

Numerical Method for Three-Dimensional Heat Conduction in Cylindrical and Spherical Coordinates

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Abstract. According to the differential equations of heat conduction on cylindrical and spherical coordinate system, numerical solution of the discrete formula on cylindrical and spherical coordinate system with high accuracy were derived. Compared with the analytical solution, this discrete formula was verified with a high degree of accuracy. To make the complex dispersion coefficient of diffusion term θ more concrete in spherical coordinates, this paper derived the discretion coefficient of diffusion term θ by the first mean value theorem of integral. The accurate schemes provide a good reference for researchers whose work in solving the equation of heat conduction of three-dimensional cylindrical coordinates and spherical coordinates, and it will provide accurate numerical schemes and the theoretical basis for solving practical engineering problems.

Keywords: Differential equation of heat conduction; Spherical coordinate; Numerical heat transfer; the first mean value theorem for integral.

1. Introduction

Numerical Heat Transfer has been widely used to solve practical complex heat transfer calculations [1-3]. The important thing to solve the problem was how to discrete the thermal conductivity differential equations. Numerical Heat Transfer in two-dimensional cylindrical coordinates and polar coordinates equation of heat conduction were applied widely. When referring to the discretion schemes of three-dimensional cylindrical and spherical coordinates, now there is not a relatively discrete format for it [4, 5].

At the present, when it comes to the questions of cylinder and sphere in numerical heat transfer, it is just simplified into the radial or two-dimensional polar coordinates, which caused a lot of inconvenience in computation, promotion and application. Therefore, it is very important for us to solve to the three-dimensional cylinder coordinates thermal conductivity partial differential equations and thermal conductivity of the sphere of partial differential equations. This paper derived high precision cylindrical coordinates and spherical coordinate's equation of heat conduction discrete schemes and compared with one-dimensional analytical solution to verify the accuracy of the numerical solution.

2. Problem description

The partial differential equations for heat conduction was established in the Cartesian coordinate system and the energy principle and Fourier's [6] law was used. Then we will have thermal conductivity of partial differential Eq. (1) as follows:

$$\frac{\partial}{\partial \tau}(\rho c T) = \lambda \frac{\partial^2 T}{\partial x^2} + \lambda \frac{\partial^2 T}{\partial y^2} + \lambda \frac{\partial^2 T}{\partial z^2} + S \quad (1)$$

Where λ is thermal conductivity, c is the heat capacity of the thermal conductivity, S is inner heat source.

The cylindrical coordinate system Eq.(2) and the spherical coordinate system Eq.(3) of the differential equations[6] are described as follows:

$$\frac{\partial}{\partial \tau}(\rho c T) = \frac{1}{r} \frac{\partial}{\partial r} \left(\lambda r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left(\lambda \frac{\partial T}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) + S, \quad 0 \leq \varphi \leq 2\pi \quad (2)$$

$$\frac{\partial}{\partial \tau}(\rho c T) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \left(\lambda \frac{\partial T}{\partial \varphi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\lambda \sin \theta \frac{\partial T}{\partial \theta} \right) + S \quad (3)$$

$$0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$$

In general, there are two methods to discrete the equation of heat conduction, that is, Taylor series expansion method and finite volume method. In order to ensure the physical significance [7, 8], this paper will derive the thermal conductivity of partial differential equations by finite volume method.

3. Literature References

First, multiply r on both sides of the Eq.(2), then the integral equation in control volume and unsteady time items are shown in Figure1. The Eq.(2) changed into Eq.(4) as follows:

$$\int_t^{t+\Delta t} \iiint_V r \frac{\partial}{\partial \tau} (\rho c T) dr d\varphi dz = \int_t^{t+\Delta t} \iiint_V \left[\frac{\partial}{\partial r} \left(\lambda r \frac{\partial T}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\lambda \frac{\partial T}{\partial \theta} \right) + r \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) \right] dr d\varphi dz \quad (4)$$

$$+ \int_t^{t+\Delta t} \iiint_V r S dr d\varphi dz$$

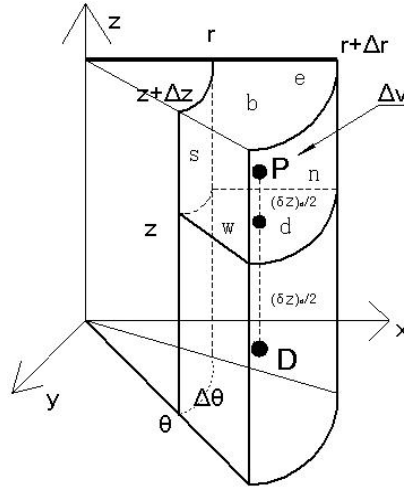


Fig.1 Cylindrical coordinates control volume

3.1 The integral of the unsteady terms

After integrating we can gain the unsteady term in the following forms:

$$\int_t^{t+\Delta t} \iiint_V r \frac{\partial T}{\partial \tau} dr d\varphi dz = \frac{(r_n + r_s) \Delta r}{2} \Delta \varphi \Delta z (T_P - T_P^0) \quad (5)$$

3.2 The integral of the diffusion terms r, φ , z

After integrating the diffusion terms into the following forms:

$$\int_t^{t+\Delta t} \iiint_V \frac{\partial}{\partial r} \left(\lambda r \frac{\partial T}{\partial r} \right) dr d\varphi dz dt = \Delta \varphi \Delta z \Delta t \int_s^n \frac{\partial}{\partial r} \left(\lambda r \frac{\partial T}{\partial r} \right) dr = \Delta \varphi \Delta z \Delta t \left[\lambda_n r_n \frac{T_N - T_P}{(\delta r)_n} - \lambda_s r_s \frac{T_P - T_S}{(\delta r)_s} \right] \quad (6)$$

$$\int_t^{t+\Delta t} \iiint_V \frac{1}{r} \frac{\partial}{\partial \theta} \left(\lambda \frac{\partial T}{\partial \theta} \right) dr d\varphi dz dt = \ln \left(\frac{r_n}{r_s} \right) \Delta z \Delta t \int_w^e \frac{\partial}{\partial \theta} \left(\lambda \frac{\partial T}{\partial \theta} \right) d\theta = \ln \left(\frac{r_n}{r_s} \right) \Delta z \Delta t \left[\lambda_e \frac{T_E - T_P}{(\delta \theta)_e} - \lambda_w \frac{T_P - T_W}{(\delta \theta)_w} \right] \quad (7)$$

$$\int_t^{t+\Delta t} \iiint_V r \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) dr d\varphi dz dt = \frac{(r_n + r_s) \Delta r}{2} \Delta \varphi \Delta t \int_d^b \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) dz \quad (8)$$

$$= \frac{(r_n + r_s) \Delta r}{2} \Delta \varphi \Delta t \left[\lambda_b \frac{T_B - T_P}{(\delta z)_b} - \lambda_d \frac{T_P - T_D}{(\delta z)_d} \right]$$

3.3 The integral of the source term

$$\int_t^{t+\Delta t} \iiint_V r S dr d\varphi dz dt = \frac{(r_n + r_s) \Delta r}{2} \Delta \varphi \Delta z \Delta t (S_C + S_P T_P) \quad (9)$$

Here, the discrete equations will be solved by the techniques for linear equation, it is sufficient to express the average value S as:

$$S = S_C + S_P T_P \quad (10)$$

where, S_C stands for the constant part of S , S_P is the coefficient of T_P .

3.4 The discrete Equation for cylinder

With the linearized source expression, the discrete equation would changed into the Eq.(4) as having the form:

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + a_B T_B + a_D T_D + b \quad (11)$$

Where:

$$\begin{aligned} a_E &= \frac{\ln(\frac{r_n}{r_s})\Delta z}{(\delta\phi)_e/\lambda_e}, \quad a_W = \frac{\ln(\frac{r_n}{r_s})\Delta z}{(\delta\phi)_w/\lambda_w}, \quad a_N = \frac{r_n\Delta\phi\Delta z}{(\delta r)_n/\lambda_n}, \quad a_S = \frac{r_s\Delta\phi\Delta z}{(\delta r)_s/\lambda_s}, \\ a_B &= \frac{(r_n + r_s)\Delta r\Delta}{2(\delta z)_b/\lambda_b}, \quad a_D = \frac{(r_n + r_s)\Delta r\Delta}{2(\delta z)_d/\lambda_d}, \quad b = S_C\Delta V + a_P^0 T_P^0 \\ a_P &= a_E + a_W + a_N + a_S + a_B + a_D + a_P^0 - S_P\Delta V, \quad \Delta V = \frac{(r_n + r_s)\Delta r}{2} \Delta \Delta z, \\ a_P^0 &= \frac{(\rho c)_P \Delta V}{\Delta t}. \end{aligned}$$

At this point, it is interesting to examine the physical significance of the various coefficients in the discrete equation. The neighbor coefficients $a_E, a_W, a_N, a_S, a_B, a_D$ represent the conductance between the point P and the corresponding neighbors. The term $a_P^0 T_P^0$ is the internal energy contained in the control volume at time t . The constant term b consists of this internal energy and the rate of heat generation in the control volume resulting from S_C . The ΔV is the volume of the control volume. The center-point coefficient a_P is the sum of all neighbor coefficients and contains a contribution from the linearized source term.

4. The discrete control equation of spherical coordinates

At this stage, multiply $r^2 \sin^2 \theta$ on both sides of the Eq.(2), then the integral equation that was in control volume and unsteady time items points are shown in Figure 2. The new equation is as follows:

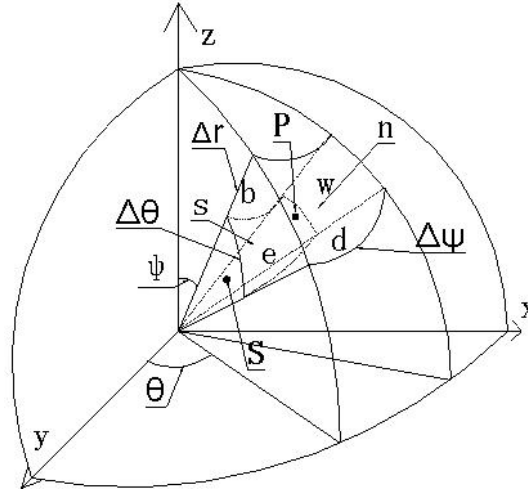


Fig.2 Spherical coordinate control volume

$$\begin{aligned} \int_t^{t+\Delta t} \iiint_V r^2 \sin^2 \theta \frac{\partial}{\partial \tau} (\rho c T) dt dr d\theta d\psi &= \int_t^{t+\Delta t} \iiint_V \left[\sin^2 \theta \frac{\partial}{\partial r} (\lambda r^2 \frac{\partial T}{\partial r}) + \frac{\partial}{\partial \theta} (\lambda \frac{\partial T}{\partial \theta}) \right. \\ &\quad \left. + \sin \theta \frac{\partial}{\partial \theta} (\lambda \sin \theta \frac{\partial T}{\partial \theta}) \right] dt dr d\theta d\psi + \int_t^{t+\Delta t} \iiint_V S \cdot r^2 \sin^2 \theta dt dr d\theta d\psi \end{aligned} \quad (12)$$

4.1 The integral of the unsteady terms

After integrating the unsteady term into the following forms:

$$\int_t^{t+\Delta t} \iiint_V r^2 \sin^2 \theta \frac{\partial}{\partial \tau} (\rho c T) dt dr d\theta d\psi = \frac{(r_n^3 - r_s^3)}{6} (\Delta\theta + \sin \theta_w \cos \theta_w - \sin \theta_e \cos \theta_e) \Delta\phi (T_P - T_P^0) \quad (13)$$

4.2 The integral of the diffusion terms r, φ

After integrating the diffusion term r, φ becomes the following forms:

$$\int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d \sin^2 \theta \frac{\partial}{\partial r} (\lambda r^2 \frac{\partial T}{\partial r}) dtdrd\theta d\varphi = \frac{1}{2} (\Delta\theta + \sin \theta_w \cos \theta_w - \sin \theta_e \cos \theta_e) \Delta\varphi \Delta t \left[r_n^2 \frac{T_N - T_P}{(\delta r)_n} - r_s^2 \frac{T_P - T_S}{(\delta r)_s} \right] \quad (14)$$

$$\int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d \frac{\partial}{\partial \theta} (\lambda \frac{\partial T}{\partial \theta}) dtdrd\theta d\varphi = \left[\frac{T_B - T_P}{(\delta \theta)_b} - \frac{T_P - T_D}{(\delta \theta)_d} \right] \Delta\theta \Delta r \Delta t \quad (15)$$

4.3 Application integral mean value theorem derive the diffusion term θ

In the derivation of the integral term, this paper use the integral mean value theorem which will derive the diffusion term θ scientifically and reasonably.

Obviously, the heat flux that leaves one control volume through a particular face must be identical to the flux that enters the next control volume through the same face. Otherwise, the overall balance would not be satisfied. It is the fact that the energy across the interface θ_e to θ_w continuously, so the heat conduction equation can be thought as a continuity equation. Because of the diffusion term $\theta \in [0, \pi]$, it happens for $\forall \theta \in [0, \pi]$, and we have $\sin \theta \geq 0$. Based on the above two conditions of continuity and $\sin \theta \geq 0$, which satisfied the integral mean value theorem[9], the diffusion term θ and Eq.(3) are changed as the following $\exists \xi \in [w, e]$ shows, so that the θ conduction term denotes:

$$\int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d \sin \theta \frac{\partial}{\partial \theta} (\lambda \sin \theta \frac{\partial T}{\partial \theta}) dtdrd\theta d\varphi = \sin \theta_\xi \int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d \frac{\partial}{\partial \theta} (\lambda \sin \theta \frac{\partial T}{\partial \theta}) dtdrd\theta d\varphi \quad (16)$$

It can be seen that when $\Delta\theta$ is much smaller, the value $\theta_\xi \approx \theta_p$.

$$\begin{aligned} \sin \theta_\xi \int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d \frac{\partial}{\partial \theta} (\lambda \sin \theta \frac{\partial T}{\partial \theta}) dtdrd\theta d\varphi &\approx \sin \theta_p \int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d \frac{\partial}{\partial \theta} (\lambda \sin \theta \frac{\partial T}{\partial \theta}) dtdrd\theta d\varphi \\ &= \sin \theta_p \left[\sin \theta_e \frac{T_E - T_P}{(\delta \theta)_e} - \sin \theta_w \frac{T_P - T_W}{(\delta \theta)_w} \right] \Delta \Delta r \Delta t \end{aligned} \quad (17)$$

4.4 The integral of the source term

After integrating the source term S becomes the following forms:

$$\int_t^{t+\Delta t} \int_s^{n_e} \int_w^b \int_d^d r^2 \sin^2 \theta \cdot S dtdrd\theta d\varphi = \frac{(r_n^3 - r_s^3)}{6} (\Delta\theta + \sin \theta_w \cos \theta_w - \sin \theta_e \cos \theta_e) \Delta\varphi \Delta t (S_C + S_P T_P) \quad (18)$$

4.5 The discrete Equation for sphere

The discrete equation can easily be seen as follows:

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + a_B T_B + a_D T_D + b \quad (19)$$

Where

$$a_E = \frac{\Delta\varphi \Delta r \sin \theta_p \sin \theta_e}{(\delta \theta)_e / \lambda}, \quad a_W = \frac{\Delta\varphi \Delta r \sin \theta_p \sin \theta_w}{(\delta \theta)_w / \lambda},$$

$$a_N = \frac{(\Delta\theta + \sin \theta_w \cos \theta_w - \sin \theta_e \cos \theta_e) \Delta \cdot r_n^2}{2(\delta r)_n / \lambda},$$

$$a_S = \frac{(\Delta\theta + \sin \theta_w \cos \theta_w - \sin \theta_e \cos \theta_e) \Delta\varphi \cdot r_s^2}{2(\delta r)_s / \lambda},$$

$$a_B = \frac{\Delta\theta \Delta r}{(\delta \theta)_b / \lambda_b}, \quad a_D = \frac{\Delta\theta \Delta r}{(\delta \theta)_d / \lambda_d}, \quad b = S_C \Delta V + a_P^0 T_P^0,$$

$$a_P = a_E + a_W + a_N + a_S + a_B + a_D + a_P^0 - S_P \Delta V,$$

$$\Delta V = \frac{(r_n^3 - r_s^3)}{6} (\Delta\theta + \sin \theta_w \cos \theta_w - \sin \theta_e \cos \theta_e),$$

$$a_P^0 = \frac{(\rho c)_P \Delta V}{\Delta t}, \quad 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi.$$

With the different express, the terms of $a_E, a_W, a_N, a_S, a_B, a_D, P, a_P^0 T_P^0$ etc. are the same as the part 3 shown.

5. Numerical value comparison

5.1 Application of discrete equations and grid independent analysis

To illustrate the workings of the discrete equations this paper gives a detailed example of cylindrical heat conduction. The inner and outer diameter of the cylinder were R_1 and R_2 . The temperature of the inside and outside of the cylinder was T_1 and T_2 and the radial heat conduction analytical solution is as follows:

$$T = T_1 + (T_2 - T_1) \frac{\ln(R/R_1)}{\ln(R_2/R_1)} \quad (20)$$

We consider it as a steady and three-dimensional cylinder, and the heat constant cylindrical boundary conditions (the problem data) are as follows:

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r} (\lambda r \frac{\partial T}{\partial r}) + \frac{1}{r^2} \frac{\partial}{\partial \theta} (\lambda \frac{\partial T}{\partial \theta}) + \frac{\partial}{\partial z} (\lambda \frac{\partial T}{\partial z}) = 0 \\ R_1 = 1, T_1 = 1 \\ R_2 = 2, T_2 = 1000 \\ 0 \leq \theta \leq 2\pi \end{cases} \quad (21)$$

The numerical computed in this paper as well as the analytical solutions are compared in Figure 3. Given the coarseness of the triangle that numerical solution gives reasonable agreement with the analytical solution, the error between the numerical and analytical solution was $8.86 \times 10^{-3}\%$.

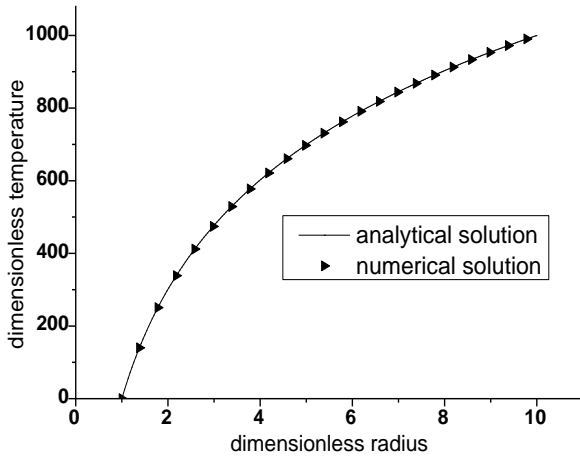


Fig.3 Analytical solution and numerical solution

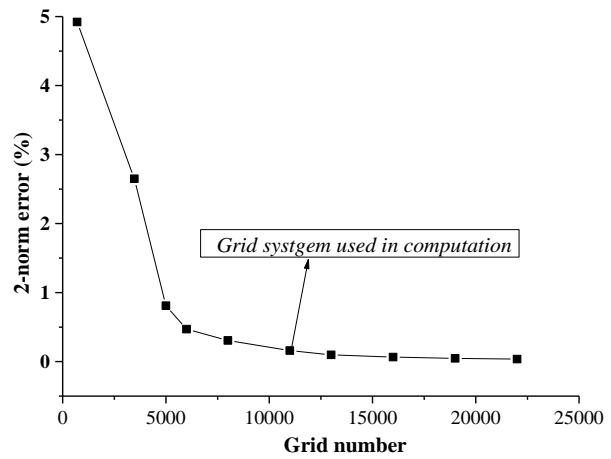


Fig.4 Grids number and error relations

To quantify the accuracy of the reconstruction formulas between numerical solution and analytical solution we define the relative error as:

$$\text{error} = \frac{\|T - T_{num}\|_2}{\|T\|_2} \times 100\% \quad (22)$$

Where the error means the error between the solutions of the governing equations, and T refers to the solutions by means of the analytical solution; T_{num} refers to the solutions by means of the numerical solution.

The error means L_2 -norm [10]: $\|T\|_2 = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$

Based on the error obtained from grid-independent solution [11], the relation between the equation error and grid number is shown in Figure 4. When the grid number reaching 5000, the calculation error control is less than 1%; with the encryption of the grid, the calculation error is reduced gradually. The analytical solution error is $8.86 \times 10^{-3}\%$, the choice of computational grid number is $37 \times 20 \times 16 = 11840$.

5.2 The validation of the thermal equation of the sphere

To illustrate the workings of the sphere equations, this paper gives a detailed example of sphere heat conduction. The inner and outer diameter were R_1 and R_2 . The temperature of the inside and outside of the sphere was T_1 and T_2 .

We consider it as a steady and three-dimensional sphere, and the heat constant boundary conditions (the problem data) are as follows:

$$\begin{cases} \frac{1}{r^2} \frac{\partial}{\partial r} (\lambda r^2 \frac{\partial T}{\partial r}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} (\lambda \frac{\partial T}{\partial \theta}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\lambda \sin \theta \frac{\partial T}{\partial \theta}) = 0 \\ R_1 = 1, T_1 = 1 \\ R_2 = 2, T_2 = 100 \\ 0 \leq r \leq 2\pi, 0 \leq \theta \leq \pi \end{cases} \quad (23)$$

The numerical computed in this paper as well as analytical solutions are compared in Figure 5. Given the coarseness of the triangle that numerical solution gives reasonable agreement with the analytical solution, the error between the numerical and analytical solution was 0.53%.

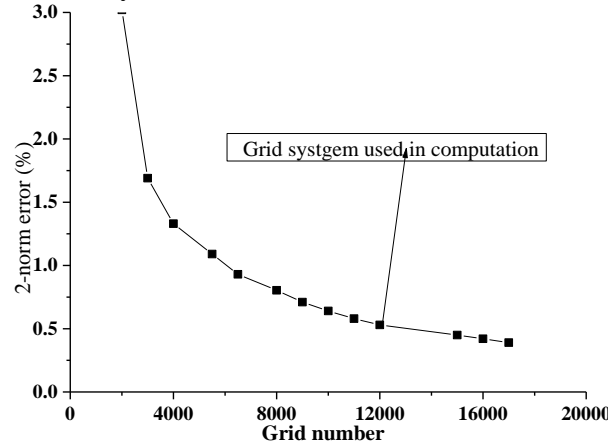
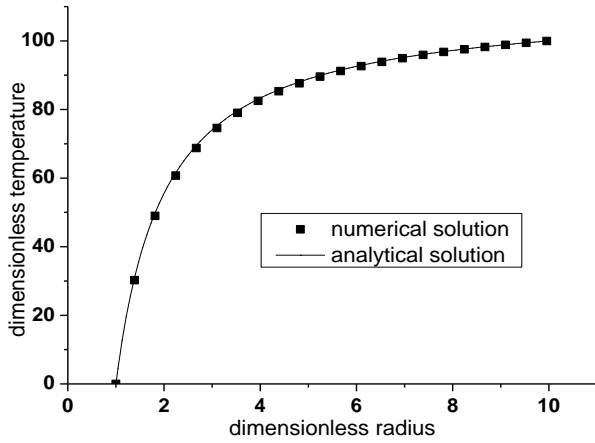


Fig.5 Analytical solution and numerical solution Fig.6 Grids number and error relations

Based on the error obtained from grid-independent solution, the relation among the equation error and grid number is shown in Figure 6. When the grid number reaching 12000, the calculation error control is less than 1%; with the encryption of the grid, the calculation error is reduced gradually. The analytical solution error is 0.5%, the choice of computational grid number is $47 \times 20 \times 12 = 11280$.

6. Conclusion

According to the three-dimensional equation of heat conduction of the cylinder and sphere, this paper derived the high-accuracy computational discrete scheme with the finite volume method. In the process of the discrete sphere, this paper uses the integral mean value theorem reasonably, which has derived the complex diffusion term more strictly. The new three-dimensional cylindrical coordinates and spherical coordinates provide numerical calculation discrete scheme, which will provide a good reference for the researchers. The application of numerical calculation discrete scheme is written in FORTRAN computer language, which is of stable operation and simple application. This paper have contrasted and analyzed the error, which was controlled within 0.5%. All the work has provided a high-precision numerical calculation discrete scheme for teaching and scientific research.

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