Hilbert Type Inequality for Finite Series

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Abstract. In this paper, by using Cauchy inequality, monotonicity of functions, Hilbert type inequality with finite series version is established.

Introduction

If $a_n \geq 0, b_n \geq 0, m, n \in \mathbb{N}$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty, 0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then the well known Hilbert inequalities are given by

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < p \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2}. \quad (1)$$

$$\sum_{n=1}^N \sum_{m=1}^N \frac{a_m b_n}{m+n} < p \left(\sum_{n=1}^N a_n^2 \right)^{1/2} \left(\sum_{n=1}^N b_n^2 \right)^{1/2}. \quad (2)$$

Hilbert type inequalities are given by

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m, n)} < 4 \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2}. \quad (3)$$

$$\sum_{n=1}^N \sum_{m=1}^N \frac{a_m b_n}{\max(m, n)} < 4 \left(\sum_{n=1}^N a_n^2 \right)^{1/2} \left(\sum_{n=1}^N b_n^2 \right)^{1/2}. \quad (4)$$

(see Hardy et al.[1]). In recently years, various improvements and extensions of the Hilbert inequality and Hilbert type inequalities appear in a great deal of papers (see [2-8]). Zhang xiaoming, Chu yuming ([2]) gave improvement of (2) as:

$$p^2 \sum_{n=1}^N a_n^2 \sum_{n=1}^N b_n^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{a_m b_n}{m+n} \right)^2 \geq \min_{1 \leq n \leq N} \{n a_n^2\} \min_{1 \leq n \leq N} \{n b_n^2\} \left[p^2 \left(\sum_{n=1}^N \frac{1}{n} \right)^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{1}{\sqrt{mn(m+n)}} \right)^2 \right] \quad (5)$$

The major objective of this paper is to formulate new inequalities, which is improvement of (5).

Some lemmas

In order to prove our main result we need some lemmas, which we present in this section.

Lemma 1 If $1 \leq n \leq N, n, N$ are positive integers, then

(i)

$$\sum_{m=1}^N \frac{1}{\sqrt{m} \cdot \max(m, n)} < 4n^{-\frac{1}{2}}. \quad (6)$$

$$(ii) \quad \sum_{m=1}^{N+1} \frac{1}{\sqrt{m} \cdot \max(m, N+1)} < 2(N+1)^{-\frac{1}{2}}. \quad (7)$$

$$(iii) \quad \sum_{m=1}^N \sum_{n=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} < 4 \sum_{n=1}^N \frac{1}{n}. \quad (8)$$

Proof. (i)

Since $\frac{1}{\sqrt{x} \cdot \max(x, n)}$ is monotone decreasing in the interval $0 < x < \infty$, then

$$\sum_{m=1}^N \frac{1}{\sqrt{m} \cdot \max(m, n)} < \int_0^N \frac{1}{\sqrt{x} \cdot \max(x, n)} dx = \int_0^n \frac{1}{n\sqrt{x}} dx + \int_n^N \frac{1}{x\sqrt{x}} dx = 4n^{-\frac{1}{2}} - 2N^{-\frac{1}{2}} < 4n^{-\frac{1}{2}}$$

(ii)

$$\sum_{m=1}^{N+1} \frac{1}{\sqrt{m} \cdot \max(m, N+1)} < \int_0^{N+1} \frac{1}{\sqrt{x} \cdot \max(x, N+1)} dx = \int_0^{N+1} \frac{1}{(N+1)\sqrt{x}} dx = 2(N+1)^{-\frac{1}{2}}.$$

$$(iii) \quad \sum_{m=1}^N \sum_{n=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} < \sum_{n=1}^N \frac{1}{\sqrt{n}} 4n^{-\frac{1}{2}} = 4 \sum_{n=1}^N \frac{1}{n}.$$

Lemma 2 Let $f(N) = 16 \left(\sum_{n=1}^N \frac{1}{n} \right)^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} \right)^2$,

then $f(N+1) > f(N)$.

Proof. $f(N+1) - f(N)$

$$\begin{aligned} &= \frac{16}{N+1} \left(2 \sum_{n=1}^N \frac{1}{n} + \frac{1}{N+1} \right) - \left(\frac{2}{\sqrt{N+1}} \sum_{n=1}^{N+1} \frac{1}{\sqrt{n} \cdot \max(N+1, n)} - \frac{1}{(N+1)^2} \right) \times \left[2 \sum_{n=1}^N \sum_{m=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} \right. \\ &\quad \left. + \frac{2}{\sqrt{N+1}} \sum_{n=1}^{N+1} \frac{1}{\sqrt{n} \cdot \max(N+1, n)} - \frac{1}{(N+1)^2} \right] \\ &= \frac{16}{N+1} \left(2 \sum_{n=1}^N \frac{1}{n} + \frac{1}{N+1} \right) \\ &\quad - \left(\frac{2}{\sqrt{N+1}} \sum_{n=1}^{N+1} \frac{1}{\sqrt{n} \cdot \max(N+1, n)} \right) \times \left[2 \sum_{n=1}^N \sum_{m=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} + \frac{2}{\sqrt{N+1}} \sum_{n=1}^{N+1} \frac{1}{\sqrt{n} \cdot \max(N+1, n)} \right] \end{aligned}$$

$$> \frac{16}{N+1} \left(2 \sum_{n=1}^N \frac{1}{n} + \frac{1}{N+1} \right) - \frac{4}{N+1} \left(8 \sum_{n=1}^N \frac{1}{n} + \frac{4}{N+1} \right) = 0.$$

Main results

Theorem 1 If $a_n \geq 0, b_n \geq 0, n = 1, 2, \dots, N$, N is positive integer, then

$$\begin{aligned} & 16 \sum_{n=1}^N a_n^2 \sum_{n=1}^N b_n^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{a_m b_n}{\max(m, n)} \right)^2 \\ & \geq \min_{1 \leq n \leq N} \{n a_n^2\} \min_{1 \leq n \leq N} \{n b_n^2\} \left[16 \left(\sum_{n=1}^N \frac{1}{n} \right)^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} \right)^2 \right] \end{aligned} \quad (9)$$

Proof. Let $c_n = \sqrt{n} a_n, d_n = \sqrt{n} b_n$, then inequality (9) is translated into

$$\begin{aligned} & 16 \sum_{n=1}^N \frac{c_n^2}{n} \sum_{n=1}^N \frac{d_n^2}{n} - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{c_m d_n}{\sqrt{mn} \cdot \max(m, n)} \right)^2 \\ & \geq \min_{1 \leq n \leq N} \{c_n^2\} \min_{1 \leq n \leq N} \{d_n^2\} \left[16 \left(\sum_{n=1}^N \frac{1}{n} \right)^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{1}{\sqrt{mn} \cdot \max(m, n)} \right)^2 \right]. \end{aligned} \quad (10)$$

By using Cauchy inequality, we have

$$\begin{aligned} & 16 \sum_{n=1}^N \frac{c_n^2}{n} \sum_{n=1}^N \frac{d_n^2}{n} - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{c_m d_n}{\sqrt{mn} \cdot \max(m, n)} \right)^2 \\ & \geq 16 \sum_{n=1}^N \frac{c_n^2}{n} \sum_{n=1}^N \frac{d_n^2}{n} - \sum_{n=1}^N \sum_{m=1}^N \frac{c_m^2}{\sqrt{mn} \cdot \max(m, n)} \sum_{n=1}^N \sum_{m=1}^N \frac{d_n^2}{\sqrt{mn} \cdot \max(m, n)}. \end{aligned} \quad (11)$$

Let $f(c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_n)$

$$= 16 \sum_{n=1}^N \frac{c_n^2}{n} \sum_{n=1}^N \frac{d_n^2}{n} - \sum_{n=1}^N \sum_{m=1}^N \frac{c_m^2}{\sqrt{mn} \cdot \max(m, n)} \sum_{n=1}^N \sum_{m=1}^N \frac{d_n^2}{\sqrt{mn} \cdot \max(m, n)}.$$

to compute the partial derivatives of a function $f(c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_n)$ with respect to c_i, d_i ,

and using lemma 1, we have

$$\begin{aligned} \frac{\partial f}{\partial c_i} &= \frac{32c_i}{i} \sum_{n=1}^N \frac{d_n^2}{n} - \sum_{n=1}^N \frac{2c_i}{\sqrt{i} \sqrt{n} \cdot \max(i, n)} \sum_{n=1}^N \frac{d_n^2}{\sqrt{n}} \left(\sum_{m=1}^N \frac{1}{\max(m, n)} \right) \\ &> \frac{32c_i}{i} \sum_{n=1}^N \frac{d_n^2}{n} - \frac{2c_i}{\sqrt{i}} \cdot 4i^{-1/2} \sum_{n=1}^N \frac{d_n^2}{\sqrt{n}} 4n^{-1/2} = 0.. \end{aligned}$$

thus $f(c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_n)$ is monotone increasing for c_i . In a similar way we can provide that

$\frac{\partial f}{\partial d_i} > 0$, and this implies $f(c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_n)$ is monotone increasing for d_i . We obtain

$$\begin{aligned} & f(c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_n) \\ & \geq f(\min_{1 \leq n \leq N} \{c_n\}, \min_{1 \leq n \leq N} \{c_n\}, \dots, \min_{1 \leq n \leq N} \{c_n\}, \min_{1 \leq n \leq N} \{d_n\}, \min_{1 \leq n \leq N} \{d_n\}, \dots, \min_{1 \leq n \leq N} \{d_n\}). \end{aligned}$$

In view of (11), (10) holds. The theorem is proved.

From theorem 1 and lemmas 2, we have

Theorem 2 If $a_n \geq 0, b_n \geq 0, n = 1, 2, \dots, N$, N is positive integer, then

$$16 \sum_{n=1}^N a_n^2 \sum_{n=1}^N b_n^2 - \left(\sum_{n=1}^N \sum_{m=1}^N \frac{a_m b_n}{\max(m, n)} \right)^2 \geq 15 \min_{1 \leq n \leq N} \{n a_n^2\} \min_{1 \leq n \leq N} \{n b_n^2\}. \quad (12)$$

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