A New Mathematical Model and its Solvability Test on Computer

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Abstract. It is interesting to test the solvability of nonlinear differential equations. In this paper, a new mathematical model passes through the test of solvability on computer. To test the solvability of the mathematical model, Mathematica 4.0 software system combined with a subsidiary equation method is utilized. At the same time, some solutions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solution. This paper shows that the Mathematica 4.0 software system combined with the subsidiary equation method can provide a preferred mathematical tool for testing the solvability of some new nonlinear differential equations.

Introduction

In 2012, Shan and Zhu [1] derived a new mathematical model:

$$u_t = 18v_{xxx} - 36(vu)_x - u_{7x} + 49u_x u_{xxx} + 14uu_{5x} + 84u_{xx} u_{xxx} - 70u_x^3 - 252uu_x u_{xx} - 56u^2 u_{xxx} + \frac{224}{3}u^3 u_x, \quad (1)$$

$$v_t = -36vv_x + v_x v_{xxx} + 3vu_{5x} - 12v_x uu_{xx} - 72vu_x u_{xx} - 36vuu_{xxx} - 6v_x u_x^2 + \frac{32}{3}u^3 v_x + 96vu^2 u_x. \quad (2)$$

To the best of our knowledge, Eqs. (1) and (2) have not been solved. It is well known that many physical phenomena can be usually described by nonlinear partial differential equations (PDEs). Researchers often investigate exact solutions of such nonlinear PDEs to gain more insight into these physical phenomena for further applications. So, solving nonlinear PDEs is interesting. Since the initial-value problem of the Korteweg-de Vries (KdV) equation was solved by Gardner, Green, Kruskal and Miura [2] in 1965, many effective methods have been proposed for nonlinear PDEs, such as Bäcklund transformation [3], Hirota's bilinear method [4], homogeneous balance method [5], tanh-function method [6], Jacobi elliptic function expansion method [7], and others [8-17]. Generally speaking, it is hard to exactly solve a nonlinear PDE because of the complexity of the PDE itself. In this paper, we shall consider a special case of Eqs. (1) and (2) when $v = 0$:

$$u_t = -u_{7x} + 49u_x u_{xxx} + 14uu_{5x} + 84u_{xx} u_{xxx} - 70u_x^3 - 252uu_x u_{xx} - 56u^2 u_{xxx} + \frac{224}{3}u^3 u_x, \quad (3)$$

for testing its solvability by using Mathematica 4.0 software system combined with a subsidiary equation method. As a result, five solutions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solution.

Solvability and Exact Solutions

Firstly, we take advantage of the following travelling wave transformation:

$$u = u(\xi), \quad \xi = kx + \omega t, \quad (4)$$

Eq. (3) becomes

$$\omega u' = -k^7 u^{(7)} + 49k^5 u' u^{(4)} + 14k^5 uu^{(5)} + 84k^5 u'' u''' - 70k^3 u'^3 - 252k^3 uu' u'' - 56k^3 u^2 u''' + \frac{224}{3}ku^3 u'. \quad (5)$$

Secondly, we suppose that Eq. (5) has a solution in the form:

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + \cdots + a_m\phi^m(\xi), \quad (6)$$

where $a_0, a_1, a_2, \dots, a_m$ are constants to be determined later, m is an undetermined integer, while $\phi(\xi)$ satisfies the subsidiary equation

$$\phi'(\xi) = \sigma + \phi^2(\xi), \quad \sigma = \text{constant}. \quad (7)$$

Balancing the highest order linear term $u^{(7)}$ with the highest order nonlinear term u^3u' in Eq. (5) yields

$$7 + m = 4m + 1, \quad (8)$$

which gives

$$m = 2. \quad (9)$$

Running Mathematica 4.0, from Eqs. (6) and (7) we have

$$\begin{aligned} u' &= (a_0 + a_1\phi + a_2\phi^2)' \\ &= a_1\phi' + 2a_2\phi\phi' \\ &= a_1(\sigma + \phi^2) + 2a_2\phi(\sigma + \phi^2) \\ &= 2a_2\phi^3 + a_1\phi^2 + 2a_2\sigma\phi + a_1\sigma, \end{aligned} \quad (10)$$

$$\begin{aligned} u'' &= 6a_2\phi^2\phi' + 2a_1\phi\phi' + 2a_2\sigma\phi' \\ &= 6a_2\sigma\phi^2 + 6a_2\phi^4 + 2a_1\sigma\phi + 2a_1\phi^3 + 2a_2\sigma^2 + 2a_2\sigma\phi^2 \\ &= 6a_2\phi^4 + 2a_1\phi^3 + 8a_2\sigma\phi^2 + 2a_1\sigma\phi + 2a_2\sigma^2, \end{aligned} \quad (11)$$

$$\begin{aligned} u''' &= 24a_2\phi^3\phi' + 6a_1\phi^2\phi' + 16a_2\sigma\phi\phi' + 2a_1\sigma\phi' \\ &= 24a_2\sigma\phi^3 + 24a_2\phi^5 + 6a_1\sigma\phi^2 + 6a_1\phi^4 + 16a_2\sigma^2\phi + 16a_2\sigma\phi^3 + 2a_1\sigma^2 + 2a_1\sigma\phi^2 \\ &= 24a_2\phi^5 + 6a_1\phi^4 + 40a_2\sigma\phi^3 + 8a_1\sigma\phi^2 + 16a_2\sigma^2\phi + 2a_1\sigma^2, \end{aligned} \quad (12)$$

$$\begin{aligned} u^{(4)} &= 120a_2\phi^4\phi' + 24a_1\phi^3\phi' + 120a_2\sigma\phi^2\phi' + 16a_1\sigma\phi\phi' + 16a_2\sigma^2\phi' \\ &= 120a_2\sigma\phi^4 + 120a_2\phi^6 + 24a_1\sigma\phi^3 + 24a_1\phi^5 + 120a_2\sigma^2\phi^2 + 120a_2\sigma\phi^4 + 16a_1\sigma^2\phi \\ &\quad + 16a_1\sigma\phi^3 + 16a_2\sigma^3 + 16a_2\sigma^2\phi^2 \\ &= 120a_2\phi^6 + 24a_1\phi^5 + 240a_2\sigma\phi^4 + 40a_1\sigma\phi^3 + 136a_2\sigma^2\phi^2 + 16a_1\sigma^2\phi + 16a_2\sigma^3, \end{aligned} \quad (13)$$

$$\begin{aligned} u^{(5)} &= 720a_2\phi^5\phi' + 120a_1\phi^4\phi' + 960a_2\sigma\phi^3\phi' + 120a_1\sigma\phi^2\phi' + 272a_2\sigma^2\phi\phi' + 16a_1\sigma^2\phi' \\ &= 720a_2\sigma\phi^5 + 720a_2\phi^7 + 120a_1\sigma\phi^4 + 120a_1\phi^6 + 960a_2\sigma^2\phi^3 + 960a_2\sigma\phi^5 \\ &\quad + 120a_1\sigma^2\phi^2 + 120a_1\sigma\phi^4 + 272a_2\sigma^3\phi + 272a_2\sigma^2\phi^3 + 16a_1\sigma^3 + 16a_1\sigma^2\phi^2 \\ &= 720a_2\phi^7 + 120a_1\phi^6 + 1680a_2\sigma\phi^5 + 240a_1\sigma\phi^4 + 1232a_2\sigma^2\phi^3 \\ &\quad + 136a_1\sigma^2\phi^2 + 272a_2\sigma^3\phi + 16a_1\sigma^3, \end{aligned} \quad (14)$$

$$\begin{aligned} u^{(6)} &= 5040a_2\phi^6\phi' + 720a_1\phi^5\phi' + 8400a_2\sigma\phi^4\phi' + 960a_1\sigma\phi^3\phi' + 3696a_2\sigma^2\phi^2\phi' \\ &\quad + 272a_1\sigma^2\phi\phi' + 272a_2\sigma^3\phi' \\ &= 5040a_2\sigma\phi^6 + 5040a_2\phi^8 + 720a_1\sigma\phi^5 + 720a_1\phi^7 + 8400a_2\sigma^2\phi^4 + 8400a_2\sigma\phi^6 \\ &\quad + 960a_1\sigma^2\phi^3 + 960a_1\sigma\phi^5 + 3696a_2\sigma^3\phi^2 + 3696a_2\sigma^2\phi^4 + 272a_1\sigma^3\phi + 272a_1\sigma^2\phi^3 \\ &\quad + 272a_2\sigma^4 + 272a_2\sigma^3\phi^2 \\ &= 5040a_2\phi^8 + 720a_1\phi^7 + 13440a_2\sigma\phi^6 + 1680a_1\sigma\phi^5 + 12096a_2\sigma^2\phi^4 + 1232a_1\sigma^2\phi^3 \\ &\quad + 3968a_2\sigma^3\phi^2 + 272a_1\sigma^3\phi + 272a_2\sigma^4, \end{aligned} \quad (15)$$

$$\begin{aligned} u^{(7)} &= 40320a_2\phi^7\phi' + 5040a_1\phi^6\phi' + 80640a_2\sigma\phi^5\phi' + 8400a_1\sigma\phi^4\phi' + 48384a_2\sigma^2\phi^3\phi' \\ &\quad + 3696a_1\sigma^2\phi^2\phi' + 7936a_2\sigma^3\phi\phi' + 272a_1\sigma^3\phi' \\ &= 40320a_2\sigma\phi^7 + 40320a_2\phi^9 + 5040a_1\sigma\phi^6 + 5040a_1\phi^8 + 80640a_2\sigma^2\phi^5 + 80640a_2\sigma\phi^7 \\ &\quad + 5040a_1\sigma^2\phi^4 + 8400a_1\sigma\phi^6 + 48384a_2\sigma^3\phi^3 + 48384a_2\sigma^2\phi^5 \end{aligned}$$

$$\begin{aligned}
& +3696a_1\sigma^3\phi^2 + 3696a_1\sigma^2\phi^8 + 7936a_2\sigma^4\phi + 7936a_2\sigma^3\phi^3 + 272a_1\sigma^4 + 272a_1\sigma^3\phi^2 \\
& = 40320a_2\phi^9 + 5040a_1\phi^8 + 120960a_2\sigma\phi^7 + 13440a_1\sigma\phi^6 + 129024a_2\sigma^2\phi^5 \\
& + 12096a_1\sigma^2\phi^4 + 56320a_2\sigma^3\phi^3 + 3968a_1\sigma^3\phi^2 + 7936a_2\sigma^4\phi + 272a_1\sigma^4. \tag{16}
\end{aligned}$$

Substituting Eqs. (6), (10)-(14) and (16) into Eq. (5) and then equating each coefficient of the same order power of ϕ^i ($i = 0, 1, 2, \dots, 9$) to zero yields a system of nonlinear equations for a_0 , a_1 , a_2 , $\dots$, a_m , k and ω :

$$\begin{aligned}
\phi^0 : & -\frac{224}{3}a_0^3a_1k\sigma + a_1\omega\sigma + 112a_0^2a_1k^3\sigma^2 + 70a_1^3k^3\sigma^3 + 504a_0a_1a_2k^3\sigma^3 - 224a_0a_1k^5\sigma^3 \\
& - 1120a_1a_2k^5\sigma^4 + 272a_1k^7\sigma^4 = 0, \tag{17}
\end{aligned}$$

$$\begin{aligned}
\phi^1 : & -224a_0^2a_1^2k\sigma - \frac{448}{3}a_0^3a_2k\sigma + 2a_2\omega\sigma + 728a_0a_1^2k^3\sigma^2 + 896a_0^2a_2k^3\sigma^2 \\
& + 924a_1^2a_2k^3\sigma^3 + 1008a_0a_2^2k^3\sigma^3 - 1344a_1^2k^5\sigma^3 - 3808a_0a_2k^5\sigma^3 - 4256a_2^2k^5\sigma^4 \\
& + 7936a_2k^7\sigma^4 = 0, \tag{18}
\end{aligned}$$

$$\begin{aligned}
\phi^2 : & -\frac{224}{3}a_0^3a_1k + a_1\omega - 224a_0a_1^3k\sigma - 672a_0^2a_1a_2k\sigma + 448a_0^2a_1k^3\sigma + 826a_1^3k^3\sigma^2 \\
& + 5544a_0a_1a_2k^3\sigma^2 - 1904a_0a_1k^5\sigma^2 + 2352a_1a_2^2k^3\sigma^3 - 18424a_1a_2k^5\sigma^3 \\
& + 3968a_1k^7\sigma^3 = 0, \tag{19}
\end{aligned}$$

$$\begin{aligned}
\phi^3 : & -224a_0^2a_1^2k - \frac{448}{3}a_0^3a_2k + 2a_2\omega - \frac{224}{3}a_1^4k\sigma - 896a_0a_1^2a_2k\sigma - 448a_0^2a_2^2k\sigma \\
& + 1904a_0a_1^2k^3\sigma + 2240a_0^2a_2k^3\sigma + 6412a_1^2a_2k^3\sigma^2 + 6832a_0a_2^2k^3\sigma^2 - 6328a_1^2k^5\sigma^2 \\
& - 17248a_0a_2k^5\sigma^2 + 1568a_2^3k^3\sigma^3 - 36176a_2^2k^5\sigma^3 + 56320a_2k^7\sigma^3 = 0, \tag{20}
\end{aligned}$$

$$\begin{aligned}
\phi^4 : & -224a_0a_1^3k - 672a_0^2a_1a_2k + 336a_0^2a_1k^3 - \frac{1120}{3}a_1^3a_2k\sigma - 1120a_0a_1a_2^2k\sigma \\
& + 1666a_1^3k^3\sigma + 10920a_0a_1a_2k^3\sigma - 3360a_0a_1k^5\sigma + 12992a_1a_2^2k^3\sigma^2 \\
& - 59864a_1a_2k^5\sigma^2 + 12096a_1k^7\sigma^2 = 0, \tag{21}
\end{aligned}$$

$$\begin{aligned}
\phi^5 : & -\frac{224}{3}a_1^4k - 896a_0a_1^2a_2k - 448a_0^2a_2^2k + 1176a_0a_1^2k^3 + 1344a_0^2a_2k^3 - 672a_1^2a_2^2k\sigma \\
& - 448a_0a_2^3k\sigma + 10948a_1^2a_2k^3\sigma + 11536a_0a_2^2k^3\sigma - 8848a_1^2k^5\sigma - 23520a_0a_2k^5\sigma \\
& + 7616a_2^3k^3\sigma^2 - 93072a_2^2k^5\sigma^2 + 129024a_2k^7\sigma^2 = 0, \tag{22}
\end{aligned}$$

$$\begin{aligned}
\phi^6 : & -\frac{1120}{3}a_1^3a_2k - 1120a_0a_1a_2^2k + 910a_1^3k^3 + 5880a_0a_1a_2k^3 - 1680a_0a_1k^5 \\
& - \frac{1568}{3}a_1a_2^3k\sigma + 20048a_1a_2^2k^3\sigma - 69608a_1a_2k^5\sigma + 13440a_1k^7\sigma = 0, \tag{23}
\end{aligned}$$

$$\begin{aligned}
\phi^7 : & -672a_1^2a_2^2k - 448a_0a_2^3k + 5460a_1^2a_2k^3 + 5712a_0a_2^2k^3 - 3864a_1^2k^5 - 10080a_0a_2k^5 \\
& - \frac{448}{3}a_2^4k\sigma + 10976a_2^3k^3\sigma - 95088a_2^2k^5\sigma + 120960a_2k^7\sigma = 0, \tag{24}
\end{aligned}$$

$$\phi^8 : -\frac{1568}{3}a_1a_2^3k + 9408a_1a_2^2k^3 - 27048a_1a_2k^5 + 5040a_1k^7 = 0, \tag{25}$$

$$\phi^9 : -\frac{448}{3}a_2^4k + 4928a_2^3k^3 - 33936a_2^2k^5 + 40320a_2k^7 = 0. \tag{26}$$

With the help of Mathematica 4.0, solving the system of Eqs. (17)-(26) we have

$$a_0 = k^2\sigma, \quad a_1 = 0, \quad a_2 = \frac{3}{2}k^2, \quad \omega = -\frac{4}{3}k^7\sigma^3, \tag{27}$$

and hence obtain

$$u = k^2 \sigma + \frac{3}{2} k^2 \phi^2(\xi), \quad \xi = kx - \frac{4}{3} k^7 \sigma^3 t. \quad (28)$$

Since the subsidiary equation $\phi' = \sigma + \phi^2$ has solutions [18]:

$$\phi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh(-\sqrt{-\sigma} \xi), & \sigma < 0, \\ -\sqrt{-\sigma} \coth(-\sqrt{-\sigma} \xi), & \sigma < 0, \\ \sqrt{\sigma} \tan(\sqrt{\sigma} \xi), & \sigma > 0, \\ -\sqrt{\sigma} \cot(\sqrt{\sigma} \xi), & \sigma > 0, \\ -\frac{1}{\xi}, & \sigma = 0. \end{cases} \quad (29)$$

We then obtain hyperbolic function solutions

$$u = k^2 \sigma - \frac{3}{2} k^2 \sigma \tanh^2(-\sqrt{-\sigma} \xi), \quad \xi = kx - \frac{4}{3} k^7 \sigma^3 t, \quad (30)$$

$$u = k^2 \sigma - \frac{3}{2} k^2 \sigma \coth^2(-\sqrt{-\sigma} \xi), \quad \xi = kx - \frac{4}{3} k^7 \sigma^3 t, \quad (31)$$

Trigonometric function solutions

$$u = k^2 \sigma + \frac{3}{2} k^2 \sigma \tan^2(\sqrt{\sigma} \xi), \quad \xi = kx - \frac{4}{3} k^7 \sigma^3 t, \quad (32)$$

$$u = k^2 \sigma - \frac{3}{2} k^2 \sigma \cot^2(\sqrt{\sigma} \xi), \quad \xi = kx - \frac{4}{3} k^7 \sigma^3 t, \quad (33)$$

and rational solution of Eq. (3):

$$u = \frac{3k^2}{2\xi^2}, \quad \xi = kx - \frac{4}{3} k^7 \sigma^3 t. \quad (34)$$

Conclusion

Since Eqs. (17)-(26) exist a set of solutions listed in Eq. (27), the mathematical model (3) has been verified to pass through the test of solvability on computer by using Mathematica 4.0 software system combined with a subsidiary equation method. Based on the Eq. (28), five solutions including hyperbolic function solutions, trigonometric function solutions and rational solution are obtained. It is shown that the Mathematica 4.0 software system combined with the subsidiary equation method can provide a preferred mathematical tool for testing the solvability of some new nonlinear differential equations.

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