The Solution Manifold and Oscillation of the Heavy Rigid System

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ABSTRACT: The analytic characteristics of invariant manifolds for quasi-homogeneous autonomous system are obtained Yu [1]. On the basis of this， this paper give the unify thoughts of several classical special solutions can be achieved by using the proposal method in the conventional heavy rigid system. In addition, a 3-dimensional invariant manifold and the accuracy solution of the heavy rigid system are obtained with the certain limiting condition in the center of mass of rigid body. The oscillation of this 3-dimensional solution manifold of the heavy rigid system is depicted in this paper.

INTRODUCTION

Paper [1] gave definitions for quasi-homogeneous according to the character of the Lie group generator admitted by some autonomous systems. Some classical mechanical systems, such as the heavy rigid system, n-body problem etc. are belong to quasi-homogeneous system. Based on the Lie group of quasi-homogeneous system, the analytical character of invariant manifolds for those systems is revealed in paper [1]. These features provide a more flexible and practical method to find the invariant manifolds of the system.

In this paper, we will realize the united solving thought of the partially known special solution for the motion problem of a heavy rigid body around fixed point by using the proposal method and the analytical character of received invariant manifolds. A 3-dimensional invariant manifold and the accurate solution for motion of the heavy rigid body will be obtained in the restrict condition (x_G=0) of the center of mass of rigid body. The oscillation of this 3-dimensional solution manifold of the heavy rigid system will also be depicted in this paper.

QUASI HOMOGENEOUS ANALYSIS OF THE HEAVY RIGID SYSTEM

The classical heavy rigid system Eq.(1) possesses the distinct quasi-homogeneous character.

$$\begin{cases} A\dfrac{dp}{dt} = (B-C)qr + Mgz_G\gamma' - Mgy_G\gamma'' \\ B\dfrac{dq}{dt} = (C-A)rp + Mgx_G\gamma'' - Mgz_G\gamma \\ C\dfrac{dr}{dt} = (A-B)pq + Mgy_G\gamma - Mgx_G\gamma' \\ \dfrac{d\gamma}{dt} = r\gamma' - q\gamma'' \\ \dfrac{d\gamma'}{dt} = p\gamma'' - r\gamma \\ \dfrac{d\gamma''}{dt} = q\gamma - p\gamma' \end{cases} \quad (1)$$

Where variables $t, p, q, r, \gamma, \gamma', \gamma''$ be regarded as the quasi-homogenous variables corresponding degrees separate are -1,1,1,1,2,2,2. Accordingly to the analytical character theorem for quasi-homogeneous polynomial invariant manifold, to determine the analytical invariant manifold about variables $p, q, r, \gamma, \gamma', \gamma''$ for the heavy rigid system, we can interchange of problem to determine the quasi-homogeneous polynomial invariant manifold about these variables. If function F is a k-dimensional quasi-homogeneous polynomial about variables $t, p, q, r, \gamma, \gamma', \gamma''$, then after the operation of the corresponding differential operator for the heavy rigid system (1), XF is a k+1-dimensioanl quasi-homogeneous polynomial in general. If we assume that quasi-homogeneous polynomial invariant manifolds F_i with the degree of: k_i , (i=1,2,…,n-m) .By the definition of invariant manifolds: $XF_i(x)=\sum_{j=1}^{q}\mu_{ij}(x)F_j(x)$, Then, corresponding quasi-homogeneous polynomial uncertain factor is $\mu_{ij}(x)$ with degree k_i+1-k_j (where i=1,2,…,n-m ; j=1,2,…,n-m). In such way, in the case of restricted the generalized degree for function F_i, we can determine the quasi-homogenous polynomial invariant manifold for system by using the method of comparing parameters.

UNIFICATION OF SEVERAL KNOWN SPECIAL SOLUTIONS

(I) With the restrictions on all quasi-homogeneous functions degree is 1 and the rigid body mass distribution in the case of $y_G=0$, $Ax_G^2(B-C)=Cx_G^2(A-B)$, then we can achieve a five-dimensional invariant manifolds that:

$$F = Ax_G p + Cz_G r \qquad (2)$$

Let invariant manifold is F=0, then we can achieve the classical case of Hess special solution (1890) [3, 4].

(II) With the restrictions on all quasi-homogeneous functions degree is 3 and the rigid body mass distribution in the case of $A=B=4C$, $y_G= z_G=0$, we can achieve a four-dimensional invariant manifolds that:

$$\begin{cases} F_1 = 4(pg + qg') + rg'' \\ F_2 = r(p^2 + q^2) - \dfrac{Mgx_G}{C} pg'' - l_0 \end{cases} \qquad (3)$$

Let invariant manifold functions is $F_1=F_2=0$, then we can achieve the classical case of Горячев-Чаплыгин special solution (1900) [3, 4].

(III) With the restrictions on the quasi-homogeneous functions F_1, F_2, F_3 degree separate is 1, 1, 2 and the rigid body mass distribution in the case of $B=2A$, $x_G=z_G=0$, we can achieve a three-dimensional invariant manifolds that:

$$\begin{cases} F_1 = r \\ F_2 = q - q_0 \\ F_2 = Apq_0 - Mgy_G g \end{cases} \qquad (4)$$

Let invariant manifold functions is $F_1=F_2=F_3=0$, then we can achieve the classical case of Бобылев-Смеклов special solution (1896) [3, 4].

We realize the unification of three classical cases, i.e. Hess cases, Горячев-Чаплыгин case and Бобылев-Смеклов case by using the analytical character of invariant manifolds for quasi-homogeneous autonomous system. Besides that, we also achieves a new special solution, which is different from the known 9 special solutions [3, 4] in the condition of restricted the center of mass of rigid body.

A NEW SPECIAL INTEGRAL CASE

While restrict that functions F_1, F_2, F_3 degree separate is 1, 1, 2 and the center of mass distribution condition for rigid body is simple limited in the x_G= 0, a new three-dimensional invariant manifold exist as:

$$\begin{cases} F_1 = p \\ F_2 = r \\ F_2 = g \end{cases} \qquad (5)$$

Let $F_1=F_2=F_3=0$ then the new integral case satisfy $q=r=\gamma=0$, with the special restricted distribution condition $x_G=0$ for the center of mass of rigid body.

We will discuss the motion of system with the corresponding integral case below in detail and give the numerical simulation accordingly.

THE MOTION OF SYSTEM WITH CORRESPONDING INTEGRAL CASE

To describe the motion of a rigid body, we usually use two systems of co-ordinates: a fixed spatial system, which is denoted as *X, Y, Z* in Figure 1, and another is rigidly fixed in the top body and to participate in its motion, which is called inertial principle axes system of a rigid body and denoted as *x, y, z* in Figure 1.

We know that the area integral $Ap\gamma+Bq\gamma'+Cr\gamma''$ of the heavy rigid system describe the component along vertical axis *Z* of the moving angular momentum for system without (imposed) any restriction conditions. The area integral equal to zero with the system variable q=r=γ=0 for the integral case (5), the geometrical interpretation is that the rigid body no longer rotate about vertical axis *Z*. Here, we take the plane of determined by arbitrary two principal axes of inertial as the plane of inertia, and then the area integral is zero, which indicate that plane of inertia *yz* lie in the same plane with vertical axis *Z*. Moreover, the restriction condition x_G=0 for the distribution of the center of mass farther assured that the center of mass of rigid body lie in the plane of inertia *yz*. Whence, the motion of the system only can be defined in the plane of inertia *yz*.

Consider that the heavy rigid system of defined by Euler-Poisson equations is Non-Hamilton system, in order to further describe the motion state of system in the plane of inertia *yz*, we chose the Euler-angle as follow: the moving *xy*-plane of inertia intersects the *X,Y*-plane in some line ON, take the angle θ between vertical axis *Z* and the principle axis of inertia *z*; the angle φ between the principle axis of inertia *X* and ON ,and the angle between the principle axis of inertia *x* and ON (Figure 1. Caption.).

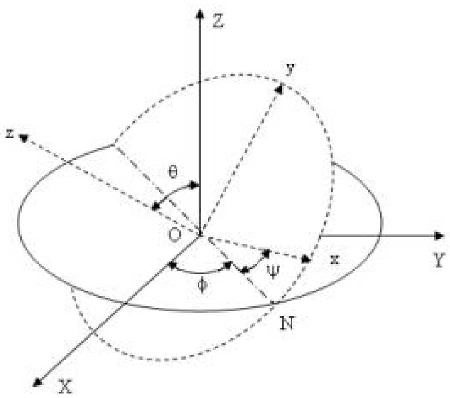


Figure 1. The position of co-ordinates and Euler-angle.

Let us now re-express system by using above given Euler-angle. With the restriction condition x_G=0 for the distribution of the center of mass in the integral case Eq.(5) and the variable q=r=γ=0, we have the Eulerian angles of φ, ψ all is zero. Whence, corresponding Euler-Poisson equations can be transformed expression by using Euler-angle θ in the integral case of Eq.(5):

$$A\ddot{\theta} = Mg(z_G \sin\theta - y_G \cos\theta) \quad (6)$$

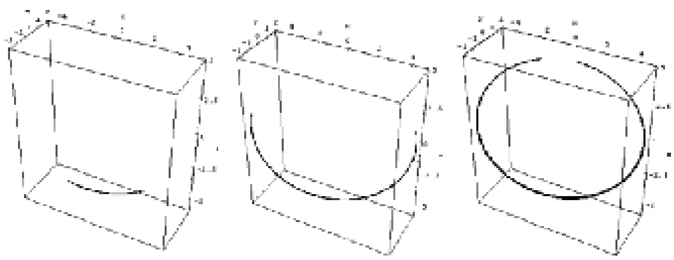
Eq.(6) is the typical Hamilton system and formally denote the single oscillation equation Accordingly, integral case Eq.(5) expresses a motion of single oscillation in plane of inertia *yz* for the heavy rigid system.

NUMERICALLY SIMULATION FOR THE MOTION OF SYSTEM IN THE INTEGRAL CASE

In order to validate and intuitionistic describe the motion of system in corresponding integral case, we will give the numerically simulation cases below. The movement of the center of mass can concentrate reect the complex motion of system, so it is very necessary to simulate the changing of the center of mass for system in spatial coordinates.

Simulation the motion of system in corresponding integral case

Assume that the rigid body total weight is Mg=1, the component of the principle momentum of inertial separate is A=1, B=2, C=3. Choosing the centre of mass of system lie in the plane of inertia yz: x_G=0, y_G=1, z_G=5 and defining the initial Eulerian angles and angular velocity satisfies the integral case of Eq.(5): ψ_0=0, θ_0=0.3π, φ_0=0, ψ'_0=0, θ'_0=0, φ'_0=0. Mathematica is used to simulate the motion of the center of mass for system in corresponding condition. Figure 2 shows the moving trend for system.



(a) Slight swing (b) Middle swing (c) Large swing

Figure 2. Free swing on invariant manifolds.

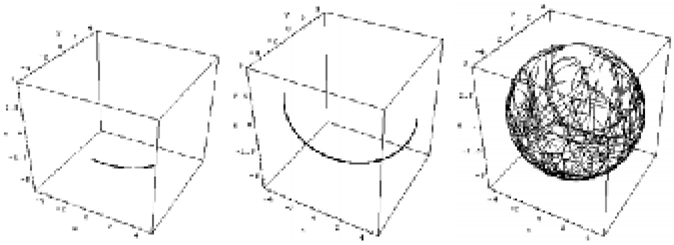
The simulation result of Figure 2 indicate that system certainly do the single oscillation movement over the plane of inertia yz in corresponding integral case. More simulation results indicates that the amplitude of the single oscillation is related to the initial angle θ and the larger initial angle φ is chosen, the lower loci for the center of mass of system is located and then the amplitude of oscillations is smaller.

The restricted condition x_G=0 for the center of mass is important assuring condition to make that system certainly do single oscillation over the plane of inertia yz in the integral case of Eq.(5). Theoretically, if the center of mass without located over the plane of inertia yz (i.e. $x_G \neq 0$), then the angle certainly exist between the plane of formed by the center of mass and the vertical axis Z and plane of inertia yz, consequently induce the different moment of inertia for the right and left hand of rigid body to lead the rotation of system. Then, the case of single oscillation shall not occur accordingly.

Though a series of numerical experimentation, we will certificate the significance of restricted condition x_G=0 of the center of mass to maintain the single oscillation of system; and then we will discuss the influence of the disturbance of initial values to stability of motion for single oscillation system as below.

Discussion of the motion of system at $x_G \neq 0$

Keep the invariable mass and principle momentum of inertia of the rigid body and choose the same initial Eulerian angles and angular velocity to i.e. ψ_0=0, θ_0=0.3π, φ_0=0, ψ'_0=0, θ'_0=0, φ'_0=0.The simulation result of the motion of the center of mass for system is shown in Figure 3.



(a) Slight swing (b) Middle swing (c) Large swing

Figure 3. When the center of gravity has a small disturbance, free swing on invariant manifolds.

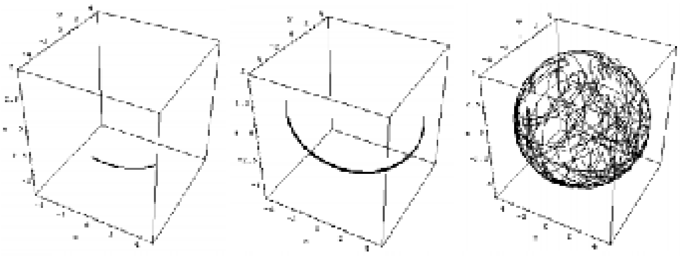
Figure 3 indicate that the motion loci of the center of mass for system is no longer kept as single oscillation, here the motion loci of system present irregular grid distribution in the nether region of sphere for chosen the lower center of mass of body.

The simulation result indicate that system impossible keep single oscillation in the plane of inertia *yz* anymore once the center of mass departure the plane of inertia *yz* ($x_G \neq 0$). We will find that the distortion degree of the motion is more distinct along with the higher loci of the center of mass of rigid body in the subsequent simulation. Even little departure of the center of mass to the plane of inertia *yz*, it will cause motion break of the system.

Inuence of initial value disturbance on motion stability

With restricted the center of mass in the plane of inertia *yz*, we discuss the inuence of initial value departure the plane of inertia *yz* on the stability of single oscillation for system in succession.

We choose system with farther distance between the center of mass to fixed point O: $x_G=0$, $y_G=1$, $z_G=17$. Let the initial loci little departure the plane of inertia *yz*, make $\psi_0=0.01$, $\theta_0=0.7\pi$, $\varphi_0=0$, $\psi'_0=0$, $\theta'_0=0$, $\varphi'_0=0$ .We simulate motion of system in the condition of above initial values and the results are shown in the Figure 4.



(a) Slight swing (b) Middle swing (c) Large swing

Figure 4. When there is small angle between the inertia surface *yz* and the *Z* axis, free swing on invariant manifolds.

Comparing the results of Figure 4 indicate that the initial value departure from the plane of inertia *yz* to destroy the single oscillation of system; being effected by the initial value disturbance, the system with more near distance of the center of mass possess better stability of motion than the system with farther distance of the center of mass.

Though above motion discussions for system in the integral case of (5), we conclude as below: the restriction condition $x_G=0$ for the distribution of the center of the mass of rigid body ensure that system do the single oscillation in the plane of inertia *yz*. The single oscillation will be destroyed once the center of mass departure the plane of inertia *yz* ($x_G \neq 0$). Being effected by the disturbance of the initial values, more near of the center of mass is, more better of stability of motion for system possesses. We will develop more rigorous theoretical analysis and mathematical certification about the stability of motion in further subsequently studies afterwards.

CONCLUSION

Appropriately utilize the analytical character of system could benefit to determine the invariant manifolds for quasi-homogeneous system by above discussion. Especially, application of the analytical character realized the unification of some classical known special solutions for the heavy rigid system. It achieved more conveniently and more realistically way to solving the special solutions for system. Additionally, a new special solution in the restricted distribution condition of the centre of mass of rigid body is given in this paper.

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REFERENCES

[1] Yu W.W. 2010. *Integrability and Integration of some Mechanical Systems*. Beijing: Capital University of Economics and Business press. (in Chinese)

[2] Arnold, V.I. 1983. *Geometrical Methods in the Theory of Ordinary Differential Equations*. New York: Springer-Verlag.

[3] Mei Feng-xiang & Liu Rui & Luo Yong.1991. *Advanced Analytical Mechanics*. Beijing: Beijing Institute of Technology Press. (in Chinese)

[4] Yusuke Hagihara. 1976. *Celestial Machaincs, Vol.I, Dynamical Principles and Transformation Theory.* Massachusetts:The MIT Press.

[5] Arnold,V. I. & Kozlov, V.V. & Neishtadt, A.I. 1997. *Mathematical Aspects of Classical and Celestial Mechanics second edition.* New York: Springer-Verlag.

[6] Olver,P.J.1986. *Application of Lie Groups to Differential Equations.* New York: Springer-Verlag.

[7] Guan Keying & Liu Sheng & Lei Jingzhi. 1998.The Lie algebra admitted by an ordinary differential equation system. *Ann. of Diff. Eqs.* 14(2):131-142.

[8] Llibre, J. & Zhang, X. 2002. Polynomial first integrals for quasi-homogeneous polynomial differential systems. *Nonlinearity* 15:1269-1280.