

Time Integration Schemes in Dynamic Problems

Effect of Damping on Numerical Stability and Accuracy

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Abstract—A great deal of progress has been made in the past several decades towards the understanding and development of time integration methods in structural dynamics. These methods involves a step by step algorithm for transient analysis of linear and non-linear dynamic problems. It is essential to provide a comprehensive survey of various methodologies used to solve second order differential equations in a single article. Broadly, the methods include direct integration, mode superposition and response spectrum methods among others. The first two methods uses an integration scheme while response spectrum method is based on extreme response analysis. Both direct integration and mode superposition method use an integration scheme but the selection of a particular method depends on the problem and frequency content of the loading. A detailed survey of various integration schemes is presented in this paper. The stability and accuracy of these integration schemes has been studied by researches in the past. However, the effect of damping on the stability and accuracy of these schemes need to be investigated. A single degree of freedom system is used to check the stability and accuracy criteria of various integration schemes. Also, the effect of damping on these parameters is studied and results are presented.

Keywords: *structural dynamics; integration schemes; damping; stability; accuracy*

I. INTRODUCTION

In many engineering applications, it is essential to perform a dynamic analysis in addition to the static check [1]. This means inclusion of the inertial term in the equation of motion. The inertia term can be neglected only if the loads or displacements are applied very slowly [2]. Some examples where inertia effects are important include the impact loading of the structures where a high intensity load is applied for a short time or seismic action where structure is analyzed for prescribed ground acceleration. Offshore structures like derricks and flare booms are very critical for dynamic loading due to the dynamic effects of wind and drilling operations [3-4]. Also, time history analysis is required for predicting the fatigue lives of structures precisely [5].

Dynamic problems can be broadly classified in two categories based on the effect of the excitation on the overall structural response: wave propagation problems and inertia problems [6]. In wave propagation problems, the behavior at the wave front is of engineering importance and intermediate to high frequency structural modes dominate the structural response in these problems [7]. Problems under this category

include shock response from nuclear weapons such as explosives or impact loading. Also, problems in which wave effects such as reflections and diffractions are important falls under the first category. The inertia problems include all other dynamic problems except wave propagation. The structural response is governed by relatively small number of low frequency modes. Problem of this type are often called structural dynamics problems. The governing equation in these problems is a second order differential equation [8-9]. Only the structural dynamics problems will be discussed in this paper.

As mentioned above, structural dynamics problems include solving a second order differential equation which is also the equation of motion. For a linear elastic system, equation of motion can be expressed as:

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = R(t) \quad (1)$$

where M is the discrete mass matrix, C is the viscous damping matrix, K is the linear stiffness matrix, R is the external load vector. In general, M , C and K are constant and symmetric. $R=R(t)$ is a given continuous function of time t . Mathematically, (1) represents a coupled system of linear ordinary differential equation of second order.

In the last several decades, significant advances have been made in the development and application of time integration methods for solving above set of coupled differential equations [10]. This was also primarily made possible due to the parallel development of high speed digital computers reducing the computational times and providing accurate results. This has resulted in the development of many commercially available software, viz. ANSYS, MARC, NONSAP and etc. [11]. All these software codes are based on various computational methods developed to solve any dynamic problem.

The objective of this paper is to present a review of various solution methods for structural dynamic problems. Solution methods like direct integration and modal superposition are based on using an integration scheme. Various conventional as well as recently developed integration schemes are discussed in this paper. The use of an integration scheme for a given problem depends on both the stability as well as the accuracy of the scheme. These criteria are discussed in detail. The presence of damping in the system can have an effect on the stability of integration scheme being used and this has also been investigated. Stability and accuracy results of various integration schemes are presented for both damped and

undamped systems. The paper concludes with discussion of the results and highlighting the importance of selecting a suitable method for a given dynamics problem.

II. SOLUTION METHODS FOR DYNAMIC PROBLEMS

Mathematically, Eq. (1) represents a system of linear differential equations of second order and can be solved by standard procedures for the solution of differential equations. However, existing standard procedures for the solution of general systems of differential equations can be very expensive especially when the order of matrices involved is large. In practical finite element analysis, only the effective methods are incorporated. These effective methods can be broadly divided in three categories: (1) Direct integration methods, (2) Mode superposition method and (3) Response spectrum method. In the following sections, the first two methods are discussed in some detail.

III. DIRECT INTEGRATION METHODS

The direct integration methods are time marching schemes where the dynamic equilibrium equation is satisfied at discrete time intervals Δt apart. Due to ease of application and their ready usability in nonlinear problems, these methods have become very popular [1]. The term 'direct' means prior to the numerical integration, no transformation of the equations into a different form is carried out. Any direct integration method is built on two basic ideas.

1. Computing the solution of equations of motion at discrete time steps. To compute the numerical solution at specific time t_i , most methods require the solution to be specified at previous time step, t_{i-1} [12].
2. Assuming a variation of displacements, velocities and accelerations within each time step, where different forms of these assumed variations give rise to different integration methods.

The available direct integration methods can be broadly subdivided into two categories: explicit methods and implicit methods [13]. Explicit methods use the equation of motion at the time(s) for which displacements are known, to obtain the solution at time $t+\Delta t$. On the other hand, implicit methods use the equations at a time for which the solution is unknown, to obtain the response at time $t+\Delta t$ [14]. The explicit methods are more appropriate to wave propagation problems, while the implicit one is used to inertia problems [10].

A. Explicit direct integration methods

These methods in general employs finite difference methods and are particularly well suited for short duration dynamical problems or wave propagation problems. In these methods, the equilibrium conditions at time t , are used to solve for the solution at time $t+\Delta t$. Hence, such methods do not involve factorization of the stiffness matrix in the step by step solution. This also means that there is no necessity to store the stiffness matrix if a diagonal mass matrix is used. These methods are computationally cost effective compared to implicit methods and less storage is required. Also, for these methods, computer operations are relatively few and are independent of the finite element mesh band or front width.

Explicit integration methods are therefore very efficient for short duration dynamic problems where stability as well as accuracy conditions are both ensured. In these problems, the contribution of intermediate to high frequency structural modes to the response is important. The stability criteria is generally governed by the highest frequency of the discrete system. These stability and accuracy criteria will be presented and compared for various methods in detail. Some of the most commonly used explicit integration methods are discussed in the next section. These are the second order central difference method and fourth order Runge-Kutte method.

Second Order Central Difference Method: The second order central difference method is one of the widely used explicit techniques. This method is said to have the maximum stability and highest accuracy among the explicit methods [15]. However, this method is only conditionally stable; that is, the chosen time step must be smaller than a critical time step to attain a stable solution. Inability to handle non-diagonal damping matrix is another shortcoming of this method.

Advances have been made and procedures have been developed to compute the diagonal mass matrix from the standard consistent mass matrix so that central difference method can be used effectively [16-18]. Also, numerical integration techniques used to compute the mass matrix are modified further to generate better and accurate diagonal mass matrices [19]. The convergence of a diagonal mass approximation has also been proved [20]. However, some errors are introduced while computing the diagonal mass matrix from standard mass matrix and have been examined [21]. According to [22], errors introduced by the lumped masses and the central difference operator tend to be compensator. So the use of diagonal mass matrices in explicit integration methods is desirable both for accuracy and computational efficiency.

Fourth Order Runge-Kutte Method: The fourth order Runge-Kutte method was proposed in the beginning of 19th century. The method has been extensively used in the past for solving ordinary differential equations. A detailed survey of this method along with others is given by [23].

This one step algorithm has several desirable features like (i) the method is self-starting (ii) the time step can be easily changed (iii) explicit in nature and hence negates the need of iteration in nonlinear problems (iv) the model is a 4th order method and possesses a weak instability only. However, in use of this method, acceleration vector must be computed four times per time step. Due to this, the computational time required for the solution of a problem can be large compared to other numerical integration methods. Also, the method does not provide any estimate of the residual errors.

This method was modified further and several modified methods were developed. Some of these methods include Runge-kutta-Fehlberg methods of order 1 to 3 [24], adaptive Runge-Kuta method [25], among others. These improved methods have better stability properties are equipped with automatic step control based on the local error estimates [26].

Recently Proposed Explicit Methods: Chang [27] recently proposed a new family of explicit methods whose numerical

properties for linear elastic systems are exactly the same as those of the Newmark family method. For this subfamily, the possibility of unconditional stability and second-order accuracy enables using a large time step and involves no iterative procedure. The method has proven very efficient for solving general structural dynamic problems where the responses are dominated by low frequency modes. The method is computationally more effective compared to other conditionally stable explicit methods where the step size is limited. This method was extended to nonlinear systems and a family of non-iterative schemes for nonlinear dynamic problems was proposed [28]. Reference [29] also proposed a family of unconditionally stable explicit direct integration algorithms with controlled numerical energy dissipation. These algorithms are unconditionally stable for linear elastic and stiffness softening –type nonlinear systems.

B. Implicit direct integration methods

The implicit methods find their strength in the areas where explicit methods are not so effective. These methods are most effective for structural dynamic problems in which structural response is controlled by a relatively small number of low frequency modes [30]. Also, problems with complex structural geometries can be solved using these algorithms. In these methods, the solution for the displacements at time $t+\Delta t$ involves solving the stiffness matrix at each time step. This may lead to high computational effort and larger storage requirement compared to explicit methods. However, unlike explicit methods, these methods are unconditionally stable and permits large time steps. With the advancement of high speed computers, the unconditionally stability criteria provide a big advantage. Some of the commonly used effective implicit direct integration methods are presented in brief.

The Newmark Family of Methods: These represent the most commonly used implicit methods for solving the equation of motion in a dynamic problem. The methods are based on the following equations.

$$\dot{u}_{i+1} = \dot{u}_i + ((1-\gamma)\Delta t)\ddot{u}_i + (\gamma\Delta t)\ddot{u}_{i+1} \quad (2)$$

$$u_{i+1} = u_i + (\Delta t)\dot{u}_i + ((0.5-\beta)(\Delta t)^2)\ddot{u}_i + (\beta(\Delta t)^2)\ddot{u}_{i+1} \quad (3)$$

The parameters β and γ define the variation of acceleration over a time step and determine the stability and accuracy characteristics of the method. Typical selection of γ is $1/2$, and $1/6 \leq \beta \leq 1/4$ is satisfactory from all points of view, including the accuracy as shown in Table I.

TABLE I. PARAMETERS FOR NERMARK METHODS

Method	Type	β	γ	Stability
Average acceleration	Implicit	1/4	1/2	unconditional
Linear acceleration	Implicit	1/6	1/2	conditional
Central difference	Explicit	0	1/2	conditional

Wilson- θ method: This method is based on the assumption that the acceleration varies linearly during the time interval t to $t+\theta\Delta t$. For $\theta = 1$, this method reduces to the linear acceleration method of the Newmark family of methods. However, unlike the former, this method is only conditionally stable. It is noted

that in linear problems, the method is unconditionally stable for $\theta \geq 1.37$. Hence, $\theta = 1.40$ is usually used.

IV. MODE SUPERPOSITION METHODS

It is seen that the number of matrix operations required in a direct integration solution is directly proportional to the number of time steps used in the solution procedure. Use of a direct integration method can be effective for a relatively short duration involving fewer time steps. For large duration problems, it is desirable to carry out the integration process by first transforming the equilibrium equations in Eq. (1) into a form in which the step-by-step solution is more effective and less costly. Mode superposition methods are such methods of transforming equilibrium equations from global coordinates to modal coordinates and then use direct integration methods to solve these simplified equations. Mode-superposition analysis is an efficient tool for the evaluation of structural response with many degrees of freedom.

The generalized equation of motion in modal coordinates for a damped system can be written as (4).

$$\ddot{X}(t) + \Phi^T C \Phi \dot{X}(t) + \Omega^2 X(t) = \Phi^T R(t) \quad (4)$$

where the columns in Φ are the mass normalized eigenvectors (free vibration modes) $\phi_1, \phi_2, \phi_3, \dots, \phi_n$ and Ω^2 is a diagonal matrix listing the eigenvalues (frequencies squared).

Equation (4) consists of n uncoupled equations which can be solved ‘exactly’ using the Duhamel integral. Alternatively, any direct integration numerical method can be used. Since the periods of vibration are known, a time step Δt can be chosen in the step-by-step integration in order to obtain a required level of accuracy.

The equilibrium equation reduces to n equations of the form:

$$\ddot{x}_i(t) + 2\omega_i \xi_i \dot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \quad (5)$$

The above equation reduces to Eq. (1) and represents governing motion of the single degree of freedom system with no damping. For the solution, as explained earlier, the response can be obtained by summation of the response in each mode.

Effectiveness of mode-superposition methods: The idea behind mode-superposition solution of a dynamic problem is that frequently only a small fraction of the total number of decoupled equations needs to be considered in order to obtain a good approximate solution of the equilibrium equation. Most frequently, only the first p equilibrium equations need to be solved i.e. only the first p modes out of n are governing where $p \ll n$. This means that only p equations out of n need to be solved and total response in the p modes can then be written as given by Eq. (6).

$$U^p(t) = \sum_{i=1}^p \phi_i x_i(t) \quad (6)$$

It is also seen that a typical finite element procedures approximates the lowest frequencies more precisely and little or no accuracy can be expected in approximating the higher

frequencies and mode shapes. Therefore, the use of lower important modes for a system is justifiable.

The fact that only few modes may need to be considered to arrive at a good approximation solution makes mode superposition method superior to direct integration methods. By solving only p equations out of n , both computational effort as well as cost can be saved. In summary, assuming that the decoupled equations have been solved accurately, the errors in a mode superposition analysis using $p < n$ are due to the fact that not enough modes have been used, whereas the errors in a direct integration analysis arise because of the use of a too large time step.

V. ANALYSIS OF INTEGRATION SCHEMES

The solution of dynamic equilibrium equations can be solved either by direct integration or mode superposition method each of which uses an integration scheme. The cost of using any integration method depends on the time step and number of steps required for solution. The chosen time step should be small enough to guarantee desirable accuracy but at the same time it should not be too small for cost reasons. Therefore, selection of time step is very important in any integration scheme. This selection is generally governed by two criteria which are namely the stability and accuracy criteria.

In case of direct integration or mode superposition, the basic equation can be denoted by (7).

$$\ddot{x}_i(t) + 2\omega_i \xi_i \dot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \quad (7)$$

It is therefore satisfactory to study the stability and accuracy criteria for above typical equation.

A. Stability Analysis

The aim in numerical integration of any dynamic problem is to find an approximate solution to the actual dynamic response of the structure. In order to predict the response accurately, the equilibrium equations (1) must be integrated to high precision. This means that all uncoupled n equations of the form of Eq. (7) need to be integrated accurately.

For the stability analysis, matrix \mathbf{A} and vector \mathbf{L} are defined as the integration approximation and load operators. These quantities can be determined for any integration method and are documented well for each integration scheme. The spectral decomposition of matrix \mathbf{A} is given by $\mathbf{A} = \mathbf{P}\mathbf{J}^n\mathbf{P}^{-1}$ where \mathbf{P} is the matrix of eigenvectors of \mathbf{A} , and \mathbf{J} is the Jordan canonical form of \mathbf{A} with eigenvalues λ_i of \mathbf{A} on its diagonal. The spectral radius of matrix \mathbf{A} is defined as $\rho(\mathbf{A})$ and is given as (8).

$$\rho(\mathbf{A}) = \max_{i=1,2,\dots} |\lambda_i| \quad (8)$$

The stability criteria is given as

- (a) If all eigenvalues are different, then $\rho(\mathbf{A}) \leq 1$.
- (b) If \mathbf{A} contains multiple eigenvalues, then all such eigenvalues should be smaller than 1.

It is noted that the spectral radii and therefore the stability of the integration methods depend on the time ratio $\Delta t/T$, the damping ratio ξ and the integration parameters used. Therefore, for a given $\Delta t/T$ and ξ , it is possible in the Wilson θ method and in the Newmark method to vary the parameters θ and α , δ respectively to obtain optimum stability and accuracy characteristics. Fig. 1 shows the stability characteristics for various integration schemes. It can be seen that the central difference method is only conditionally stable and the Newmark, Wilson θ and Houbolt methods are unconditionally stable.

The stability characteristics of Wilson integration scheme with varying values of its parameter θ is shown in Fig. 2. It is noted that the scheme is only conditionally stable for θ values of 0.5 and 1.0 while it is unconditionally stable for values 1.40 and more. It is therefore desirable to study the stability of the Wilson operator, as a function of θ for different values of $\Delta t/T$ for both damped and undamped cases. The results are shown in Fig. 3.

It can be seen that for an undamped system, the method is unconditionally stable for any $\Delta t/T$ ratio provided $\theta \geq 1.37$. However, presence of damping in the system allows choosing a lower θ value and method is unconditionally stable for $\theta \geq 0.5$.

The Wilson scheme works well with $\theta=1.40$ as shown above. The method is found to retain its unconditional stability when damping is present as shown in Fig. 4.

The central difference method is a conditionally stable method; that is, the chosen time step must be smaller than a critical time step to attain a stable solution. The stability condition and critical time step is given by (9).

$$\Delta t \leq \Delta t_{cr} = \frac{T}{\pi} \quad (9)$$

where T is the smallest natural period of the structure corresponding to the highest frequency mode.

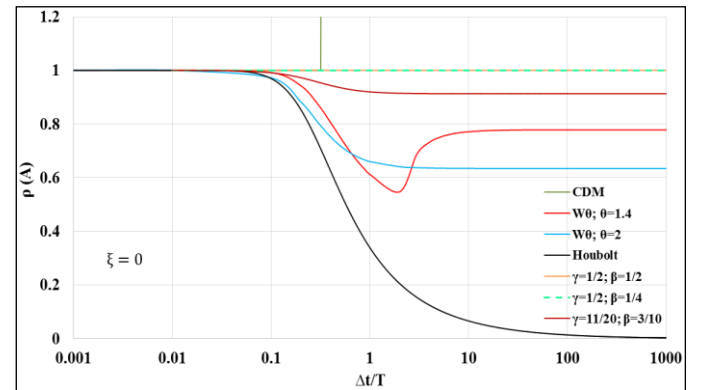


Fig. 1. Spectral radii of integration schemes, $\xi = 0$

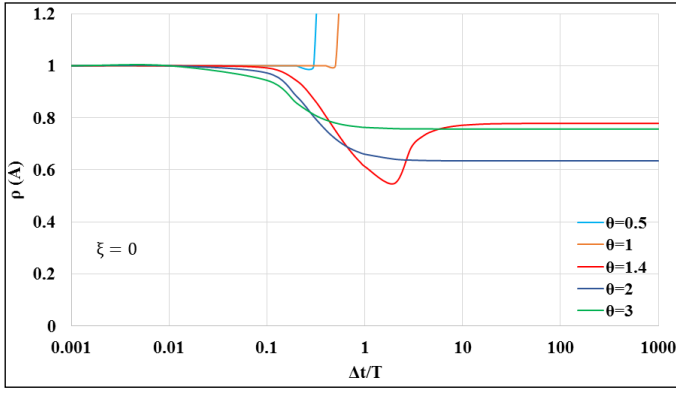


Fig. 2. Spectral radii of Wilson- θ for varying θ , $\xi = 0$

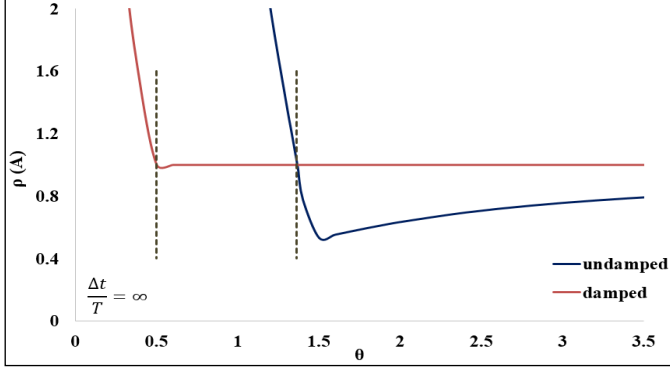


Fig. 3. Spectral radii of Wilson- θ scheme as a function of θ

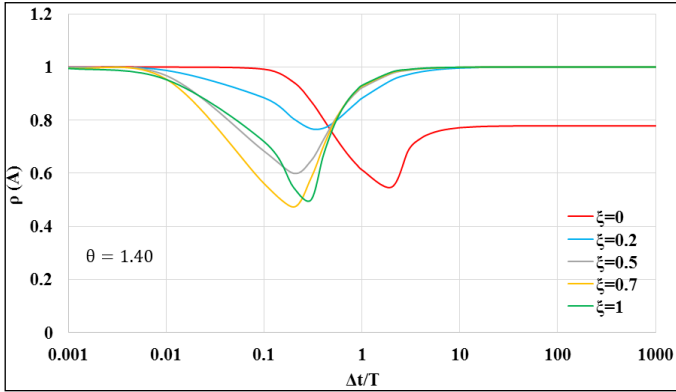


Fig. 4. Spectral radii of Wilson- θ for damped cases, $\theta = 1.40$

For undamped system, the method is stable for $\Delta t/T = 1/\pi = 0.318$ as shown in Fig. 5. It is also shown that the method remains stable when damping is present in the system. For the Newmark method, the two parameters γ and β can be varied to obtain optimum stability and accuracy. The integration scheme is unconditionally stable provided that $\gamma \geq 0.5$ and $\beta \geq 0.25(\gamma + 0.5)^2$. The method corresponding to $\gamma = 0.5$ and $\beta = 0.25$ has the most desirable accuracy characteristics. The stability characteristics of this methods with different sets of parameters is shown in Fig. 6.

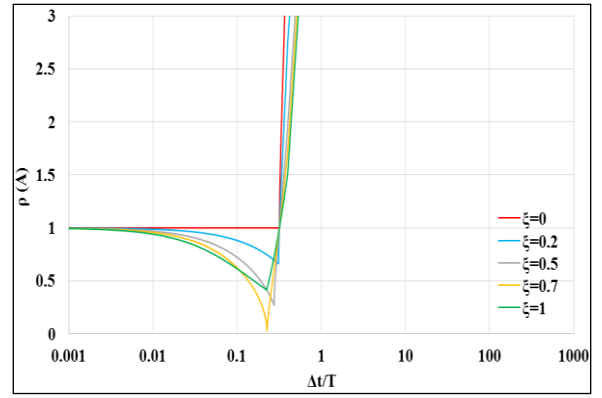


Fig. 5. Spectral radii of central difference method for damped cases

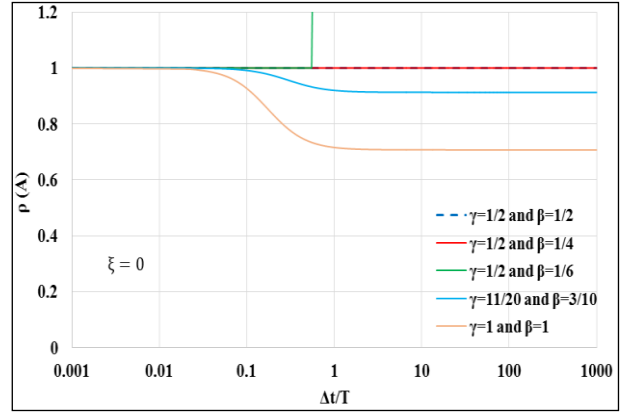


Fig. 6. Spectral radii of Newmark method for various sets of parameters

It is also desirable to observe the stability characteristic of this integration scheme when damping is present in the system. The effect of damping on numerical stability for some sets of parameters is shown in Fig. 7 to Fig. 9. The results are shown for three sets of the parameters. It is concluded that the Newmark integration scheme retain its unconditional stability under damping effects. It is therefore suitable to use this integration scheme for damped systems as well.

The integration schemes discussed so far showed that the stability characteristics for an integration scheme are retained under the damping effects. The stability characteristics of Houbolt method is shown in Fig. 10. It can be observed that like other implicit integration schemes, this method is also unconditionally stable for undamped system.

However, it is observed that the presence of damping introduces instability in the method for a smaller time step Δt . The method remains stable for higher time step values. It is therefore recommended not to use this integration scheme in damped systems for solving short duration dynamic problems as it might give inaccurate results.

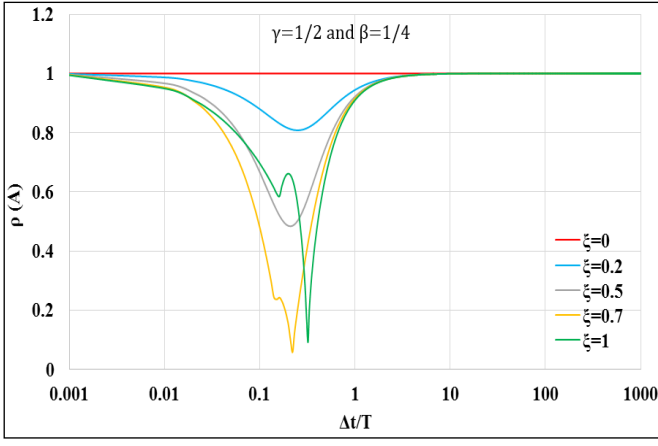


Fig. 7. Spectral radii of Newmark method: damped $\gamma = 1/2$ and $\beta = 1/4$

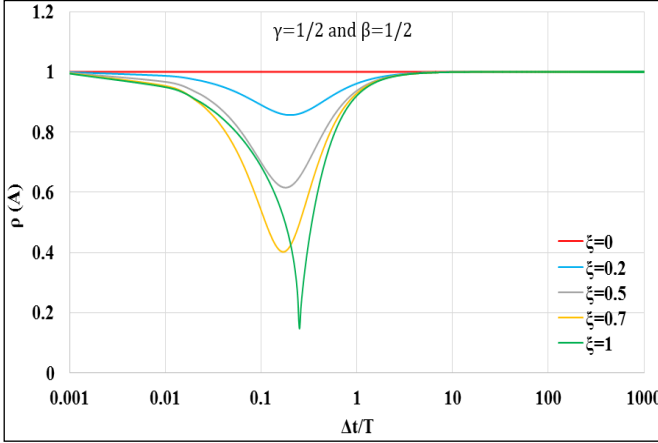


Fig. 8. Spectral radii of Newmark method: damped $\gamma = 1/2$ and $\beta = 1/2$

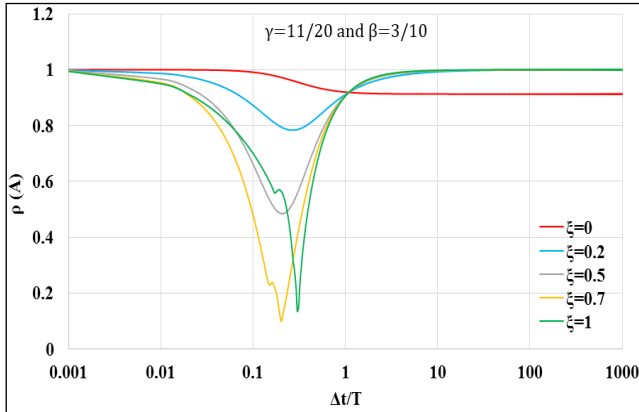


Fig. 9. Spectral radii of Newmark method: damped $\gamma = 11/20$, $\beta = 3/10$

B. Accuracy Analysis

Along with the stability, the accuracy of the solution is very important. The choice of an integration scheme is governed by the cost of the solution which in turn depends on the number of steps required in the integration. The direct integration of the equilibrium equations in Eq. (1) is equivalent to integrating simultaneously all n decoupled equations of the form of Eq. (7).

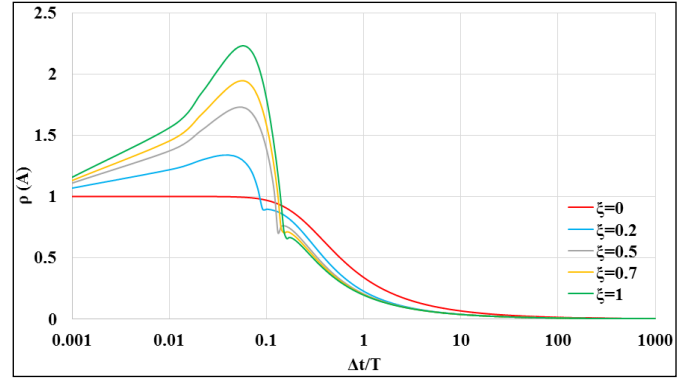


Fig. 10. Spectral radii of Houbolt method for damped cases

Therefore, accuracy of solution of (1) can be studied by assessing the accuracy obtained in the solution of (7) as a function of $\Delta t/T$. The accuracy analysis is explained on a simple initial value problem defined by

$$\left. \begin{aligned} \ddot{x} + \omega^2 x &= 0 \\ {}^0x &= 1.0; {}^0\dot{x} = 0.0; {}^0\ddot{x} = -\omega^2 \end{aligned} \right\} \quad (10)$$

The exact solution for Eq. (10) is given as $x = \cos \omega t$. The Newmark and Wilson θ methods can be directly used with the initial conditions given in Eq. (10). The numerical solution obtained using these integration schemes is shown in Fig. 11 and Fig. 12 respectively. The natural period of the system, T is fixed as 10 seconds and the time step used in the integration scheme is varied. The results obtained using Newmark scheme shows increase in the time period with increase in the step size (or $\Delta t/T$). Also, a very slight decrease in the amplitude is also observed. However both the period elongation and amplitude decay are significant when Wilson integration scheme is used.

From above two figures, it is demonstrated that the errors in any integration scheme can be measured in terms of two parameters namely the period elongation and amplitude decay. Fig. 13 and Fig. 14 shows the percentage period elongations and amplitude decays in the implicit integration schemes as a function of $\Delta t/T$.

It can be seen that the numerical integration using any of the methods is accurate when $\Delta t/T$ is smaller than about 0.01. However, when this ratio is higher, various integration schemes exhibit different characteristics. It can be seen that for a given ratio $\Delta t/T$, the Wilson θ method with $\theta = 1.4$ introduces less amplitude decay and period elongation than the Houbolt method and the Newmark constant average acceleration method introduces only period elongation and no amplitude decay.

While using one of the unconditionally stable schemes, the time step Δt can be much larger (compared to central difference method) and should only be small enough that the response in all modes that significantly contribute to the total structural response is calculated accurately. In this way, stability as well as accuracy criteria are both met. The other modal responses of higher frequency are not evaluated accurately, but the errors are unimportant because the response measured in those modes is negligible and does not grow artificially.

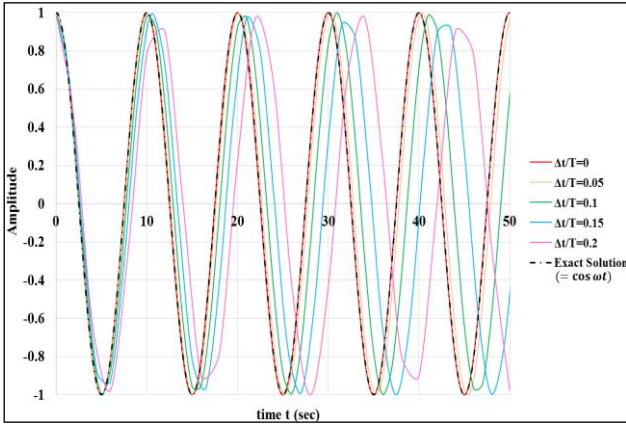


Fig. 11. Numerical solution obtained for Eq. (10) using Newmark scheme

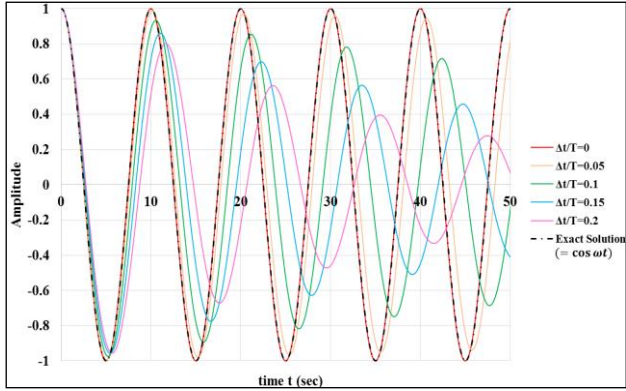


Fig. 12. Numerical solution obtained for (10) using Wilson scheme, $\theta=1.40$

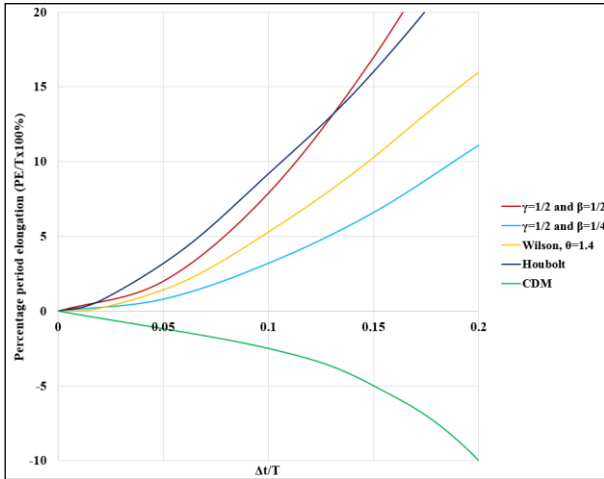


Fig. 13. Percentage period elongation for various integration schemes

VI. DIRECT INTEGRATION VERSUS MODE SUPERPOSITION

A dynamic equation represented in Eq. (1) can be solved either using direct integration methods or mode superposition method. Both of these methods use an integration scheme in which the high frequency response is filtered out of the solution. The direct integration method is equivalent to a mode superposition analysis in which all eigenvalues and vectors have been calculated and uncoupled equations in Eq. (7) are integrated with a common time step Δt .

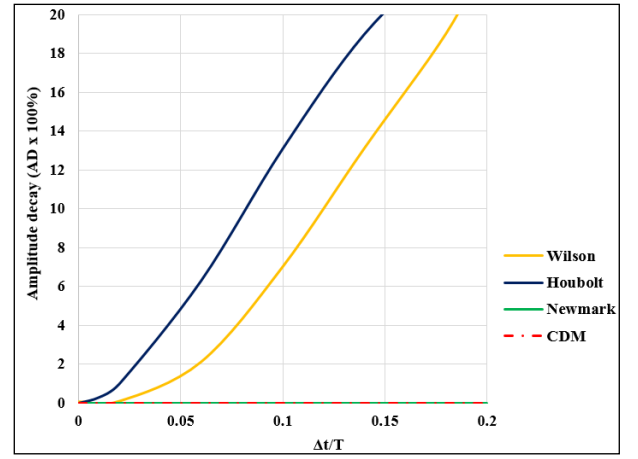


Fig. 14. Percentage amplitude decay for various integration schemes

For this method, the integration is accurate for those modes for which $\Delta t/T$ is small, but the response in the modes for which $\Delta t/T$ is large is eliminated by the artificial damping. Therefore, the direct integration is quite equivalent to a mode superposition analysis in which only the lowest modes of the system are considered. The number of modes to be included in the analysis depends on the time step Δt and the distribution of the periods. It is noted that the direct integration method is the most effective when all important periods of the system are clustered together, when time step that is based on the smallest natural period is chosen. For system with natural periods far apart, it is recommended to use mode superposition method. In this case, a separate suitable time step can be chosen for each of the n uncoupled equations. The number of modes to be considered in mode superposition method depends on the load distribution and frequency content of the loading.

VII. DISCUSSION AND CONCLUSIONS

A comprehensive survey of various methods used to solve a dynamic problem is presented in this article. The methods used to solve an inertial dynamic problem, namely, the direct integration methods, mode-superposition method and the response spectrum method are reviewed. The first two methods along with their advantages and disadvantages are discussed in this paper. Both methods use an integration scheme for solving differential equations. The stability and accuracy of some of these integration schemes is discussed.

The direct integration methods are the most general solution methods for dynamic analysis and equilibrium equations are solved using a step-by-step procedure. Mode superposition method is another powerful tool to solve dynamic problems by reduced computational effort compared to direct integration methods. The reduction in the computational effort/ cost is due to the transformation of equilibrium equations from the global coordinate system to modal coordinate system. Using this transformation, uncoupled equations of equilibrium are obtained which can be easily solved similar to a single DOF system.

The stability and accuracy of these integration schemes are also discussed. The most important step in any scheme is the selection of the time step. Too small time step can lead to huge

computation costs whereas a too large time step causes inaccuracies. It is therefore important to choose the correct time step in order to ensure stability, accuracy as well as economy of a solution. The effect of damping on the stability of an integration scheme is also studied in detail. It is concluded that most of the implicit integration schemes retain their unconditional stability criteria when damping is present.

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