Impulsive Control and Synchronization of Time-delay Chaotic System

Yan Yan\textsuperscript{1,}\textsuperscript{a}, Zhanji Gui\textsuperscript{2,}\textsuperscript{b} and Kaihua Wang\textsuperscript{3,}\textsuperscript{c}

\textsuperscript{1} Haikou College of Economics, Hainan,571127, China
\textsuperscript{2} Hainan College of Software Technology, Qionghai 571400, China
\textsuperscript{3} Hainan Normal University, Haikou, Hainan, 571158 ,China
\textsuperscript{a}20689967@qq.com, \textsuperscript{b}zhanjigui@sohu.com, \textsuperscript{c} Corresponding Author: 35225454@qq.com

\textbf{Keywords:} Impulsive system, Impulsive synchronization.

\textbf{Abstract.} With the increasingly perfecting and development of chaos theory, this paper considers the impulsive control and synchronization under the influence of time-delay chaotic system. It will utilize Lyapunov—krasovskii function and linear matrix inequality to research the asymptotic stability of chaotic deviation dynamical system between receiving system and sending system, and several new criteria will be obtained to ensure the impulsive control and synchronization of receiving and sending system.

\textbf{Introduction}

The research of chaos theory is a challenging and cross-subjects project with great potential applications and broad developing prospect. Especially since the 1990s, the breaking progress of chaos synchronization and control promote the research of chaos theory and also bring some opportunities and bright future of chaos application.

With the increasingly perfecting and development of chaos theory application, One new area comes out, that is the research of chaotic system under the Impulsive control and synchronization (e.g[1]-[7]). The Pulse design is established through Discrete time sequence driving system’s state variable sample, and results the discrete of receiving system in the time sequence. After limited time, the path of two chaotic system will approach together, therefore these two chaotic system synchronize. However, in the Communication security system, the transmission lag in the application of the impulsive control and synchronization is unavoidable. It is necessary to analysis the robustness of Impulsive control and synchronization under the time-delay chaotic system. Therefore, this paper will consider the chaotic system with time-delay influences:

All manuscripts must be in English, also the table and figure texts, otherwise we cannot publish your paper.

\begin{equation}
\dot{y}(t) = Ay(t) + By(t-\tau) + \sum_{i=1}^{n} f_i(y(t),t) + g(y(t-\tau),t) + u(t) \tag{1}
\end{equation}

Where $y(t) \in R^n$ is state variable. $u(t) \in R^n$ is control feedback in put variable, $A, B \in R^{nxn}$ is constant matrix and $f_i, g \in R^n$ is nonlinear continuous function which ensure solution exists and is unique. $\tau > 0$ is time-delay constant. This paper assume system (1) satisfies initial condition

\begin{equation}
y(t) = \phi(t) \quad t \in [t_0-\tau, t_0] \tag{2}
\end{equation}

Where $\phi(t)$ in $[t_0-\tau, t_0]$ is bounded continuous

Assumesystem has unstable equilibrium or unstable periodic orbit $\tilde{y}(t)$, the purpose of feedback control is to ensure that $\tilde{y}(t)$ is asymptotic convergence.

This kind of control is a standard feedback control way, control input vector is:

$u(t) = K(\tilde{y}(t) - y(t))$

where $K$ at here is adjustable matrix coefficient of controller.

And then, it apply the pulse control system in Chaotic system (1):
\[
\begin{align*}
\dot{y}(t) &= Ay(t) + By(t - \tau) + \sum_{j=1}^{n} f_j(y(t), t) + g(y(t - \tau), t) + u(t) \quad t \neq t_i \\
\Delta y(t) &= y(t^+) - y(t^-) = D_i y(t^-) \quad t = t_i
\end{align*}
\] (3)

where \( y(t_i^+) = \lim_{t \to t_i^+} y(t), y(t_i^-) = \lim_{t \to t_i^-} y(t) \). \( D_0 = 0 \).

Pulse time satisfy: \( t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots \), and when \( n \to \infty, t \to \infty \).

Assume \( y(t_i^-) = y(t_i)(i=1,2,\ldots) \). The receiving end of driving system (1) is
\[
\tilde{y}(t) = A\tilde{y}(t) + B\tilde{y}(t - \tau) + \sum_{j=1}^{n} f_j(\tilde{y}(t), t) + g(\tilde{y}(t - \tau), t)
\] (4)

\( \tilde{y}(t) = \varphi(t), t \in [t_0 - \tau, t_0] \) is initial condition where \( \varphi(t) \) in \([t_0 - \tau, t_0]\) is bounded continuous.

In the discontinuity point \( t_i(i=1,2,\ldots) \) the status which delivers from driving system will occur jump discontinuity. In this situation, the impulse model of driving system is:
\[
\begin{align*}
\tilde{y}(t) &= A\tilde{y}(t) + B\tilde{y}(t - \tau) + \sum_{j=1}^{n} f_j(\tilde{y}(t), t) + g(\tilde{y}(t - \tau), t) \\
\Delta \tilde{y}(t) &= \tilde{y}(t^+) - \tilde{y}(t^-) = D_i \tilde{y}(t^-)
\end{align*}
\] (5)

Where \( \tilde{y}(t_i^+) = \lim_{t \to t_i^+} \tilde{y}(t), \tilde{y}(t_i^-) = \lim_{t \to t_i^-} \tilde{y}(t), D_0 = 0 \), and \( \tilde{y}(t) \) assume is left continuous, then \( y(t_i^-) = y(t_i)(i=1,2,\ldots) \).

According (3) and (5), it can obtain the error impulsive differential system is:
\[
\begin{align*}
\dot{e}(t) &= Ae(t) + Be(t - \tau) + \sum_{j=1}^{n} F_j(e(t), t) + G(e(t - \tau), t) - Ke(t) \quad t \neq t_i \\
\Delta e(t) &= e(t^+) - e(t^-) = D_i e(t^-) \quad t = t_i
\end{align*}
\] (6)

Where
\[
e(t) = y(t) - \tilde{y}(t), F_i[e(t), t] = f_i[y(t), t] - f_i[\tilde{y}(t), t], G[e(t - \tau), t] = g[y(t - \tau), t] - g[\tilde{y}(t - \tau), t].
\]

The initial condition of error system (6) satisfy:
\[
e(t) = \varphi(t) - \tilde{\varphi}(t) = \phi(t) \quad t \in [t_0 - \tau, t_0]
\] (7)

since \( \varphi(t), \tilde{\varphi}(t) \) is bounded continuous, therefore \( \phi(t) \) is bounded, \( \| \phi \| = \| \phi - \tilde{\phi} \| = \sup_{t_0 - \tau \leq t \leq t_0} \{ \| \varphi(t) - \tilde{\varphi}(t) \| \} \)

For the error system (6), due to the \( F_j(0, t) \equiv 0, G(0, t) \equiv 0 \), therefore it utilize Taylor expansion:
\[
\sum_{j=1}^{n} F_j(e(t), t) + G(e(t - \tau), t) = \sum_{j=1}^{n} A_j(t) e(t) + o(e(t)) + A_0(t) e(t - \tau) + o(e(t - \tau))
\] (8)

Here
\[
A_j(t) = \left. \frac{\partial F_j[e(t), t]}{\partial e(t)} \right|_{e(t) = 0}, A_0(t) = \left. \frac{\partial G[e(t - \tau), t]}{\partial e(t - \tau)} \right|_{e(t - \tau) = 0}.
\]

Let
\[
o(e(t)) + o(e(t - \tau)) = H[e(t), e(t - \tau), t] = o(\| e(t) \| + \| e(t - \tau) \|)
\] (9)

According to the infinitesimal of higher order conception, for all \( \alpha_i > 0 \) \( (i=1,2) \), all exist \( M > 0 \), when \( \| e(t) \| \leq M \) and \( \| e(t - \tau) \| \leq M \), then:
\[
H[e(t), e(t - \tau), t] \leq \alpha_1 \| e(t) \| + \alpha_2 \| e(t - \tau) \|
\] (10)

Assume \( A_j(t) = A_{i_1} + A_{i_2}(t)(t = 0, 1, 2, \ldots) \), where \( A_{i_1} \) is constant matrix, \( A_{i_2}(t) \) is time-varying uncertain matrix, which satisfy:
\[
\| A_{i_2}(t) \| \leq \beta_i
\] (11)

where \( \beta_i(i=0, 1, 2, \ldots) \) is constant.
According to the above assumption, system (6) can rewrite as:

\[
\begin{align*}
\dot{e}(t) &= Ee(t) + Fe(t - \tau) + \sum_{i=1}^n A_{i2}(t)e(t) + A_{i0}(t)e(t - \tau) \\
&\quad + H[e(t), e(t - \tau), t] \\
\Delta e(t) &= e(t^+) - e(t^-) = D_1e(t^-) \\
e(t) &= \phi(t)
\end{align*}
\]

where \( E = A - k + \sum_{i=1}^n A_{i1} \), \( F = B + A_{01} \), \( D_0 = 0 \).

Due to \( F(t - \tau) = Fe(t) - Fe(t - \tau) \), then system (12) can be changed as:

\[
\begin{align*}
\left[ \begin{array}{c}
\dot{e}(t) + F\int_{t-\tau}^t e(s)ds \\
\end{array} \right] &= (E + F)e(t) + \sum_{i=1}^n A_{i2}(t)e(t) + A_{i0}(t)e(t - \tau) \\
&\quad + H[e(t), e(t - \tau), t] \\
\Delta e(t) &= e(t^+) - e(t^-) = D_1e(t^-) \\
e(t) &= \phi(t)
\end{align*}
\]

For the sake of convenient, it define \( L : C([-\tau, 0], R^n) \rightarrow R^n \) as:

\[
L(e) = e(t) + F\int_{t-\tau}^t e(s)ds
\]

Before this paper put forward the main conclusion, there introduce three lemmas:

**Lemmas 1:** For the vector \( a, b \in R^n \) and symmetric positivematrix \( Q \), there is:

\[
2a^TB \leq a^TQ^{-1}a + b^TQB
\]

**Lemmas 2:** For any positive matrix \( N \in R^{n\times n} \), it exists \( \gamma > 0 \), and vector function \( \omega : [0, \gamma] \rightarrow R^n \), there is:

\[
\left( \int_0^\gamma \omega(s)ds \right)^T N\left( \int_0^\gamma \omega(s)ds \right) \leq \gamma\int_0^\gamma \omega^T(s)N\omega(s)ds
\]

**Lemmas 3:** (Schur Complement) exists constant matrix \( Z_1, Z_2, Z_3 \), where \( Z_1 = Z_1^T \), \( Z_2 = Z_2^T \), there is:

\[
\begin{bmatrix}
Z_1 & Z_3^T \\
Z_3 & Z_2
\end{bmatrix} < 0, \text{ if and only if } Z_1 < 0, Z_2 - Z_1^{-1}Z_3^T < 0; \text{ or } Z_2 < 0, Z_2 - Z_2^{-1}Z_3^T < 0.
\]

**Impulsive control and synchronization with time-delay and chaotic system**

**Theorem 1:** For system (13), if it satisfy \( \|F\| < 1 \), and exist positivesymmetric matrix \( P > 0, Q > 0 \), which make the linear matrix inequality satisfy:

\[
N =\begin{pmatrix}
(E + F)^T P + P(E + F) + L_1I + L_2I + L_3I + L_4I + 4Q\tau & \tau(E + F)^TPF \\
\tau FP^T (E + F) & -\tau Q
\end{pmatrix} < 0
\]

\[
\|1 + D_1\| \sqrt{\frac{(1 + \|F\|)^2}{1 - \|F\|}} \lambda_{\max}(P) + \|L_1 + L_2\| + 2\tau^2 \lambda_{\max}(P) < 1
\]

Then the zero-solution of error system(13) is exponentially stable, that is mean the pulsedindex of system (1) and system (4) is synchronous, where \( L_1, L_2, L_3, L_4 \), can be viewed respectively as:

\[
L_1 = \tau\beta_0^2 \|PFQ^{-1}FP\| + \beta_0 \|P\|, \quad L_2 = (\alpha_1 + \alpha_2)\alpha_2 \|PFQ^{-1}FP\| + \alpha_2 \|P\|
\]
\[ L_2 = 2 \sum_{i=1}^{n} \beta_i \| P \| + \tau \sum_{i=1}^{n} \beta_i^2 \| PFQ^{-1}F^T P \| + \beta_0 \| P \| \\
L_4 = (\alpha_1 + \alpha_2) \alpha_i \| PFQ^{-1}F^T P \| + \alpha_2 \| P \| + 2\alpha_i \| P \| \]

**Proof:**

According to the definition’s condition (15), for \( \varepsilon > 0 \), let:

\[ N + (e^{\varepsilon t} - 1) \left( L_1 + L_2 + 4Q\tau \right) e^{\varepsilon t} \left( \tau Q + \tau^2 F^T PF \right) + e \left( P F^T P Q^{-1} \right) \leq 0 \] (17)

Let Lyapunov-Krasovskii function as:

\[ V(e) = V_1(e) + V_2(e) \]

\[ V_1(e) = \int_{t-\tau}^{t} e^T (\rho) [(4 + \varepsilon Q + \varepsilon \tau F^T PF) e(\rho) e^{(\rho + \varepsilon t)} \rho d\rho , V_2(e) = (L_1 + L_2) \int_{t-\tau}^{t} e(s) \| e(s) \|^2 e^{(\varepsilon + \varepsilon t)} ds . \]

Through system (13), to make a derivative of Lyapunov-Krasovskii function \( V_1(e) \), obtain:

\[ \frac{dV_1(e)}{dt} = 2e^{\varepsilon t} L^T(e)P \frac{dL(e)}{dt} + e\varepsilon e^{\varepsilon t} L^T(e)PL(e) \]

\[ = e^{\varepsilon t} \left[ e^T(t)(P(F + E + \sum_{i=1}^{n} A_i(t))) + (F + E + \sum_{i=1}^{n} A_i(t))^T P \right] e(t) \]

\[ + 2e^{\varepsilon t}(t) A_{02}(t) e(t - \tau) + 2e^{\varepsilon t}(t)(E + F)^T PF \int_{t-\tau}^{t} e(s) ds \]

\[ + 2e^{\varepsilon t}(t - \tau) A_{02}^T(t) PF \int_{t-\tau}^{t} e(s) ds + 2e^{\varepsilon t}(t) \left( \sum_{i=1}^{n} A_i(t) \right)^T PF \int_{t-\tau}^{t} e(s) ds \]

\[ + 2e^{\varepsilon t}(t) PH[t(e(t), e(t-\tau), t) + 2H^T [e(t), e(t-\tau), t] PF \int_{t-\tau}^{t} e(s) ds \]

\[ + \varepsilon [e^T(t) Pe(t) + 2 e^T(t) PF \int_{t-\tau}^{t} e(s) ds + \int_{t-\tau}^{t} e^T(s) ds F^T PF \int_{t-\tau}^{t} e(s) ds] \}

To apply Lemmas 1 and system (11), it can be obtained:

\[ \frac{dV_1(e)}{dt} \leq e^{\varepsilon t} \left[ e^T(t)(P(F + E + \sum_{i=1}^{n} \beta_i I)) + (F + E + \sum_{i=1}^{n} \beta_i I)^T P \right] e(t) \]

\[ + 2e^{\varepsilon t}(t) A_{02}(t) e(t - \tau) + 2e^{\varepsilon t}(t)(E + F)^T PFQ^{-1}F^T P(E + F) e(t) \]

\[ + \tau\beta_0^2 e^{\varepsilon t}(t - \tau) PFQ^{-1}F^T Pe(t - \tau) + 4 \int_{t-\tau}^{t} e^T(s) Q e(s) ds + \tau \sum_{i=1}^{n} \beta_i^2 e^{\varepsilon t}(t) PFQ^{-1}F^T Pe(t) \]

\[ + 2e^{\varepsilon t}(t) PH[t(e(t), e(t-\tau), t) + 2H^T [e(t), e(t-\tau), t] PFQ^{-1}F^T PH[t(e(t), e(t-\tau), t)] \]

\[ + e[e^T(t) Pe(t) + 2 e^T(t) PF \int_{t-\tau}^{t} e(s) ds + \int_{t-\tau}^{t} e^T(s) ds F^T PF \int_{t-\tau}^{t} e(s) ds] \}

When \( \| \phi \| < M \), there exist \( l > 0 \), which make \( \forall t \in [t_0, t_0 + l] \) has \( e(t) \| \leq M, \| e(t - \tau) \| \leq M \), now satisfy system (10), and then apply Lemma 2, here it is:

\[ \frac{dV_1(e)}{dt} \leq e^{\varepsilon t} \left[ e^T(t)(P(F + E + \sum_{i=1}^{n} \beta_i I)) + (F + E + \sum_{i=1}^{n} \beta_i I)^T P \right] e(t) \]

\[ + 4 \int_{t-\tau}^{t} e^T(s) Q e(s) ds + \tau e^{\varepsilon t}(t)(E + F)^T PFQ^{-1}F^T P(E + F) e(t) \]

\[ \tau \sum_{i=1}^{n} \beta_i^2 \| PFQ^{-1}F^T P \| e(t)^2 + \tau \beta_0^2 \| PFQ^{-1}F^T P \| e(t - \tau)^2 \]

\[ + 2 \beta_0 \| P \| \| e(t) \| \| e(t - \tau) \| + 2\alpha_i \| P \| \| e(t) \| \]

© 2016. The authors — Published by Atlantis Press
Therefore, to make a derivative of function $V_1(e_t)$ with system (13), it obtain:
\[
\frac{dV_1(e_t)}{dt} \leq e^{\alpha_1 t} \{e^T(t)[P(F + E) + (F + E)^T P + (E + F)^T PFQ^{-1}F^T P(E + F)] \\
+ eP + ePFQ^{-1}F^T P + L_1 I + L_2 I\} e(t) + e^{\alpha_2} \int_{t-\tau}^t e^T(s) Pe(s) ds \\
+ (4 + e) \int_{t-\tau}^t e^T(s) Qe(s) ds + (L_1 + L_2) \| e(t - \tau) \| \}
\]
(18)

For
\[
\frac{dV_2(e_t)}{dt} \leq e^{\alpha_1 t} \{e^T(t)[(4 + e)Q + \varepsilon \tau F^T PF] e(t) - e^{\alpha_1 t} \int_{t-\tau}^t e^T(s)[(4 + e)Q + \varepsilon \tau F^T PF] e(s) ds
\]
(19)

\[
\frac{dV_3(e_t)}{dt} = (L_1 + L_2) \| e(t) \|^2 e^{\alpha_2(t-\tau)} - (L_1 + L_2) \| e(t-\tau) \|^2 e^{\alpha_2 t}
\]
(20)

Accord to system (18), (19), (20), there is:
\[
\frac{dV(e_t)}{dt} \leq e^{\alpha_1 t} \{e^T(t)(P(F + E) + (F + E)^T P + \tau (E + F)^T PFQ^{-1}F^T P(E + F)) \\
+ (L_1 I + L_2 I)e^{\alpha_1 t} + L_1 I + L_4 I + (4 + e)e^{\alpha_2 t} Q + \varepsilon \tau^2 e^{\alpha_1 t} F^T PF + eP + PFQ^{-1}F^T P)e(t)\}
\]
For $\forall t \in [t_0, t_0 + I]$, according to the inequation (17), there is:
\[
V'(e_t) \leq 0
\]
(21)

for $\forall t \in [t_0, t_1]$, there is:
\[
V(e_t) \leq V(e_{t_0}) \leq \lambda_{\max}(P)(1 - \tau \| F \|)^2 \| \phi \|^2 e^{\alpha_1 t} + (L_1 + L_2)e^{\alpha_2 t} \| \phi \|^2 e^{\alpha_2 t} \\
+ \lambda_{\max}[(4 + e)Q + \varepsilon \tau F^T PF] \frac{\tau^2}{2} e^{\alpha_2 t} \| \phi \|^2 e^{\alpha_2 t} = Ce^{\alpha_1 t}\| \phi \|^2 e^{\alpha_2 t},
\]
where
\[
C = \lambda_{\max}(P)(1 - \tau \| F \|)^2 \| \phi \|^2 e^{\alpha_1 t} + (L_1 + L_2)e^{\alpha_1 t} + \lambda_{\max}[(4 + e)Q + \varepsilon \tau F^T PF] \frac{\tau^2}{2} e^{\alpha_2 t}.
\]

From literature (8), it is:
\[
(1 - \tau \| F \|)^2 \| e(t) \|^2 \leq L(e_t) \| e(t) \|^2,
\]
and then
\[
(1 - \tau \| F \|)^2 \lambda_{\min}(P) \| e(t) \|^2 e^{\alpha_1 t} \leq L(e_t) PL(e_t) e^{\alpha_1 t} \leq V(e_t) \leq C \| \phi \|^2 e^{\alpha_1 t}.\]

That is
\[
\| e(t) \| \leq \sqrt{\frac{C}{(1 - \tau \| F \|)^2 \lambda_{\min}(P)}} \| \phi \|^2 e^{-\frac{\tau^2}{2}(t-t_0)} = \Phi \| \phi \|^2 e^{-\frac{\tau^2}{2}(t-t_0)} t \in [t_0, t_1),
\]
where
\[
\Phi = \sqrt{\frac{C}{(1 - \tau \| F \|)^2 \lambda_{\min}(P)}},
\]
according to the system (13), it can deduct that:
\[
\| e(t^*_i) \| \leq \| (1 + D_1) e(t_i) \| \leq \| (1 + D_2) \| \Phi \| \phi \|^2 e^{-\frac{\tau^2}{2}(t-t_0)}
\]
according to system (21), $\forall t \in [t_0, t_1]$, there is
\[
V(e_t) \leq V(e_{t_0}) \leq C \| e(t^*_i) \|^2 e^{\alpha_1 t_i} \leq C \| (1 + D_2) \| \Phi \| \phi \|^2 e^{\alpha_1 t} e^{-\frac{\tau^2}{2}(t-t_0)} e^{\alpha_1 t},
\]
for
\[
(1 - \tau \| F \|^2 \lambda_{\text{min}}(P) \| e(t) \|^2 e^{\epsilon t} \leq L^T (e_t) P L(e_t) e^{\epsilon t} \leq V(e_t) \leq C (1 + D_t) \| \Phi \|_2 \| e^{\epsilon t} e^{-\epsilon(t_{k-1})} e^{\epsilon t} ,
\]
so \( \| e(t) \| \leq (1 + D_t) \| \Phi \|_2 \| e^{\epsilon t} e\frac{\epsilon}{2(t_{k-1})} \) \( t \in (t_k, t_{k+1}] \).

by utilizing mathematical induction \( \forall t \in (k, k+1] \), there is:
\[
\| e(t_{k+1}) \| \leq \prod_{i=1}^{k+1} \| (1 + D_i) \| \Phi \|_2 \| e^{\epsilon t} e\frac{\epsilon}{2(t_{k-1})} \),
\]
when \( \epsilon > 0 \), according to the condition (16) can be known:
\[
\| 1 + D_i \| \frac{C}{(1 - \tau \| F \|^2 \lambda_{\text{min}}(P)) e^{\frac{\epsilon}{2(t_{k-1})}} = \| 1 + D_i \| \Phi e^{\frac{\epsilon}{2(t_{k-1})}} \leq 1,
\]
if \( d = \max \{1, \Phi \}, \forall t \geq t_0 \) all have:
\[
\| e(t) \| \leq d \| \Phi \| e^{\frac{\epsilon}{2(t_{k-1})}} (22)
\]

When \( \| \Phi \| \leq \frac{M}{d} \), it is easy to get \( \forall t \in [t_0, \infty) \) , and all \( \| e(t) \| \leq M, \| e(t-\tau) \| \leq M \), therefore, for \( \forall t \in [t_0, \infty) \), inequation (22) is established. therefore, the zero solution of error system (13) is exponentially stable, thus the pulsed index of system (1) and system (4) is synchronous.

Acknowledgements

This work is supported jointly by the Natural Sciences Foundation of China under Grant No. 60963025, Natural Sciences Foundation of Hainan Province under Grant No. 110007, Haikou College of Economics No.hjky16-17 and the Start-up fund of Hainan Normal University under Project No. 00203020201.

References


