteq h_e \subseteq ge \subseteq \text{aprr}_p(ge). \]

Thus \( \text{aprr}_p(ge) = \text{int}(ge) \).

(3) By Proposition 2.17 and Proposition 6.1,

\[ \text{aprr}_p(ge) = \text{int}(\tilde{U} - ge) = \tilde{U} - \text{int}(\tilde{U} - ge) = \text{cl}(ge). \]

Theorem 6.10 Let \( R \) be reflexive and transitive on \( \tilde{U} \) and \( \tau_R \) the soft topology induced by \( R \) on \( U \). Then \( \forall ge \in S(U,E), ge \in \tau_R \Rightarrow ge \in \tau_R \).

Proof. Necessity. Let \( ge \in \tau_R \). Then \( \text{aprr}_p(ge) = ge \). By Proposition 6.1 and Remark 2.9, \( \text{aprr}_p(ge)' = \tilde{U} - \text{aprr}_p(ge) = \tilde{U} - ge = ge \). By Theorem 6.9, \( ge = \text{aprr}_p(ge') \in \tau_R \). Thus \( ge \in \tau_R \).
Sufficiency. Let \( g_E \in \tau_R \). Then \( g'_E \in \tau_R \) and 
\[
\overline{apr}_p(g'_E) = g'_E.
\]
By Proposition 6.1 and Remark 2.9, 
\[
\overline{apr}_p(g_E) = U - \overline{apr}_p(g'_E) = g_E.
\]
By Theorem 6.5, 
\[
g_E = \overline{apr}_p(g_E) \in \tau_R. \tag*{\Box}
\]

**Definition 6.11** Let \( \tau \) be a topology on \( U \). \( \tau \) is called a pseudo-discrete topology on \( U \), if \( A \subseteq U \) is open in \( U \) if and only if \( A \) is closed in \( U \).

**Theorem 6.12** Let \( R \) be reflexive and transitive on \( U \). Then \( \tau_R \) is a pseudo-discrete topology over \( U \).

**Proof.** This holds by Theorem 6.7 and Theorem 6.10. \( \Box \)

7. Conclusions

In this paper, the fact that soft sets can be translated into soft point sets has been proved. Thus, we may expediently handle soft set like ordinary sets. We have proposed \( s \)-relations on soft sets. By means of soft points and \( s \)-relations, a pair of soft rough approximate operations has been defined. Serial, reflexive, symmetric, transitive and Euclidean \( s \)-relations have been characterized by using soft rough approximate operations. In addition, we have investigated soft topologies induced by a reflexive \( s \)-relation on a special soft set and given their structure. In the future, we will investigate the axiomatization of the proposed soft rough approximate operations and consider some concrete applications of our proposed notions.

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