Linear Integral Boundary Value Problem of a Class of Evolution Inclusions

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Abstract. In this paper, we establish the existence of solutions of a class of evolution inclusions with a multivalued perturbation under linear integral boundary value conditions. By applying the Leray-Schauder’s theorem we can investigate the existence of solutions for the nonconvex cases and convex cases on the multivalued perturbation.

Introduction

In recent years, the solvability problem for integral boundary value conditions of evolution equation has been one of the important research field at home and abroad, and a number of outstanding research results have been produced, see [1-4, 7-10]. However, relevant results of the problem of solvability for integral boundary value conditions of evolution inclusions are very few. Therefore, this paper studies the solvability for integral boundary value conditions of the evolution inclusions: 

\[ y'(t) + By \in F(t, y), \]

where \( F \) is a multifunction and \( B \) is a bounded linear operator.

Preliminaries

For convenience, we present some notations and basic concepts as follows. In Euclidean space \( R^N \), \((,)\)expresses an inner product, and \(|:|\)expresses the Euclidean norm. Let \( L = [0, 2\pi] \), we denote the set of the map \( y: L \to R^N \) which satisfies 

\[ \int_0^{2\pi} |y|^2 \, dt < \infty \quad \text{in} \quad K^2(L, R^N), \]

and the norm is denoted by 

\[ \|y\|_2 = \left( \int_0^{2\pi} |y|^2 \, dt \right)^{1/2}. \]

Let

\[ C_{2\pi} = \{ y|L \to R^N \mid y(0) = \alpha \int_0^{2\pi} y(s) \, ds, \quad 0 < \alpha < 1 \}, \]  

(1)

\[ V^{1,2}(L, R^N) = \left\{ y \in C_{2\pi} : \int_0^{2\pi} \left( |y|^2 + |y'(t)|^2 \right) \, dt < \infty \right\}, \]  

(2)

where \( y' \) is the derivative of \( y \). \( C_{2\pi} \) and \( V^{1,2}(L, R^N) \) are Banach space, which the norm are

\[ \|y\|_2 = \max_{t \in L} |y| \quad \text{and} \quad \|y\|_{1,2} = \left( \int_0^{2\pi} \left( |y|^2 + |y'(t)|^2 \right) \, dt \right)^{1/2}. \]  

(3)

Let \([L, \Sigma]\) be the Lebesgue measurable space and \( Y \) be a separable Banach space.
Let \( M \subseteq Y \) and \( Y \) is closed, \( y \in Y \), then the distance form \( y \) to \( M \) is defined as
\[
d(y,M) = \inf \left\{ \left| y - m \right| : m \in M \right\}.
\]
(4)

We define \( P_f(Y) = \{ M \subseteq Y : \text{nonempty and closed} \} \), a multifunction \( F : L \rightarrow P_f(Y) \) is measurable if and only if, for every \( z \in Y \), the function \( t \rightarrow d(z, F(t)) = \inf \{ \| z - y \| : y \in F(t) \} \) is measurable. A multifunction \( G : L \rightarrow 2^\Y \backslash \{ \emptyset \} \) is said to be graph measurable if \( GrG = \{(t,y) : y \in G(t) \} \in \Sigma \times N(Y) \) with \( N(Y) \) being the Borel \( \sigma \)-field of \( Y \). On \( P_f(Y) \) we define a generalized metric as the ‘Hausdorff metric’, by setting
\[
h(M,N) = \max \left\{ \sup_{m \in M} d(m,N), \sup_{n \in N} d(n,M) \right\},
\]
for all \( M,N \in P_f(Y) \). As we all known, \( (P_f(Y),h) \) is a complete metric space.

Let \( X,Z \) be Hausdorff topological spaces and \( G : X \rightarrow 2^Z \backslash \{ \emptyset \} \). \( G() \) is ‘upper semicontinuous’ (resp. ‘lower semicontinuous’) , if for all \( C \subseteq Z \) nonempty and closed, \( \overline{G^-} = \{ x \in X : G(x) \cap C \neq \emptyset \} \) (resp. \( \overline{G^+} = \{ x \in X : G(x) \subseteq C \} \)) is closed in \( X \).

**Lemma 2.1.** Consider the equation
\[
\begin{align*}
y'(t) + By &= f(t), \\
y(0) &= \alpha \int_0^{2\pi} y(s)ds,
\end{align*}
\]
(6)

Where \( B : R^N \rightarrow R^N \) is a bounded linear operator, \( 0 < \alpha < 1 \), and there exists \( c \in R^+ \) such that \( \langle By, y \rangle \geq c \| y \| \) where \( c > \frac{1}{\alpha} \) for all \( y \in R^N \), then the problem (6) has a unique solution and satisfies \( \| y \|_2 \leq C\| f \| \), where \( C \) is a constant.

**Proof:** First, we consider the following Cauchy problem
\[
\begin{align*}
y'(t) + By &= f(t), \\
y(0) &= \lambda,
\end{align*}
\]
(7)

Where \( \lambda \in R^N \). It is easy to know that the problem (7) has a unique solution written as follows:
\[
y(t) = e^{Bt}\lambda + \int_0^t e^{B(t-s)} f(s)ds
\]
(8)

Integrating the equation (8) from 0 to \( 2\pi \), we have that
\[
\int_0^{2\pi} y(t)dt = \int_0^{2\pi} e^{Bt}\lambda dt + \int_0^{2\pi} \int_0^t e^{B(t-s)} f(s)dsdt
\]
(9)

Let \( y(0) = \alpha \int_0^{2\pi} y(t)dt \), so \( \lambda = \alpha \int_0^{2\pi} y(t)dt \), by (9), we have that
\[
\frac{1}{\alpha} \int_0^{2\pi} e^{Bt} dt \lambda + \int_0^{2\pi} \int_0^t e^{B(t-s)} f(s)dsdt = \lambda
\]
(10)

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And by $\langle By, y \rangle \geq c \| y \|^2$, then there exists $\left( \frac{1}{\alpha} - \int_0^{2\pi} e^{Bt} dt \right)^{-1}$, thus we obtain that

$$
\hat{\lambda} = \left( \frac{1}{\alpha} - \int_0^{2\pi} e^{Bt} dt \right)^{-1} \times \int_0^{2\pi} \int_0^t e^{B(t-s)} f(s) ds dt
$$

(11)

So we get the following unique solution (12) by taking (11) into (8)

$$
y(t) = e^{Bt} \left( \frac{1}{\alpha} - \int_0^{2\pi} e^{Bt} dt \right) \times \int_0^{2\pi} \int_0^t e^{B(t-s)} f(s) ds dt + \int_0^t e^{B(t-s)} f(s) ds
$$

(12)

Taking the norm on both sides of (12), we have

$$
|y(t)| \leq e^{Bt} \left( \frac{1}{\alpha} - \int_0^{2\pi} e^{Bt} dt \right) \cdot 2\pi \left| \int_0^{2\pi} e^{B(t-s)} f(s) ds \right| + \left| \int_0^{2\pi} e^{B(t-s)} f(s) ds \right|
$$

$$
\leq (M_1 2\pi + 1) \int_0^{2\pi} \left| f(s) \right| ds
$$

$$
= M_2 \int_0^{2\pi} \left| f(s) \right| ds
$$

Where $M_1$ and $M_2$ are constants. Therefore, there exists a constant $C > 0$ for $|y| \leq C \| f \|$. This completes the proof of lemma 2.1.

**Main Results**

Consider the following evolution inclusion problem:

$$
\begin{cases}
    y'(t) + By \in F(t, y), \\
    y(0) = \alpha \int_0^{2\pi} y(s) ds,
\end{cases}
$$

(13)

Where $B : R^N \rightarrow R^N$ is a bounded linear operator, $0 < \alpha < 1$, and there exists $c \in R^+$ such that $\langle By, y \rangle \geq c \| y \|^2$ where $c > \frac{1}{\alpha}$ and $F : L \times R^N \rightarrow 2^{R^N}$ is a multifunction.

**Definition 3.1.** $y$ is called the solution of the problem (13), if a function $y \in V^{1,2}(L, R^N)$ and there exists a function $f(t) \in F(t, y(t))$ such that

$$
\langle y'(t), w \rangle + \langle By, w \rangle = \langle f(t), w \rangle
$$

(14)

For all $w \in R^N$ and almost $t \in L$.

Next, we prove the existence of solutions of the problem (13) under the following hypothesis $G$, when the multivalued $F$ is nonconvex-valued.

$$
G_i : F : L \times R^N \rightarrow R^N
$$

is a multifunction with compact such that
(i) \( (t, y) \rightarrow F(t, y) \) is graph measurable;
(ii) For almost all \( t \in L, y \rightarrow F(t, y) \) is lower semicontinuous;
(iii) There exists an nonnegative function \( n(\cdot) \in K^2(L) \) and a constant \( c_1 > 0 \) such that
\[
\|F(t, y)\| = \sup \left\{ \|f\| : f \in F(t, y) \right\} \leq n(t) + c_1 |y|^\beta, \quad \text{for all } y \in R^N, \quad \text{where } \beta < 1.
\]

**Theorem 3.1.** If hypothesis \( G_1 \) hold, then the problem (13) has a solution \( y \in V^{1,2}(L, R^N) \).

*Proof* Let \( Ky = y' + By \), for all \( y \in V^{1,2}(L, R^N) \), then \( K : V^{1,2}(L, R^N) \rightarrow K^2(L, R^N) \) is a linear operator. By Lemma 2.1, one priori estimate to the solution of equation and \( V^{1,2}(L, R^N) \) embeds compactly into \( K^2(L, R^N) \), \( K : V^{1,2}(L, R^N) \) then \( K^2(L, R^N) \) is one to one, and because \( K \) is continuous, so we have \( K^{-1} : K^2(L, R^N) \rightarrow V^{1,2}(L, R^N) \) is a continuous operator. Thanks to \( V^{1,2}(L, R^N) \) embedding compactly into \( K^2(L, R^N) \), so \( K^{-1} : K^2(L, R^N) \rightarrow K^2(L, R^N) \) is continuously continuous.

Next, let \( Q : K^2(L, R^N) \rightarrow 2^{K^2(L, R^N)} \) be the multivalued Nemitsky operator corresponding to be defined by \( Q(y) = \{ w \in K^2(L, R^N) : w(t) \in F(t, y(t)) \} \). By Theorem 3.1 of [5], \( Q(\cdot) \) has nonempty, closed, decomposable values and is lower semicontinuous.

By continuous selection theorem of [6] and obtain a continuous map \( f : K^2(L, R^N) \rightarrow K^2(L, R^N) \) such that \( f(y) \in F(t, y) \). To finish our proof, we only need to solve the fixed point problem: \( y = K^{-1} \circ f(y) \). We apply Leray-Schauder’s alternative theorem to prove the fixed point problem. It is equivalent to test the set \( W = \{ y \in K^2(L, R^N) : y = \sigma K^{-1} f(y), \sigma \in (0, 1) \} \) is bounded. \( y \in W \), then \( y = \sigma K^{-1} \circ f(y) \). By hypothesis \( G_1(iii) \), we can derive \( |f(y)| \leq n(t) + c_1 |y|^\beta \), then

\[
\|f(y)\|_2 \leq \|n\|_2 + \|c_1 |y|^\beta\|_2
\]

\[
\leq \|n\|_2 + \left( \|c_1 \|_1 \|y\|_2^{\beta/2} \right)^{1/2} \geq \|n\|_2 + \|c_1 \|_1 \|y\|_2^{\beta/2} \geq \|n\|_2 + c_1 (2 \pi)^{1/2} \|y\|_2^{\beta/2}.
\]

By Lemma 2.1, we get that

\[
\|y\|_2 \leq c_2 \|f\|_2
\]

for some constant \( c_2 > 0 \). So, we have that

\[
\|y\|_2 \leq c_2 \|n\|_2 + c_1 c_2 (2 \pi)^{1-\beta} \|y\|_2^{\beta/2}.
\]

Due to \( 0 < \beta < 1 \), there exists a constant \( M > 0 \) such that \( \|y\|_2 \leq M \). If the constant \( M \) does not exist, by (15), we have
has a closed graph. According to Leray-Schauder)

\[ N \subset y \subset R \quad \text{and the Dunford-Pettis theorem,} \]

\[ \alpha N \] is completely continuous, \[ \sigma \] is a multifunction with compact and convex value such that\n
\[ t \quad \text{a constant } \alpha N \quad \text{such that} \]

\[ N \land \sigma \subset y \subset R, \]

\[ ii \quad \text{We need to show that set } N \]

\[ \land \sigma \subset y \subset R, \]

\[ iii \quad \text{As before, we have} \]

\[ y \subset R, \quad \text{Let} \]

\[ y \subset R, \quad \text{and the} \]

\[ is upper semicontinuous from } \subset R, \text{ and maps bounded sets into relatively compact sets.} \]

\[ 2 \quad \text{is USC} \quad \text{and} \]

\[ K^{-1} : K^2(L, R^n) \rightarrow K^2(L, R^n) \]

\[ \text{nonempty, closed and convex}. \]

Next, we prove that \( Q(\cdot) \) is upper semicontinuous from \( V^{1,2}(L, R^n) \) to \( K^2(L, R^n)_w \). Let \( Q_n \) be a nonempty and weakly closed subject of \( K^2(L, R^n) \). We need to show that set \( Q^{-1}(Q_n) = \{ y \in E(K) : Q(y) \cap Q_n \neq \emptyset \} \) is closed. We assume that there is function sequence \( \{ y_n \}_{n=1}^{\infty} \subset Q^{-1}(Q_n) \) satisfies \( y_n \rightarrow y \) in \( V^{1,2}(L, R^n) \). So there is a convergent subsequence, we can get \( y_n(t) \rightarrow y(t) \). Let \( f_n \subset Q(y_n) \cap Q_n, \) \( n \geq 1 \). Due to hypothesis \( G_2(iii) \) and the Dunford-Pettis theorem, then \( f_n \rightarrow f \) weakly in \( K^2(L, R^n) \). As before, we have

\[
\begin{aligned}
f(t) & \in \ conv \lim_{n \geq 1} \{ f_n(t) \} \\
& \subseteq \ conv \lim_{n \geq 1} F(t, y) \subseteq F(t, y),
\end{aligned}
\]

For almost all \( t \in L \), where \( conv \) expresses closed convex inclusion of set, then \( f \in Q(y) \cap Q_n \), so \( Q^{-1}(Q_n) \) is closed in \( V^{1,2}(L, R^n) \). Therefore, \( Q(\cdot) \) is upper semicontinuous from \( V^{1,2}(L, R^n) \) into \( K^2(L, R^n)_w \). This ends the proof.

The problem (13) is equivalent to following fixed point problem:

\[
y \subset K^{-1} \circ Q(y).
\]

Recalling that \( K^{-1} : K^2(L, R^n) \rightarrow K^2(L, R^n) \) is completely continuous, \( K^{-1} \circ Q : K^2(L, R^n) \rightarrow K^2(L, R^n) \) is USC and maps bounded sets into relatively compact sets. as in the proof of Theorem 3.1. We easily check that the set

\[
\Gamma_1 = \left\{ y \subset K^{-1}(L, R^n) : y \subset \sigma K^{-1} Q(y), \sigma \subset (0,1) \right\}
\]
Is bounded. Invoking set value Leray-Schauder’s alternative theorem, there exists \( y \in V^{1,2}(L,R^N) \) such that \( y \in K^{-1} \circ Q(y) \). Evidently, \( y \) is a solution of the problem (13).

Let \( S_p \) is the solution set of the problem (13). As in the proof of Theorem 3.1, we get a priori estimate to the solution. Then we have that \( |S_p| = \sup \{ \|u\|_{L^2} : u \in S_p \} \leq M \), where \( M > 0 \). By hypothesis \( G_2(iii) \) and Dunford-Pettis theorem, we may assume that \( y_n \to y \) in \( V^{1,2}(L,R^N) \). As before, for almost all \( t \in L \), we have

\[
K_y \in \text{conv} \lim \left\{ K_{y_n} \right\}_{n \geq 1} \subseteq \text{conv} \lim F(t, y_n) \subseteq F(t, y)
\]

Then \( y \in S_p \), hence \( S_p \) is weakly compact in \( V^{1,2}(L,R^N) \). The proof is completed.

References