

# A novel analytical solution to the problem of thin-walled hollow pier and water interaction in a three-dimensional flow

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**Abstract.** An analytical algorithm for solving the nonlinear bending deformation of a thin-walled hollow bridge pier under the action of water flow considering the fluid-structure coupling effect, is proposed. Initially, The kinematic equation and the dynamic equation on the wetted surface of the structure are established by using the united Lagrangian-Eulerian(ULE) method. Afterwards, based on nonlinear bending theory of elastic shell and potential flow theory, the coupled mathematical model of a flexible pier and surrounding water, is presented. Finally, in a specific example, comparative study is conducted between the analytical solutions and the numerical solutions computed by ANSYS combined with CFX (two-way coupling) in ANSYS Workbench, and the analytical algorithm is verified to be feasible.

## Introduction

With the development of transportation, many deep-water bridges have been built in recent years. Whether these flexible piers submerged in deep-water can perfectly withstand complex water flow loads, will have a direct effect on the success or failure of bridge design. However, Wang et al. [1] discovered that the results obtained from calculation using formulas specified in design codes of different countries are either too large or too small by comparison with more advanced coupled fluid-structure finite element solutions. So it is necessary to seek an advanced analytical method.

Originally, neglecting the effect of structure deformation on fluid field, Morison [2] proposed a formula for calculating the hydrodynamic pressure exerted by surface wave on piles. It is also known as Morison hydrodynamic equation by which Huang [3] analyzed the effect of hydrodynamic force on the deep-water piers. Latter, the Morison hydrodynamic formula is improved by Yang [4] to consider the inside and outside hydrodynamic pressure of hollow piers. Although widely used, The Morison equation did not theoretically reveal the essence of FSI.

In the ULE method, each material is described in the preferred reference frame, e.g., Lagrangian for solid, Eulerian for fluid, Lagrangian combined with Eulerian for the interfaces, and which is implemented in the work of Ilgamov [5]. This approach can directly employ the existing equations and known solutions in solid mechanics and fluid mechanics. Utilizing the ULE method, Hao and Bai [6] solved the bending deformation and stress of elastic thin plate in a two-dimensional flow.

In the present paper, a new analytical algorithm based on the ULE method is proposed to solve the problem of pier-water interaction, in a three-dimensional flow.

## Contact Conditions on the Interfaces

Let us consider a circular thin-walled hollow pier with elastic modulus  $E$ , Poisson's ratio  $\mu$ , outer radius  $R$ , wall thickness  $t$  and height  $h$ , in a three-dimensional flow field with water depth  $d$ , mass density  $\rho$ , velocity  $V_\infty$  and pressure  $P_\infty$  at infinity. The fluid is assumed to be incompressible and inviscid, and the fluid motion is irrotational, so that the flow can be described by a velocity potential  $\phi$ , which satisfies the three-dimensional Laplace equation with respect to the original cylindrical coordinates  $(z, \theta, r)$  as follows,

$$\nabla^2 \phi = 0 \quad (1)$$

According to the theory of the ULE method[5], and the principle of simplification in reference [7], when the thin-walled hollow pier undergoes small bending deformation, the kinematic and dynamic contact conditions can be written respectively as

$$\frac{\partial \phi}{\partial r} + u_r \frac{\partial^2 \phi}{\partial r^2} + \frac{u_\theta}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} + u_z \frac{\partial^2 \phi}{\partial r \partial z} = \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \frac{\partial \phi}{\partial \theta} + \frac{\partial u_r}{\partial z} \frac{\partial \phi}{\partial z} \quad (r = R) \quad (2)$$

$$Z_3 = p + u_z \frac{\partial p}{\partial z} + \frac{u_\theta}{r} \frac{\partial p}{\partial \theta} + u_r \frac{\partial p}{\partial r} \quad (r = R) \quad (3)$$

In the type:  $u_z$ ,  $u_\theta$ ,  $u_r$  denote the axial, circumferential, and radial displacement components at a point on the middle surface of pier, respectively;  $p$  stands for pressure.

### Analytical Model and Formulation

According to the basis equations in fluid mechanics, we can readily obtain the following relationship

$$p = P_\infty + \frac{\rho}{2} (V_\infty^2 - (\nabla \phi)^2) \quad (4)$$

With the help of elastic mechanics [8], constructing an auxiliary displacement function  $\varphi = \varphi(\theta, z)$ , we can obtain

$$\nabla^8 \varphi + \frac{Et}{R^2 D} \frac{\partial^4 \varphi}{\partial z^4} = -\frac{Z_3}{RD} \quad (5)$$

Where  $D = Et^3/(1-\mu^2)$  is called the bending stiffness and  $Z_3$  represents the water flow loads along the inner normal direction. As long as  $\varphi$  is determined, the displacements and internal forces can be obtained, so Eq. (5) is referred to as the coupled governing equation herein. Let us set

$$\varphi(\theta, z) = \psi_0(z) + \sum_{m=1}^M \psi_m(z) \cos(m\theta) \quad (6)$$

In which  $\psi_m(z)$  for  $m=0 \cdots M$  are arbitrary functions of  $z$ .

By separation of variables, applying boundary conditions, there follows

$$\varphi(\theta, r, z) = \cos \theta \sum_{s=1,3,5,\dots}^{\infty} \cos\left(\frac{2s-1}{2d}\pi z\right) [E_s I_1\left(\frac{2s-1}{2d}\pi r\right) + F_s K_1\left(\frac{2s-1}{2d}\pi r\right)] \quad (7)$$

Where  $E_s$  and  $F_s$  are arbitrary constants,  $I_1(\cdot)$  and  $K_1(\cdot)$  denote the first order modified Bessel function of the first kind and second kind respectively.

Substituting Eq. (4), and Eq. (7) into (5), separating variables, we obtain a eighth-order ordinary differential equation, is of the form

$$\left[ \left( \frac{d^2}{dz^2} - \frac{m^2}{R^2} \right)^4 + \frac{Et}{R^2 D} \frac{d^4}{dz^4} \right] \psi_m = p_0 + p_1 \cos \frac{\pi z}{d} + p_2 \cos \frac{2\pi z}{d} + p_3 \cos \frac{3\pi z}{d} + p_5 \cos \frac{5\pi z}{d} \quad (m = 1, 5, 9, \dots) \quad (8)$$

where are constants about geometrical and physical parameters.

The auxiliary equation of Eq. (8) is

$$\left( \lambda^2 - \frac{m^2}{R^2} \right)^4 + \frac{Et}{R^2 D} \lambda^4 = 0 \quad (9)$$

Since  $Et/R^2 D$  is always positive, Eq. (9) has four pairs of complex roots. Assuming that they are  $a_m \pm ib_m$ ,  $-a_m \pm ib_m$ ,  $c_m \pm id_m$  and  $-c_m \pm id_m$ , the solution of the complementary equation is

$$\begin{aligned} \psi_m = & c_{1m} \cosh(a_m z) \sin(b_m z) + c_{2m} \cosh(a_m z) \cos(b_m z) + \\ & c_{3m} \sinh(a_m z) \cos(b_m z) + c_{4m} \sinh(a_m z) \sin(b_m z) + \\ & c_{5m} \cosh(c_m z) \sin(d_m z) + c_{6m} \cosh(c_m z) \cos(d_m z) + \\ & c_{7m} \sinh(c_m z) \cos(d_m z) + c_{8m} \sinh(c_m z) \sin(d_m z), \end{aligned} \quad (10)$$

In order to determine  $c_{im}$  ( $i=1\cdots 8$ ), eight boundary conditions are needed. There are the displacement boundary conditions for the fixed end and the force boundary conditions for the free end [8]. Also, we seek a particular solution.

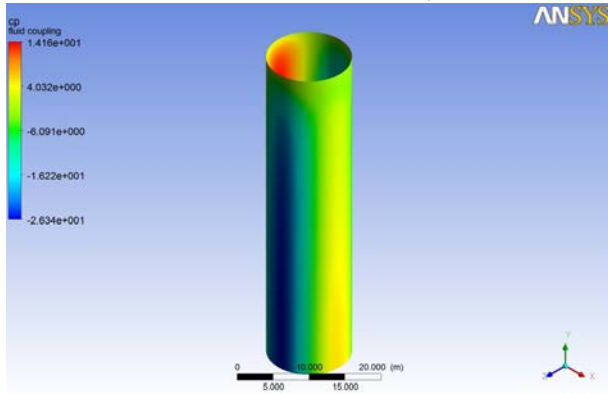
Finally, it should be pointed out that the above complex calculation is executed by the Maple code. The results are not shown here.

## Comparison of Results and Analysis

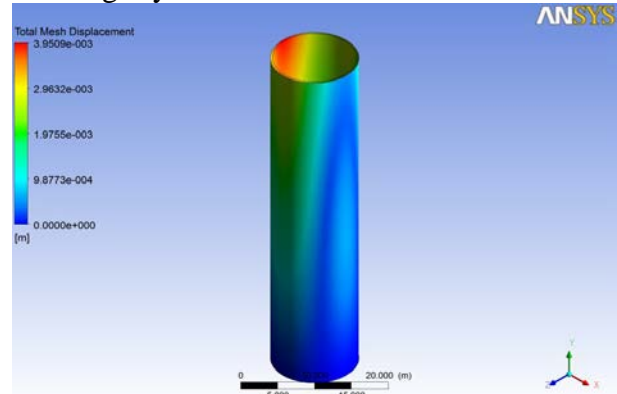
In order to validate the proposed analytical method, a fluid-structure coupled comparative model is built up by the advanced numerical method, which is conducted by ANSYS combined with CFX in ANSYS Workbench platform.

The parameters are set as follows:  $V_\infty=3\text{m/s}$ ,  $P_\infty=0\text{Pa}$ ,  $\rho=1000\text{kg/m}^3$ ,  $d=h=50\text{m}$ ,  $R=6\text{m}$ ,  $t=0.3\text{m}$ ,  $E=200\times 10^9\text{Pa}$  and  $\mu=0.3$ .

The coefficient of dynamic pressure is defined by  $CP=0.5(P-P_\infty)/\rho V_\infty^2$ , whose contour is depicted in Fig. 1(a). It is shown that the thin-walled hollow pier with a range of  $CP$  from -26.34 to 14.16, is subjected to both positive pressure and negative pressure. Comparison between the values of  $CP$  calculated by the proposed method and the ones computed by the ANSYS procedure, is displayed in Fig. 2(a). We find that the change trends of curves derived from the two methods are consistent with each other. For  $\theta=90^\circ$  and  $\theta=270^\circ$ ,  $CP$  increases gradually as height of bridge pier increases. For  $\theta=0^\circ$ ,  $CP$  decreases progressively with the increase of height of bridge pier. For  $\theta=180^\circ$ , at first  $CP$  increases slowly and then decreases quickly. In general, theoretical solutions are close to numerical solutions. Here, the latter are greater slightly than the former.

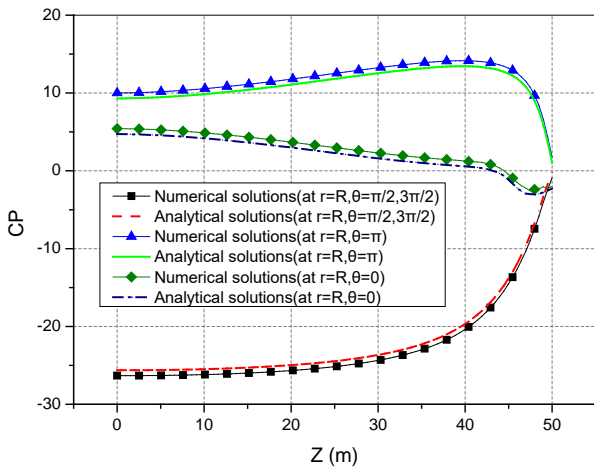


(a) dynamic pressure coefficient

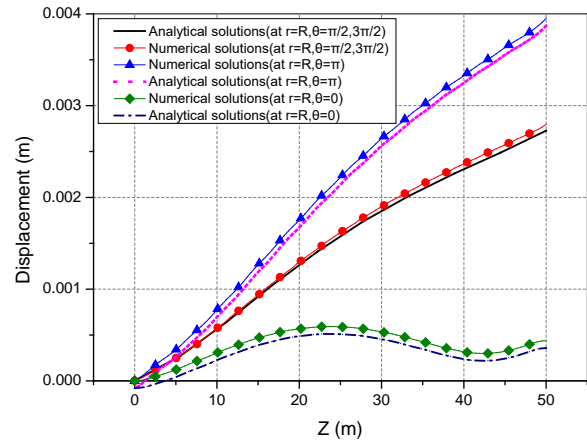


(b) displacement

Fig. 1. Contour of results simulated results



(a) dynamic pressure coefficient



(b) displacement

Fig. 2. Comparison of calculation results between the proposed method and ANSYS procedure  
The contour of displacement of the pier is depicted in Fig. 1(b). As can be seen from the one, the

displacement value varies from 0 to 3.95mm. For  $\theta=90^\circ$  and  $\theta=270^\circ$ , the displacement field is symmetric while symmetry does not exist for  $\theta=90^\circ$  and  $\theta=270^\circ$ . Fig. 2(b) shows the displacement results, respectively, obtained from the proposed method and the ANSYS procedure and variation of the displacement with respect to height of the bridge pier for different angular positions. It could be observed that the displacement increases progressively with the increase of height of bridge pier for  $\theta=180^\circ$ ,  $\theta=90^\circ$  and  $\theta=270^\circ$ , while the characters of the curve for  $\theta=0^\circ$  are relatively complicated.

Although there exists numerical deviation between the two solutions, their distribution regularities coincide well. Consequently, the analytical algorithm based on the ULE method is verified to be feasible.

## Conclusion

The analytical algorithm based on the ULE method for the problem of pier-water coupling is feasible. The present method can multi-directionally analyze the response of bridge pier under the action of water flow by a mathematical procedure.

The proposed method will also serve as a theoretical basis for a new numerical study on the issue of FSI with the ULE technology. Some work along this line is already underway and will be reported in a subsequent paper.

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