A generalized asymmetry model for square contingency tables with ordered categories

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For square contingency tables with ordered categories, the present paper proposes an asymmetry model with m-additional parameters, which indicates (1) the generalized marginal homogeneity and (2) the structure of quasi-symmetry for cumulative probabilities. The proposed model includes a modified palindromic symmetry model by Iki, Oda and Tomizawa [7]. Also the present paper gives the decomposition of the symmetry model using the proposed model. Examples are given.

Keywords: Cumulative probability; Decomposition; Ordered category; Palindromic symmetry; Square contingency tables.

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1. Introduction

Consider the square contingency tables with same row and column classifications. For example, consider the data in Table 1. The data in Table 1, taken from Goodman [5], are constructed from occupational status of 2391 farther-son pairs in Denmark. The row is the father's status category and column is the son's status category. The categories are ordered from (1) to (5) (high to low). In Table 1, many observations concentrate on the main diagonal cells. Therefore for these data, the model of independence does not hold. Namely, the father's status is strongly associated with the son's status. Instead, we are interested in whether or not a father's status is symmetric to his son's status.

Table 1. Occupational status for Danish father-son pairs; from Goodman [5]. (Upper and lower parenthesized values are the maximum likelihood estimates of expected frequencies under the PS(3) and AS(3) models, respectively.)

Father's status	Son's status					Total
	(1)	(2)	(3)	(4)	(5)	
(1)	18	17	16	4	2	57
	(18.00)	(18.48)	(14.60)	(4.41)	(3.57)	
	(18.00)	(18.21)	(14.07)	(4.09)	(4.05)	
(2)	24	105	109	59	21	318
	(22.75)	(105.00)	(105.59)	(59.24)	(17.29)	
	(22.98)	(105.00)	(104.83)	(56.46)	(18.62)	
(3)	23	84	289	217	95	708
	(24.69)	(87.14)	(289.00)	(220.85)	(98.02)	
	(25.44)	(87.79)	(289.00)	(221.45)	(95.50)	
(4)	8	49	175	348	198	778
	(7.44)	(48.56)	(171.11)	(348.00)	(195.96)	
	(7.90)	(51.37)	(170.60)	(348.00)	(196.33)	
(5)	6	8	69	201	246	530
	(5.01)	(10.98)	(65.98)	(203.34)	(246.00)	
	(4.20)	(9.70)	(68.51)	(202.91)	(246.00)	
Total	79	263	658	829	562	2391

Note: Status (1) is High professionals, (2) White-collar employees of higher education, (3) White-collar employees of less high education, (4) Upper working class, and (5) Unskilled workers.

For an $r \times r$ square contingency table with the same row and column ordinal classifications, let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). The symmetry (S) model is defined by

$$p_{ij} = p_{ji} \quad (i < j);$$

see Bowker [3], Bishop, Fienberg and Holland [2, p. 282] and Agresti [1, Chap. 11]. For the data in Table 1, the S model indicates that the probability that a father's status is i and his son's status is j, is equal to the probability that the father's status is j and his son's status is i. Namely, this model describes a structure of symmetry of the probabilities $\{p_{ij}\}$ with respect to the main diagonal of the table. Let X_1 and X_2 denote the row and column variables, respectively, and let

$$G_{ij} = \sum_{s=1}^{i} \sum_{t=j}^{r} p_{st} = P(X_1 \leq i, X_2 \geq j) \quad (i < j),$$

and

$$G_{ji} = \sum_{s=j}^{r} \sum_{t=1}^{i} p_{st} = P(X_1 \geq j, X_2 \leq i) \quad (i < j).$$

The S model may be expressed as

$$G_{ij} = G_{ji} \quad (i < j).$$

For the data in Table 1, this indicates that the cumulative probability that a father's status is i or below and his son's status is j or above, is equal to the cumulative probability that the father's status is j or above and his son's status is i or below. The marginal homogeneity (MH) model (Stuart [12]) is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, \ldots, r),$$

where $p_{i\cdot} = \sum_{t=1}^{r} p_{it}$ and $p_{\cdot i} = \sum_{s=1}^{r} p_{si}$. The MH model may be expressed as

$$G_{i,i+1} = G_{i+1,i} \quad (i = 1, \ldots, r-1).$$

Tahata and Tomizawa [13] considered the *m*-additional parameters marginal homogeneity (MH(*m*)) model which is a generalization of the MH model. For a given m $(m = 1, \ldots, r-1)$, the MH(*m*) model is defined by

$$G_{i,i+1} = \Delta_i^{(m)} G_{i+1,i} \quad (i = 1, \ldots, r-1),$$

where

$$\Delta_i^{(m)} = \prod_{k=0}^{m-1} \Delta_k^{i^k}.$$

A special case of the MH(*m*) model obtained by putting $\{\Delta_k = 1\}$ is the MH model. When $m = 1, 2$ and $r - 1$, the MH(*m*) models are the extended marginal homogeneity model in Tomizawa [16], the generalized marginal homogeneity model in Tomizawa [17] and saturated model, respectively.

Caussinus [4] proposed the quasi-symmetry (QS) model for cell probabilities, defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji}$ (see also Goodman [6]; Kateri and Agresti [8]). Denote the odds ratio for rows i and s $(> i)$, and columns j and t $(> j)$ by $\theta_{is;jt}$. Thus $\theta_{is;jt} = (p_{ij}p_{st})/(p_{it}p_{sj})$. For the data in Table 1, the $\theta_{is;jt}$ $(= (p_{ij}/p_{sj})/(p_{it}/p_{st}))$ indicates that the ratio of the odds that the father's status is i instead of s when the son's status is j to the odds that the father's status is i instead of s when the son's status is t. Using odds ratios, the QS model is also expressed as

$$\theta_{is;jt} = \theta_{jt;is} \quad (i < s; j < t).$$

Therefore this model has characterization in terms of symmetry of odds ratios (though the S model has characterization in terms of symmetry of cell probabilities). The QS model may be expressed as

$$p_{ij} = \xi_i \phi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\phi_{ij} = \phi_{ji}$. Caussinus [4] also gave the theorem that the S model holds if and only if the QS and MH models hold. Miyamoto, Ohtsuka and Tomizawa [10] proposed the cumulative quasi-symmetry (CQS) model for cumulative probabilities $\{G_{ij}\}$, defined by

$$G_{ij} = \gamma_i \Psi_{ij} \quad (i \neq j), \quad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. This model may be expressed as

$$\frac{G_{ij}}{G_{ji}} = \frac{\gamma_i}{\gamma_j} \quad (i < j);$$

see also Tahata and Tomizawa [14].

McCullagh [9] considered the palindromic symmetry (PS) model, defined by

$$G_{ij} = \begin{cases} \Delta \alpha_i \Psi_{ij} & (i < j), \\ \alpha_{i-1} \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$ and $\alpha_1 = 1$ without loss of generality. A special case of the PS model obtained by putting $\Delta = 1$ and $\{\alpha_i = 1\}$ is the S model. Note that the PS model with Δ replaced by Δ_i is the generalized palindromic symmetry model (McCullagh [9]). The PS model is also expressed as

$$\frac{G_{ij}}{G_{ji}} = \Delta \frac{\alpha_i}{\alpha_{j-1}} \quad (i < j).$$

Saigusa, Tahata and Tomizawa [11] considered the *m*-additional parameters palindromic symmetry (PS(*m*)) model. For a given m ($m = 1, \dots, r-1$), the PS(*m*) model is defined by

$$\frac{G_{ij}}{G_{ji}} = \Delta_i^{(m)} \frac{\alpha_i}{\alpha_{j-1}} \quad (i < j),$$

where

$$\Delta_i^{(m)} = \prod_{k=0}^{m-1} \Delta_k^{i^k}.$$

When $m = 1$ (with $\Delta_i^{(1)} = \Delta_0$), the PS(1) model is identical to the PS model, and when $m = r-1$, the PS($r-1$) model is the generalized palindromic symmetry model. Iki, Oda and Tomizawa [7] considered the modified palindromic symmetry (MPS) model. The MPS model is defined by

$$G_{ij} = \begin{cases} \beta_i \Psi_{ij} & (i < j; j \neq i+1), \\ \Gamma \beta_i \Psi_{ij} & (i < r; j = i+1), \\ \beta_{i-1} \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$ and $\beta_1 = 1$ without loss of generality. A special case of this model obtained by putting $\Gamma = 1$ and $\{\beta_i = 1\}$ is the S model. The MPS model is also expressed as

$$\frac{G_{i,i+1}}{G_{i+1,i}} = \Gamma \quad (i = 1, \dots, r-1),$$

and

$$\frac{G_{ij}}{G_{ji}} = \frac{\beta_i}{\beta_{j-1}} \quad (i < j; j \neq i+1).$$

The PS(*m*) model is the PS model with the *m*-additional parameters $\Delta_i^{(m)}$ instead of one parameter Δ. So, we are interested in a model which is the MPS model with the *m*-additional parameters instead of one parameter Γ.

In the present paper, Section 2 proposes a new model which is the MPS model with the *m*-additional parameters. Section 3 gives the decomposition of the S model using the proposed model. Section 4 describes the goodness-of-fit test and Section 5 gives examples. Section 6 provides some concluding remarks.

2. An asymmetry model with *m*-additional parameters

Consider a model defined by, for a given m $(m = 1, \ldots, r-1)$,

$$G_{ij} = \begin{cases} \beta_i \Psi_{ij} & (i < j; j \neq i+1), \\ \Gamma_i^{(m)} \beta_i \Psi_{ij} & (i < r; j = i+1), \\ \beta_{i-1} \Psi_{ij} & (i > j), \end{cases} \quad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$, $\beta_1 = 1$ without loss of generality and

$$\Gamma_i^{(m)} = \prod_{k=0}^{m-1} \Gamma_k^{i^k}.$$

We shall refer to this model as the *m*-additional parameters asymmetry (AS(*m*)) model. Especially, the AS(1) model is identical to the MPS model. The AS(*m*) model is also expressed as

$$\frac{G_{i,i+1}}{G_{i+1,i}} = \Gamma_i^{(m)} \quad (i = 1, \ldots, r-1), \tag{2.1}$$

and

$$\frac{G_{ij}}{G_{ji}} = \frac{\beta_i}{\beta_{j-1}} \quad (i < j; j \neq i+1). \tag{2.2}$$

From the equation (2.1), the parameter $\Gamma_i^{(m)}$ indicates that the ratio of the cumulative probability that an observation will fall in row category i or below and column category $i+1$ or above (i.e., $G_{i,i+1}$) to the cumulative probability that the observation falls in row category $i+1$ or above and column category i or below (i.e., $G_{i+1,i}$). In addition, the equation (2.1) is identical to the MH(*m*) model. Namely, the log-odds of cumulative probabilities are expressed as the polynomial function of category indicator i $(i = 1, \ldots, r-1)$, i.e.,

$$\log\left(\frac{G_{i,i+1}}{G_{i+1,i}}\right) = \log \Gamma_0 + i \log \Gamma_1 + \cdots + i^{m-1} \log \Gamma_{m-1}.$$

The equation (2.2) states that the cumulative probability that an observation will fall in row category i or below and column category j $(i < j; j \neq i+1)$ or above, is β_i/β_{j-1} times higher than the cumulative probability that the observation falls in row category j or above and column category i or below. From $\beta_1 = 1$, the parameters β_{j-1} indicates the ratio of G_{j1} to G_{1j} $(j = 3, \ldots, r)$. The AS(*m*) model is expressed as the equations (2.1) and (2.2), the interpretation of the parameters $\{\Psi_{ij}\}$ is not essential.

Define the odds ratio based on $\{G_{ij}\}$, $i \neq j$, by $\Theta_{is;jt} = (G_{ij}G_{st})/(G_{it}G_{sj})$ for $1 \leq i < s < j < t \leq r$. For the data in Table 1, the $\Theta_{is;jt}$ $(= (G_{ij}/G_{sj})/(G_{it}/G_{st}))$ indicates that the ratio of the odds that the father's status is i or below instead of s or below when the son's status is j or above to the odds that the father's status is i or below instead of s or below when the son's status is t or above.

The PS(m) model implies

$$\Theta_{is;jt} = \Theta_{jt;is} \quad (1 \le i < s < j < t \le r).$$

The AS(m) model also implies

$$\Theta_{is;jt} = \Theta_{jt;is} \quad (1 \le i < s < j < t \le r; j \ne s+1),$$

and

$$\Gamma_s^{(m)}\Theta_{is;jt} = \Theta_{jt;is} \quad (1 \le i < s < j < t \le r; j = s+1).$$

Therefore, PS(m) model implies the symmetry of odds ratios based on $\{G_{ij}\}$, $i \ne j$; however, the AS(m) model implies the symmetry of odds ratios with the asymmetry partially. Note that the PS(m) and AS(m) models have the structure of constant of odds $\{G_{i,i+1}/G_{i+1,i}\}$, $i = 1, \ldots, r-1$.

3. Decomposition of symmetry model

Tomizawa, Miyamoto and Ouchi [18] proposed the cumulative subsymmetry (CS) model, defined by

$$G_{i,i+2} = G_{i+2,i} \quad (i = 1, \ldots, r-2).$$

For a given positive integer s, we consider the sth moment equality (ME(s)) model defined by

$$\mathrm{E}(X_1^s) = \mathrm{E}(X_2^s),$$

where

$$\mathrm{E}(X_1^s) = \sum_{i=1}^{r} i^s p_{i\cdot}, \quad \mathrm{E}(X_2^s) = \sum_{i=1}^{r} i^s p_{\cdot i}.$$

We obtain the decomposition of the S model as follows:

Theorem 3.1. *For a given m ($m = 1, \ldots, r-1$), the S model holds if and only if all the AS(m), CS and ME(s) ($s = 1, \ldots, m$) models hold.*

Proof. If the S model holds, then for a given m ($m = 1, \ldots, r-1$), the AS(m), CS and ME(s) ($s = 1, \ldots, m$) models hold. Assuming that all the AS(m), CS and $\{$ME(s)$\}$ models hold, then we shall show that the S model holds. From the AS(m) and CS models, we see

$$\frac{G_{i,i+2}}{G_{i+2,i}} = \frac{\beta_i}{\beta_{i+1}} = 1 \quad (i = 1, \ldots, r-2).$$

Since $\beta_1 = 1$, thus we see $\{\beta_i = 1\}$. From the AS(m) model holds, we see

$$\frac{G_{i,i+1}}{G_{i+1,i}} = \Gamma_i^{(m)} \quad (i = 1, \ldots, r-1).$$

A structure of $\{G_{i,i+1}/G_{i+1,i}\}$ in the AS(m) model is identical to that of the MH(m) model. Tahata and Tomizawa [13] showed that, for a given m ($m = 1, \ldots, r-1$), the MH model holds if and only if all the MH(m) and ME(s) ($s = 1, \ldots, m$) models hold. Therefore, the model which satisfies all

constraints of $\{G_{i,i+1}/G_{i+1,i}\}$ in the AS(m) model and ME(s) ($s = 1, \ldots, m$) model is identical to the MH model. Namely,

$$\frac{G_{i,i+1}}{G_{i+1,i}} = 1 \quad (i = 1, \ldots, r-1).$$

Thus, we obtain $\Gamma_k = 1$ for $k = 0, 1, \ldots, m-1$. Therefore, the S model holds. The proof is completed.
□

4. Goodness-of-fit test

Let n_{ij} denote the observed frequency in the ith row and jth column of the $r \times r$ table ($i = 1, \ldots, r$; $j = 1, \ldots, r$) with $n = \sum\sum n_{ij}$. Assume that a multinomial distribution applies to the $r \times r$ table. The maximum likelihood estimates of expected frequencies under models could be obtained by using the Newton-Raphson method in the log-likelihood equation. We can test each model for goodness-of-fit by the likelihood ratio chi-squared statistic (denoted by G^2) with the corresponding degrees of freedom. The test statistic G^2 is defined by

$$G^2 = 2\sum_{i=1}^{r}\sum_{j=1}^{r} n_{ij} \log\left(\frac{n_{ij}}{\widehat{m}_{ij}}\right),$$

where $\widehat{m}_{ij}$ is the maximum likelihood estimate of expected frequency m_{ij} under the model.

For nested models, the conditional test statistic would be the G^2 value for the model with fewer free parameters minus the G^2 value for the model with more free parameters, which has a chi-square distribution with degrees of freedom equal to the distance between numbers of parameters in the two models.

The number of degrees of freedom for the AS(m) model is $\frac{1}{2}(r^2 - 3r + 4) - m$ for $m = 1, 2, \ldots, r-1$. This number is identical to that of the PS(m) model.

5. Examples

Example 1. Consider the data in Table 1 again. Table 2 gives the values of likelihood ratio statistic G^2 for testing the goodness-of-fit of some models. The S model fits these data poorly. Also, each of the PS(1), PS(2), AS(1), AS(2), CS and ME(s), $s = 1, 2, 3, 4$ models fits poorly, but each of the PS(3), PS(4), AS(3) and AS(4) models fits the data well.

According to the test based on the difference between the likelihood ratio chi-squared values G^2 for the AS(3) and AS(4) models at the 0.05 significance level, we obtain the AS(3) model is preferable to the AS(4) model. In the similar way, the PS(3) model is preferable to the PS(4) model.

Under the AS(3) model, the maximum likelihood estimates of parameters Γ_0, Γ_1 and Γ_2 are $\widehat{\Gamma}_0 = 0.299$, $\widehat{\Gamma}_1 = 2.614$ and $\widehat{\Gamma}_2 = 0.853$, respectively. The cumulative probability that the father's status is i or below and the son's status is $i+1$ or above, is estimated to be $\widehat{\Gamma}_0\widehat{\Gamma}_1^i\widehat{\Gamma}_2^{i^2}$ times higher than the cumulative probability that the father's status is $i+1$ or above and the son's status is i or below. When $i = 1$, the cumulative probability that the father's status is '(1)' and his son's status is '(2)' to '(5)', is estimated to be $\widehat{\Gamma}_0\widehat{\Gamma}_1\widehat{\Gamma}_2 = 0.668$ times higher than the cumulative probability that the father's status is '(2)' to '(5)' and his son's status is '(1)'. Besides, the maximum likelihood estimates of $\{\beta_i\}$ are $\widehat{\beta}_2 = 1.690$, $\widehat{\beta}_3 = 1.486$ and $\widehat{\beta}_4 = 1.036$, with $\widehat{\beta}_1 = 1$. For $i < j$ with $j - i \geq 2$, the probability that the father's status is i or below and the son's status is j or above is estimated to be $\widehat{\beta}_i/\widehat{\beta}_{j-1}$ times higher than the probability that the father's status is j or above and the son's status

Table 2. Likelihood ratio chi-squared values G^2
for models applied to the data in Table 1.

Models	Degree of freedom	G^2
S	10	24.80*
PS(1)	6	15.07*
PS(2)	5	14.42*
PS(3)	4	3.82
PS(4)	3	2.60
AS(1)	6	15.40*
AS(2)	5	14.35*
AS(3)	4	3.99
AS(4)	3	3.21
CS	3	12.07*
ME(1)	1	5.73*
ME(2)	1	6.86*
ME(3)	1	6.58*
ME(4)	1	5.80*

*means significant at the 0.05 level.

Table 3. Values of $\{\widehat{G}_{ij}/\widehat{G}_{ji}\}$, $i \neq j$, under the AS(3)
and PS(3) models applied to Table 1.

(a) Under the AS(3) model

	$j = 1$	2	3	4	5
$i = 1$	-	0.668	0.592	0.673	0.965
2	-	-	1.084	1.138	1.631
3	-	-	-	1.281	1.434
4	-	-	-	-	1.102
5	-	-	-	-	-

(b) Under the PS(3) model

	$j = 1$	2	3	4	5
$i = 1$	-	0.686	0.608	0.641	0.712
2	-	-	1.114	1.174	1.305
3	-	-	-	1.305	1.450
4	-	-	-	-	1.103
5	-	-	-	-	-

is i or below. When $i = 1$ and $j = 3$, the cumulative probability that the father's status is '(1)' and his son's status is '(3)' to '(5)', is estimated to be $\widehat{\beta}_1/\widehat{\beta}_2 = 0.592$ times higher than the cumulative probability that the father's status is '(3)' to '(5)' and his son's status is '(1)'. Under the PS(3) model, the maximum likelihood estimates of parameters Δ_0, Δ_1 and Δ_2 are $\widehat{\Delta}_0 = 0.305$, $\widehat{\Delta}_1 = 2.651$ and $\widehat{\Delta}_2 = 0.849$, respectively. Besides, the maximum likelihood estimates of $\{\alpha_i\}$ are $\widehat{\alpha}_2 = 1.127$, $\widehat{\alpha}_3 = 1.070$ and $\widehat{\alpha}_4 = 0.963$, with $\widehat{\alpha}_1 = 1$. From Table 3, we see that the values of $\{G_{ij}/G_{ji}\}$, $i \neq j$, under the AS(3) model are close to those under the PS(3) model.

Furthermore, under the AS(3) model, the odds ratio $\theta_{34;12}$ is estimated to be $\widehat{\Gamma}_2^{(3)} = 1.084$ times greater than the odds ratio $\theta_{12;34}$. Hence the ratio of the odds that the son's status is '(1)' instead of '(1)' or '(2)' when the father's status is '(3)' or '(4)' to the odds when the father's status is '(4)', is estimated to be $\widehat{\Gamma}_2^{(3)} = 1.084$ times greater than the ratio of the odds that the father's status is '(1)' instead of '(1)' or '(2)' when the son's status is '(3)' or '(4)' to the odds when the son's status is

'(4)'. Also, the odds ratio $\theta_{45;23}$ is estimated to be $\widehat{\Gamma}_3^{(3)} = 1.281$ times greater than the odds ratio $\theta_{23;45}$, and interpreted similarly.

Example 2. The data in Table 4, taken from Stuart [12], are constructed from unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946.

Table 4. Unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946; from Stuart ([12]. (The upper, middle and lower parenthesized values are the maximum likelihood estimates of expected frequencies under the AS(1), AS(2) and AS(3) models, respectively.)

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1520	266	124	66	1976
	(1520.00)	(266.46)	(129.43)	(64.22)	
	(1520.00)	(266.55)	(129.47)	(64.21)	
	(1520.00)	(266.00)	(129.49)	(64.10)	
Second (2)	234	1512	432	78	2256
	(233.52)	(1512.00)	(430.86)	(82.71)	
	(233.42)	(1512.00)	(430.84)	(82.68)	
	(234.00)	(1512.00)	(432.00)	(82.62)	
Third (3)	117	362	1772	205	2456
	(110.99)	(363.15)	(1772.00)	(205.63)	
	(110.94)	(363.17)	(1772.00)	(205.55)	
	(110.94)	(362.00)	(1772.00)	(205.00)	
Worst (4)	36	82	179	492	789
	(39.22)	(76.47)	(178.34)	(492.00)	
	(39.22)	(76.51)	(178.42)	(492.00)	
	(39.28)	(76.58)	(179.00)	(492.00)	
Total	1907	2222	2507	841	7477

Table 5 gives the values of likelihood ratio statistic G^2 for testing the goodness-of-fit of some models. The S, PS(2), PS(3) and ME(s), $s = 1, 2, 3$ models fit poorly, but each of the PS(1), AS(m), $m = 1, 2, 3$ and CS models fits the data well. Using Theorem 3.1, it is inferred that, e.g., when $m = 3$, the poor fit of S model is caused by the influence of the lack of structure of the ME(s), $s = 1, 2, 3$ models rather than those of the CS and AS(3) models.

From Table 5, we can see the likelihood ratio chi-squared values G^2 of AS(m), $m = 1, 2, 3$ models are almost the same. Under the AS(1) model, the maximum likelihood estimate of parameter Γ_0 is $\widehat{\Gamma}_0 = 1.199$. Under the AS(2) model, the maximum likelihood estimates of parameters Γ_0 and Γ_1 are $\widehat{\Gamma}_0 = 1.201$ and $\widehat{\Gamma}_1 = 0.999$, respectively. Besides, $\widehat{\Gamma}_0\widehat{\Gamma}_1 = 1.200$, $\widehat{\Gamma}_0\widehat{\Gamma}_1^2 = 1.199$ and $\widehat{\Gamma}_0\widehat{\Gamma}_1^3 = 1.198$. Under the AS(3) model, the maximum likelihood estimates of parameters Γ_0, Γ_1 and Γ_2 are $\widehat{\Gamma}_0 = 1.173$, $\widehat{\Gamma}_1 = 1.027$ and $\widehat{\Gamma}_2 = 0.993$, respectively. Besides, $\widehat{\Gamma}_0\widehat{\Gamma}_1\widehat{\Gamma}_2 = 1.196$, $\widehat{\Gamma}_0\widehat{\Gamma}_1^2\widehat{\Gamma}_2^4 = 1.203$ and $\widehat{\Gamma}_0\widehat{\Gamma}_1^3\widehat{\Gamma}_2^9 = 1.193$.

According to the test based on the difference between the likelihood ratio chi-squared values G^2 for AS(1) and AS(2) models (also for AS(2) and AS(3) models), the AS(1) model is preferable to

Table 5. Likelihood ratio chi-squared values G^2
for models applied to the data in Table 4.

Models	Degree of freedom	G^2
S	6	19.25*
PS(1)	3	6.24
PS(2)	2	6.24*
PS(3)	1	6.18*
AS(1)	3	1.55
AS(2)	2	1.55
AS(3)	1	1.54
CS	2	5.00
ME(1)	1	11.98*
ME(2)	1	11.30*
ME(3)	1	10.00*

*means significant at the 0.05 level.

the AS(2) and AS(3) models. Under the AS(1) model, for example, the cumulative probability that a woman's right eye grade is Best (1) and her left eye grade is Second (2) to Worst (4), is estimated to be $\widehat{\Gamma}_0 = 1.199$ times higher than the cumulative probability that the woman's right eye grade is Second (2) to Worst (4) and her left eye grade is Best (1).

6. Concluding remarks

We have proposed the AS(m) model for $m = 1, 2, \ldots, r-1$. Especially when $m = 1$, the AS(1) model is identical to the MPS model. Compare the AS(m) and the PS(m) models. Both of the AS(m) and PS(m) models have the structure of MH(m) model. The AS(m) model has the structure of CQS model for $\{G_{ij}\}$ with $|j - i| \geq 2$, although the PS(m) model does not have the such structure. As seen in Example 1, when we analyze the data using the MPS model (i.e., AS(1) model), the structure of asymmetry in MPS model with $G_{i,i+1}/G_{i+1,i} = \Gamma_0$ may be strong restriction. The addition of parameters in the AS(m) model relative to the S model improves the fit. However, the more we use the parameters in the AS(m) model, the more complicated the interpretation for the AS(m) model becomes. The AS(m) model implies the structure of symmetry of odds ratios $\{\theta_{ij;st}\}$ based on $\{G_{ij}\}$, $i \neq j$, with the structure of asymmetry partially (being $\Gamma_j^{(m)} \theta_{ij;st} = \theta_{st;ij}$ for $1 \leq i < j < s < t \leq r$ and $s = j + 1$). However, PS(m) model has the structure of symmetry of odds ratio. Therefore the AS(m) model would be useful for seeing the mixed structure of symmetry plus asymmetry of odds ratios for cumulative probabilities $\{G_{ij}\}$, $i \neq j$.

The decomposition of the S model into the AS(m), CS and ME(s) ($s = 1, 2, \ldots, m$) models, given by Theorem 3.1, would be useful for seeing the reason for its poor fit when the S model fits the data poorly (see Example 2).

Acknowledgments

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