Positive solutions of the fourth-order boundary value problem with dependence on the first order derivative

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**Abstract:** In this paper, By the use of a new fixed point theorem and the Green function. The existence of at least one positive solutions for the fourth-order boundary value problem with the first order derivative

\[
\begin{aligned}
\dot{u}^{(4)}(t) + Au^{(3)}(t) &= \lambda f(t, u(t), u'(t)) & 0 < t < 1 \\
u(0) &= u(1) = u''(0) = u''(1) = 0
\end{aligned}
\]

is considered, where \( f \) is a nonnegative continuous function and \( \lambda > 0, 0 < A < \pi^2 \).

1. Introduction

Recently, there has been much attention focused on the question of positive solution of fourth-order differential equation with one or two parameters. For example, astronomy, biology, physics, chemical engineering and information science and other fields. So, the fourth-order boundary value problems has very important in real life applications, see for example [1-4, 6-9].

Li [6] investigated the existence of positive solutions for the fourth-order boundary value problem. All the above works were done under the assumption that the first order derivative \( \dot{u} \) is not involved explicitly in the nonlinear term \( f \). In this paper, we are concerned with the existence of positive solutions for the fourth-order boundary value problem

\[
\begin{aligned}
\dot{u}^{(4)}(t) + Au^{(3)}(t) &= \lambda f(t, u(t), u'(t)) & 0 < t < 1 \\
u(0) &= u(1) = u''(0) = u''(1) = 0
\end{aligned}
\]  

(1)

The following conditions are satisfied throughout this paper:

(H\(_1\)) \( \lambda > 0, 0 < A < \pi^2 \);

(H\(_2\)) \( f : [0,1] \times [0, \infty) \times R \rightarrow [0, \infty) \) is continuous.

2. The preliminary lemmas

Suppose \( Y = C[0,1] \) be the Banach space equipped with the norm \( \| u \| = \max_{t \in [0,1]} | u(t) | \).
Let $\lambda_1, \lambda_2$ be the roots of the polynomial $P(\lambda) = \lambda^2 + A\lambda$, namely, $\lambda_1 = 0, \lambda_2 = -A$. By (H1) it is easy to see that $-\pi^2 < \lambda_2 < 0$.

Let $G_i(t, s) (i = 1, 2)$ be the Green’s function of the linear boundary value problem:

$-u^{(4)}(t) + \lambda u(t) = 0, u(0) = u(1) = 0$. Then, carefully calculation yield:

$G_i(t, s) = \begin{cases} 
  s(1-t), & 0 \leq s \leq t \leq 1 \\
  t(1-s), & 0 \leq t \leq s \leq 1
\end{cases}$

$G_2(t, s) = \begin{cases} 
  \frac{\sin \sqrt{As} \sin \sqrt{A}(1-t)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq s \leq t \leq 1 \\
  \frac{\sin \sqrt{At} \sin \sqrt{A}(1-s)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq t \leq s \leq 1
\end{cases}$

**Lemma 2.1:** Suppose (H1) (H2) hold. Then for any $g(t) \in C[0,1]$, BVP

\[
\begin{cases} 
  u^{(4)}(t) + Au^{(3)}(t) = g(t), & 0 < t < 1 \\
  u(0) = u(1) = u'(0) = u'(1) = 0 
\end{cases}
\] (2)

the unique solution $u(t) = \int_0^1 \int_0^1 G_1(t, s)G_2(s, \tau)g(\tau)d\tau ds$. (3)

where

$G_i(t, s) = \begin{cases} 
  s(1-t), & 0 \leq s \leq t \leq 1 \\
  t(1-s), & 0 \leq t \leq s \leq 1
\end{cases}$

$G_2(s, \tau) = \begin{cases} 
  \frac{\sin \sqrt{At} \sin \sqrt{A}(1-s)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq \tau \leq s \leq 1 \\
  \frac{\sin \sqrt{As} \sin \sqrt{A}(1-\tau)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq s \leq \tau \leq 1
\end{cases}$

**Lemma 2.2** [5]: Assume (H1) (H2) hold. Then one has:

(i) $G_i(t, s) \geq 0, \forall t, s \in [0,1]$;

(ii) $G_i(t, s) \leq C_iG_i(s, s), \forall t, s \in [0,1]$;

(iii) $G_i(t, s) \geq \delta_iG_i(t,t)G_i(s, s), \forall t, s \in [0,1]$.

Where: $C_1 = 1, \delta_1 = 1; C_2 = \frac{1}{\sin \sqrt{A}}, \delta_2 = \sqrt{A} \sin \sqrt{A}$.

**Lemma 2.3:** Assume (H1) (H2) hold and are given as above, Then one has:

$$
\min_{\|u\| \leq \frac{\pi}{4}} u(t) \geq d \|u\|,
$$
where: \( d = \frac{\sqrt{A \sin^2 \sqrt{AC_0G_0}}}{M_1} \), \( C_0 = \int_0^1 G_1(s, s)G_2(s, s)ds \), \( M_1 = \int_0^1 G_1(s, s)ds \), \( G_0 = \min_{t \in \mathbb{R}} G_t(t, t) \).

**Proof:** By (3) and (ii) of Lemma 2.2, we get:

\[
u(t) \leq C_1C_2\int_0^1\int_0^1 G_1(s, s)G_2(t, t)g(\tau)g(\tau)d\tau ds \leq C_1C_2M_1\int_0^1 G_2(t, t)g(\tau)d\tau
\]

Therefore, \( \|\nu\|_0 \leq C_1C_2M_1\int_0^1 G_2(t, t)g(\tau)d\tau \)

By (iii) of Lemma 2.2, we have:

\[
u(t) \geq \delta_1\delta_2\int_0^1\int_0^1 G_1(s, s)G_2(t, t)g(\tau)g(\tau)d\tau ds
\]

\[
= \delta_1\delta_2C_0G_1(t, t)\int_0^1 G_2(\tau, \tau)g(\tau)d\tau
\]

\[
\geq \frac{\delta_1\delta_2C_0}{C_1C_2M_1} G_1(t, t)\|\nu\|_0
\]

Let \( G_0 = \min_{t \in \mathbb{R}} G_t(t, t) \), we have:

\[
\min_{t \in \mathbb{R}} u(t) \geq \frac{\delta_1\delta_2C_0G_0}{C_1C_2M_1} \|\nu\|_0
\]

\[
= \frac{\sqrt{A \sin^2 \sqrt{AC_0G_0}}}{M_1} \|\nu\|_0
\]

\[
= d \|\nu\|_0
\]

**Theorem 2.1**[10]: Let \( r_2 > r_1 > 0, L > 0 \) be constants and

\[
\Omega_i = \{u \in X : \alpha(u) < r_i, \beta(u) < L\}, i = 1, 2
\]

two bounded open sets in \( X \). Set \( D_i = \{u \in X : \alpha(u) = r_i\}, i = 1, 2; \)

Assume \( \mathcal{T} : K \to K \) is a completely continuous operator satisfying:

(A1) \( \alpha(Tu) < r_1, u \in D_1 \cap K; \alpha(Tu) > r_2, u \in D_2 \cap K; \)

(A2) \( \beta(Tu) < L, u \in K; \)

(A3) there is \( \exists p \in (\Omega_2 \cap K) \setminus \{0\} \),

such that \( \alpha(p) \neq 0 \) and \( \alpha(u + \lambda p) \geq \alpha(u) \), for all \( \forall u \in K, \lambda \geq 0 \).

Then \( \mathcal{T} \) has at least one fixed point in \( (\Omega_2 \setminus \overline{\Omega_1}) \cap K \).
3. The main results

Let $X = C^1[0,1]$ be the Banach space equipped with the norm $\|u\| = \max_{t \in [0,1]} |u(t)| + \max_{t \in [0,1]} |u'(t)|$, and $K = \left\{ u \in X : u \geq 0, \min_{t \in [\frac{1}{4}, 1]} u(t) \geq d \|u\| \right\}$ is a cone in $X$.

Define functionals $\alpha(u) = \max_{t \in [0,1]} |u(t)|$, $\beta(u) = \max_{t \in [0,1]} |u'(t)|$, $\forall u \in X$.

then, $\|u\| \leq 2 \max \{ \alpha(u), \beta(u) \}$, $\alpha(\lambda u) = |\lambda| \alpha(u)$, $\beta(\lambda u) = |\lambda| \beta(u)$, $\forall u \in X, \lambda \in R$, $\alpha(u) \leq \alpha(v)$, $\forall u, v \in K, u \leq v$.

Assume (H1) hold, the green's function of the problem (2) $G_i(t,s) \geq 0$. let $g(t) = 1$, we have

$$\int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)d\tau ds = \frac{\sin \sqrt{A}(1-t) + \sin \sqrt{A}t}{A^2 \sin \sqrt{A}} + \frac{t^2 - t}{2A} - \frac{1}{A^2}$$

we denote:

$M = \max_{t \in [0,1]} \left[ \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)d\tau ds \right]$, $m = \max_{t \in [\frac{1}{4}, 1]} \left[ \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)d\tau ds \right]$

$Q = \frac{2A^2 \sin \sqrt{A}}{[6\sqrt{A} - (1 - \cos \sqrt{A}) - 3\sin \sqrt{A}]}$

We will suppose that there are $\exists L > b > db > c > 0$, such that $f(t,u,v) f(t,u,v)$

satisfies the following growth conditions:

(H3) $f(t,u,v) < \frac{c}{\lambda M}$, $\forall (t,u,v) \in [0,1] \times [0,c] \times [-L,L]$;

(H4) $f(t,u,v) \geq \frac{b}{\lambda m}$, $\forall (t,u,v) \in [\frac{1}{4}, \frac{3}{4}] \times [db,b] \times [-L,L]$;

(H5) $f(t,u,v) < \frac{L}{\lambda Q}$, $\forall (t,u,v) \in [0,1] \times [0,b] \times [-L,L]$.

Let

$$f^*(t,u,v) = \begin{cases} f(t,u,v), (t,u,v) \in [0,1] \times [0,b] \times (-\infty, \infty) \\
 f(t,b,v), (t,u,v) \in [0,1] \times (b, \infty) \times (-\infty, \infty) \\
 f^*(t,u,v), (t,u,v) \in [0,1] \times (0, \infty) \times [-L,L] \\
 f^*(t,u,-L), (t,u,v) \in [0,1] \times [0, \infty) \times (-\infty, -L] \\
 f^*(t,u,L), (t,u,v) \in [0,1] \times [0, \infty) \times [L, \infty) \end{cases}$$

Define:

$$(Tu)(t) = \lambda \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds$$

$$(Tu)'(t) = \lambda \int_0^1 \int_0^1 G_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds - \int_0^1 \int_0^1 sG_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds$$

Lemma 3.1: Suppose (H1) (H2) hold, then $T : K \rightarrow K$ is completely continuous.

Proof: For $\forall u \in K$, by (5) and Lemma 2.2, there is $Tu \geq 0$. 

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so,
\[
\|Tu\|_0 = \max_{t \in [0,1]} \left| \lambda \int_0^1 G_i(t,s)G_j(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds \right|
\]
\[
\leq \lambda \int_0^1 \int_0^1 C_iC_j G_i(t,s)G_j(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds
\]
\[
\leq \lambda C_iC_j M_1 \int_0^1 G_j(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau
\]
we have:
\[
\min_{[0,1]} (Tu)(t) = \min_{[0,1]} \lambda \int_0^1 G_i(t,s)G_j(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds
\]
\[
\geq \lambda \delta \delta_1 \int_0^1 G_i(t,t)G_i(s,s)G_j(s,\tau)G_j(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds
\]
\[
\geq \lambda \delta \delta_1 C_0 G_i(t,t) \int_0^1 G_j(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau
\]
\[
\geq \lambda \delta \delta_1 C_0 G_0 \int_0^1 G_j(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau
\]
\[
\geq \frac{2 \delta \delta_1 C_0 G_0}{C_iC_j M_1} \|Tu\|_0
\]
\[
= d \|Tu\|_0
\]
Therefore, we get \(T(K) \subset K\).

So we can get \(T(K) \subset K\). Let \(B \subset K\) is bounded, it is clear that \(T(B)\) is bounded. Using \(f,G_i(t,s),G_j(t,s)\) is continuous, we show that \(T(B)\) is equicontinuous. By the Arzela-Ascoli theorem, a standard proof yields \(T : K \to K\) is completely continuous.

**Theorem 3.1:** Suppose condition (H1)—(H5) hold, Then BVP (1) has at least one positive solution \(u(t)\) satisfying:
\[
c < \alpha(u) < b, |u'(t)| < L
\]

**Proof:** Take \(\Omega_1 = \{u \in X : |\mu(t)| < c, |u'(t)| < L\}, \Omega_2 = \{u \in X : |\mu(t)| < b, |u'(t)| < L\}\) two bounded open sets in \(X\) and \(D_1 = \{u \in X : \alpha(u) = c\}, \ D_2 = \{u \in X : \alpha(u) = b\}\) such that \(\alpha(u + \lambda p) \geq \alpha(u), \forall u \in K, \lambda \geq 0, \ \forall u \in D_1 \cap K, \alpha(u) = c\),

From (H3) we have:
\[
\alpha(Tu) = \max_{\tau \in [0,1]} \left| \lambda \int_0^1 G_i(t,s)G_j(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds \right|
\]
\[
< \max_{\tau \in [0,1]} \lambda \int_0^1 G_i(t,s)G_j(s,\tau) \frac{c}{\lambda M} d\tau ds
\]
\[
= \frac{c}{M} \max_{\tau \in [0,1] \cup \{t\}} \int_0^1 G_i(t,s)G_j(s,\tau)d\tau ds
\]
\[
= c
\]
\(\forall u \in D_2 \cap K, \alpha(u) = b\) . From Lemma 2.3, we have \(u(t) \geq d \alpha(u) = db, t \in [\frac{1}{4}, \frac{3}{4}]\),

so, from (H3) we get:
\[
\alpha(Tu) = \max_{\tau \in [0,1]} \left| \lambda \int_0^1 G_i(t,s)G_j(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds \right|
\]
\[
> \max_{\tau \in [\frac{1}{4}, \frac{3}{4}]} \lambda \int_0^1 G_i(t,s)G_j(s,\tau) \frac{b}{\lambda M} d\tau ds
\]
\[
= b \max_{m \in \{1, 2\}} \int_0^t G_1(t, s) G_2(s, \tau) \, d\tau \, ds
= b \\
\forall u \in K, \text{ from (H5) we get:}
\beta(Tu) = \max_{r \in [0, 1]} |\lambda \int_0^1 G_2(s, \tau) f_1(\tau, u(\tau), u'(\tau)) \, d\tau \, ds - \lambda \int_0^1 s G_2(s, \tau) f_1(\tau, u(\tau), u'(\tau)) \, d\tau \, ds |
< d \lambda \int_0^1 Q(s, s) Q(s, \tau) f_1(\tau, u(\tau), u'(\tau)) \, d\tau \, ds
= \left( \frac{6\sqrt{A}(1 - \cos \sqrt{A}) - 3A \sin \sqrt{A}}{2A^2 \sin \sqrt{A}} \right) \times \frac{L}{Q} = L
\]

Theorem 2.1 implies there is \( u \in (\Omega_2 \setminus \Omega_1) \cap K \), such that \( u = Tu \), so \( u(t) \) is a positive solution for BVP(1), satisfying:
\[
c < \alpha(u) < b, |u'(t)| < L
\]
Thus, Theorem 3.1 is completed.

References