Exp-Function Method with Computerized Symbolic Computation for New Exact Solution of Burger's Equation

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Abstract. It is well known that computer applications are ubiquitous. In this paper, the exp-function method with computerized symbolic computation is improved to exactly solve a (2+1)-dimensional Burger's equation. As a result, a new and more general exact solution is obtained. It is shown that the improved exp-function method can be used for other nonlinear partial differential equations (PDEs).

Introduction

Exactly solving nonlinear PDEs plays an important role in the study of nonlinear physical phenomena in many fields of nonlinear science. In the past several decades, some effective methods equipped with the powerful computerized symbolic computation systems like *Mathematica* or *Maple* have been proposed for nonlinear PDEs [1-8]. It is shown that these symbolic computation systems become indispensable auxiliary tools for solving complex PDEs or constructing more general solutions of PDEs. Among the existing methods, the exp-function method [1] proposed by He and Wu in 2006 is a simple and direct method for nonlinear PDEs. More importantly, the exp-function method and its improvements are available for many PDEs such as those in [9-16], this is due to the method possesses a more general ansatz [17]:

$$u = \frac{\sum_{n=-f}^g a_n \exp(n\xi)}{\sum_{m=-p}^q b_m \exp(m\xi)}, \quad \xi = \sum_{i=1}^s k_i x + wt, \quad (1)$$

Supposed for a given nonlinear PDE with independent variables $t, x_1, x_2, \dots, x_s$ and dependent variable u :

$$F(u, u_t, u_{x_1}, u_{x_2}, \dots, u_{x_m}, u_{x_1 t}, u_{x_2 t}, \dots, u_{x_s t}, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, \dots, u_{x_s x_s}, \dots) = 0. \quad (2)$$

In Eq. (1), a_n and b_m are undetermined constants, integers f , p , g and q can be determined by balancing the highest order linear term with the highest order nonlinear term in Eq. (2). In general, the final solution does not strongly depend on the choices of values of f , p , g , q and $f = p = g = q = 1$ is usually the simplest choice. In [17], Zhang et al. proposed a direct algorithm of the exp-function method for simplifying the computation so that the so-called middle expression expansion problem can be effectively solved. In this paper, we would like to improve the exp-function method by introducing the following general ansatz:

$$u = \frac{\sum_{n=-f}^g a_n(y, z, \dots, t) \exp(n\xi)}{\sum_{m=-p}^q b_m(y, z, \dots, t) \exp(m\xi)}, \quad \xi = kx + \eta(y, z, \dots, t), \quad (3)$$

where $a_n(y, z, \dots, t)$, $b_m(y, z, \dots, t)$ and $\eta(y, z, \dots, t)$ are undetermined differentiable functions of the indicated variables. To test the effectiveness and advantage of ansatz (3), in the next section we consider the following (2+1)-dimensional Burger's equation [18]:

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0. \quad (4)$$

Exact Solutions

We suppose that

$$u = \frac{a_1(y, t) \exp(n\xi) + a_0(y, t) + a_{-1}(y, t) \exp(-n\xi)}{b_1(y, t) \exp(n\xi) + b_0(y, t) + b_{-1}(y, t) \exp(-n\xi)}, \quad (5)$$

where $\xi = kx + \eta(y, t)$, $a_n(y, t)$, $b_m(y, t)$ and $\eta(y, t)$ are all differentiable functions to be determined later. Substituting Eq. (5) into Eq. (4) and then equating each coefficient of the same order power of $\exp(j\xi)$ ($j = 1, 2, \dots, 8$) to zero yields a system of nonlinear PDEs for $a_1(y, t)$, $a_0(y, t)$, $a_{-1}(y, t)$, $b_1(y, t)$, $b_0(y, t)$, $b_{-1}(y, t)$ and $\eta(y, t)$. From this derived system of nonlinear PDEs, we can see that if the solution (5) is a non-trivial one then the following equations must hold simultaneously

$$a_{1yy}(y, t) = 0, \quad a_{-1yy}(y, t) = 0. \quad (6)$$

When $a_0(y, t) = 0$, $b_1(y, t) = b_1$, $b_0(y, t) = 0$ and $b_{-1}(y, t) = b_{-1}$, here b_1 and b_{-1} are arbitrary constants, the system of nonlinear PDEs are reduced as follows:

$$\begin{aligned} & -8b_1^2b_{-1}k^3a_1(y, t) - 4b_1b_{-1}k^2a_1^2(y, t) + 8b_1^3k^3a_{-1}(y, t) + 4b_1^2k^2a_1(y, t)a_{-1}(y, t) \\ & + 2b_1^2b_{-1}ka_{1t}(y, t) - 2b_1^3ka_{1t}(y, t) - 4b_1^2b_{-1}ka_1(y, t)\eta_t(y, t) + 4b_1^3ka_{-1}(y, t)\eta_t(y, t) \\ & + 4b_1^2b_{-1}a_{1y}(y, t)\eta_y(y, t) - 4b_1^3a_{-1y}(y, t)\eta_y(y, t) - 4b_1^2b_{-1}a_1(y, t)\eta_y^2(y, t) \\ & + 4b_1^3a_{-1}(y, t)\eta_y^2(y, t) + 2b_1^2b_{-1}a_1(y, t)\eta_{yy}(y, t) - 2b_1^3a_{-1}(y, t)\eta_{yy}(y, t) = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} & 32b_1^2b_{-1}k^3a_1(y, t) + 8b_{-1}^2k^2a_1^2(y, t) - 32b_1^2b_{-1}k^3a_{-1}(y, t) - 16b_1b_{-1}k^2a_1(y, t)a_{-1}(y, t) \\ & + 8b_1^2ka_{1t}^2(y, t) + 4b_1b_{-1}^2ka_{1t}(y, t) - 4b_1^2b_{-1}ka_{-1t}(y, t) + 4b_1^2b_{-1}ka_{-1t}(y, t) \\ & + 8b_1^2b_{-1}a_{1y}(y, t)\eta_y(y, t) + 4b_1b_{-1}^2a_1(y, t)\eta_{yy}(y, t) - 4b_1^2b_{-1}a_{-1}(y, t)\eta_{yy}(y, t) = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} & -8b_{-1}^3k^3a_1(y, t) + 8b_1b_{-1}^2k^3a_{-1}(y, t) + 4b_{-1}^2ka_{1t}(y, t) - 4b_1b_{-1}k^2a_{-1}^2(y, t) \\ & + 2b_1^3ka_{1t}(y, t) - 2b_1b_{-1}^2ka_{1t}(y, t) + 4b_{-1}^3ka_1(y, t)\eta_t(y, t) - 4b_1b_{-1}^2ka_{-1}(y, t)\eta_t(y, t) \\ & + 4b_{-1}^3a_{1y}(y, t)\eta_y(y, t) - 4b_1b_{-1}^2a_{-1y}(y, t)\eta_y(y, t) + 4b_{-1}^3a_1(y, t)\eta_y^2(y, t) \\ & - 4b_1b_{-1}^3a_{-1}(y, t)\eta_y^2(y, t) + 2b_{-1}^3a_1(y, t)\eta_{yy}(y, t) - 2b_1b_{-1}^2a_{-1}(y, t)\eta_{yy}(y, t) = 0. \end{aligned} \quad (9)$$

Solving Eqs. (6)-(9), we have

$$a_1(y, t) = -b_1(1 + 2k + \frac{f'(t)}{k}), \quad a_{-1}(y, t) = -b_{-1}(1 - 2k + \frac{f'(t)}{k}), \quad \eta(y, t) = ly + f(t), \quad (10)$$

where l is an arbitrary constant, k is a non-zero constant, and $f(t)$ is an arbitrary differentiable function of t .

With the help of Eqs. (5) and (10), we obtain a new exact and explicit solitary wave solution of Eq. (4):

$$u = \frac{[-b_1(1 + 2k + \frac{f'(t)}{k})]\exp(\xi) + [-b_{-1}(1 - 2k + \frac{f'(t)}{k})]\exp(-\xi)}{b_1 \exp(\xi) + b_{-1} \exp(-\xi)}, \quad (11)$$

where $\xi = kx + ly + f(t)$. It is easy to see that solution (11) cannot be constructed by the ansatz (1) used in the exp-function method [1]. Since the arbitrary function $f(t)$ is contained, solution (11) is more general than the one constructed by the exp-function method [1]. Besides, if we set $f(t) = wt$, here w is a constant, then solution (11) becomes

$$u = \frac{[-b_1(1 + 2k + \frac{w}{k})]\exp(\xi) + [-b_{-1}(1 - 2k + \frac{w}{k})]\exp(-\xi)}{b_1 \exp(\xi) + b_{-1} \exp(-\xi)}, \quad (12)$$

where $\xi = kx + ly + wt$. Because solution (12) is a special case of solution (11), the improved exp-function method equipped with ansatz (3) has advantage over the exp-function method [1].

With the arbitrary function $f(t)$, solution (11) possesses rich spatial structures. For example, we set $f(t) = 1 - \operatorname{sn}(1+t^2, m)$, $k=1$, $b_1=1$, and $b_{-1}=1$, a spatial structure of solution (11) is shown in Fig. 1. For the comparison between solutions (11) and (12), we use Fig. 2 to show a spatial structure of solution (12) by selecting $w=1$ and letting the other parameters be same as those in Fig. 1.

Fig. 1. Spatial structure solution (11)

Fig. 2. Spatial structure of solution (12)

From Figs. 1 and 2, we can see that solutions obtained by the improved exp-function method in present paper may have new spatial structures besides the spatial structures determined by the solutions constructed by the exp-function method [1].

Conclusion

In summary, we have improved the exp-function method by introducing ansatz (3) which includes the one used in the exp-function method [1] as a special case. Due to the generality of ansatz (3), new and more general solutions with arbitrary functions can be obtained. The effectiveness and advantage of the improved exp-function method have been tested through the (2+1)-dimensional Burger's equation. Applying the improved exp-function method presented in this paper to other nonlinear PDEs is worthy of study.

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