

Split General Mixed Variational Inequality Problem

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Abstract. In this paper, we introduce a split general mixed variational inequality problem which is a natural extension of a split variational inequality problem, mixed variational and variational inequality problems in Hilbert spaces. Using the resolvent operator technique, we propose an iterative algorithm for a split general mixed variational inequality problem and discuss some special cases. Further, we discuss the convergence criteria of these iterative algorithms. The results presented in this paper generalize, unify and improve many previously known results for mixed variational and variational inequality problems.

Introduction

It is well known that the mixed variational inequality problem is a generalized form of a variational inequality problem, having applications in different areas of optimization, optimal control, operation research, economics equilibrium and free boundary value problems. The mixed variational inequality has been extensively studied including its various generalizations in a general setting. In recent years, considerable interest has been shown in developing various extensions and generalizations of split variational inequality problem. By using the projection method, the authors of [2] introduced and studied split general quasi-variational inequality problem. [1, 3] introduced gap function and global error bounds for generalized mixed quasivariational inequalities. [6] studied the split common null point problem. [8] introduced and studied algorithms for the split variational inequality problems. Kazmi and Rizvi [12] introduced the iterative approximation of a common solution of a split generalized equilibrium problem and a fixed point problem for nonexpansive semigroup. Some split variational inequality problems and some examples, see the references therein.

Inspired and motivated by the above work, in this paper, we introduce and study a split general mixed variational inequality problem (in short, SpGMVIP) which is a natural extension of a split variational inequality problem (in short SpVIP), mixed variational and variational inequality problems in Hilbert spaces. By using the resolvent operator technique, we propose an iterative algorithm for a split general mixed variational inequality problem and discuss some special cases. Furthermore, we discuss the convergence criteria of these iterative algorithms. It is of further research effort to extend the iterative method presented here to solving split variational inclusions [11], the split equilibrium problem [12] and split general quasi-variational inequality problem [2].

Throughout the paper unless stated otherwise, for each $i \in \{1, 2\}$, let H_i be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $f_i : H_i \rightarrow H_i, g_i : H_i \rightarrow H_i$ be nonlinear mappings and $A : H_1 \rightarrow H_2$ a bounded linear operator with its adjoint operator A^* . Let $\varphi_i : H_i \rightarrow R \cup \{+\infty\}$ with $\text{Im } g_i \cap \text{dom } \partial \varphi_i \neq \emptyset$, $\partial \varphi_i$ denotes the subdifferential of a proper, convex and lower semi-continuous function $\varphi_i : H \rightarrow R \cup \{+\infty\}$.

An important generalization of the variational inequality is the mixed variational inequality problem. The mixed variational inequality problem (in short, MVIP) is to find $x_1 \in H_1$ such that

$$\langle f_1(x_1), y_1 - x_1 \rangle + \varphi(x_1) - \varphi(y_1) \geq 0, \forall y_1 \in H_1. \quad (1)$$

In this paper, we consider the following split general mixed variational inequality problem (in short, SpGMVIP): Find $x_1^* \in H_1$ such that $g_1(x_1^*) \in \text{dom} \partial \varphi_1$ and

$$\langle f_1(x_1^*), x_1 - g_1(x_1^*) \rangle + \varphi(g_1(x_1^*)) - \varphi(x_1) \geq 0, \quad \forall x_1 \in H_1, \quad (2)$$

and such that $x_2^* = Ax_1^*$ and $g_2(x_2^*) \in \text{dom} \partial \varphi_2$ solves

$$\langle f_2(x_2^*), x_2 - g_2(x_2^*) \rangle + \varphi(g_2(x_2^*)) - \varphi(x_2) \geq 0, \quad \forall x_2 \in H_2. \quad (3)$$

Now, we observe a special case of SpGMVIP (2)-(3).

If we set $g_i = I_i$, where I_i is an identity operator on H_i , then SpGMVIP (2)-(3) is reduced to the following split mixed variational inequality problem (in short, SpMVIP): Find $x_1^* \in H_1$ such that $x_1^* \in \text{dom} \partial \varphi_1$ and

$$\langle f_1(x_1^*), x_1 - x_1^* \rangle + \varphi(x_1^*) - \varphi(x_1) \geq 0, \quad \forall x_1 \in H_1, \quad (4)$$

and such that $x_2^* = Ax_1^*$ and $x_2^* \in \text{dom} \partial \varphi_2$ solves

$$\langle f_2(x_2^*), x_2 - x_2^* \rangle + \varphi(x_2^*) - \varphi(x_2) \geq 0, \quad \forall x_2 \in H_2. \quad (5)$$

Which appears to be new.

If $\varphi_i: H \rightarrow R \cup \{+\infty\}$ is the indicator function of closed convex set $C_i \subset H_i$, then the split mixed variational inequality problem is reduced to split general variational inequality problem (in short, SpGVIP): Find $x_1^* \in C_1$ such that

$$\langle f_1(x_1^*), x_1 - g_1(x_1^*) \rangle \geq 0, \quad \forall x_1 \in C_1, \quad (6)$$

and such that $x_2^* = Ax_1^* \in C_2$ solves

$$\langle f_2(x_2^*), x_2 - g_2(x_2^*) \rangle \geq 0, \quad \forall x_2 \in C_2. \quad (7)$$

If $\varphi_i: H \rightarrow R \cup \{+\infty\}$ is the indicator function of closed convex set $C_i \subset H_i$, and $g_i = I_i$, where I_i is an identity operator on H_i , then the split mixed variational inequality problem is reduced to split variational inequality problem (in short, SpVIP): Find $x_1^* \in C_1$ such that

$$\langle f_1(x_1^*), x_1 - x_1^* \rangle \geq 0, \quad \forall x_1 \in C_1,$$

and such that $x_2^* = Ax_1^* \in C_2$ solves

$$\langle f_2(x_2^*), x_2 - g_2(x_2^*) \rangle \geq 0, \quad \forall x_2 \in C_2.$$

If we set $H_1 = H_2$, $f_1 = f_2$, $g_i = I_i$, then SpGMVIP (2)-(3) is reduced to MVIP (1).

Iterative algorithms

For each $i \in \{1, 2\}$, it is well known that

$$-f_i(x_i^*) \in \partial \varphi_i(g_i(x_i^*)).$$

Further, it is easy to see that the following is true:

$$g_i(x_i^*) - \rho_i f_i(x_i^*) \in (I + \rho_i \partial \varphi_i)g_i(x_i^*).$$

We have

$$g_i(x_i^*) = J_{\rho_i}^{\partial \varphi_i}(g_i(x_i^*) - \rho_i f_i(x_i^*)),$$

for $\rho_i \geq 0$, where $J_{\rho_i}^{\partial \varphi_i}: H_i \rightarrow H_i$ defined as $J_{\rho_i}^{\partial \varphi_i}(\cdot) = (I - \rho_i \partial \varphi_i)^{-1}(\cdot)$ is a resolvent operator of $\partial \varphi_i$.

Based on the above arguments, we propose the following iterative algorithm for approximating a solution to SpGMVIP (2)-(3). Let $\{\alpha^n\} \subseteq (0, 1)$ be a sequence such that $\sum_{n=1}^{\infty} \alpha^n = +\infty$.

Algorithm 1. Given $x_1^0 \in H_1$, compute the iterative sequences $\{x_1^n\}$ defined by the following of

iterative schemes:

$$g_1(y^n) = J_{\rho}^{\partial\varphi_1}(g_1(x_1^n) - \rho_1 f_1(x_1^n)), \quad (12)$$

$$g_2(z^n) = J_{\rho}^{\partial\varphi_2}(g_2(Ay^n) - \rho_2 f_2(Ay^n)), \quad (13)$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)] \quad (14)$$

For all $n = 0, 1, 2, \dots, \rho_1, \rho_2, \gamma > 0$.

If $g_i = I_i$, then Algorithm 1 is reduced to the following iterative algorithm for SpMVIP (4)-(5):

Algorithm 2. Given $x_1^0 \in H_1$, compute the iterative sequences $\{x_1^n\}$ defined by the following of iterative schemes:

$$g_1(y^n) = J_{\rho}^{\partial\varphi_1}(x_1^n - \rho_1 f_1(x_1^n)),$$

$$g_2(z^n) = J_{\rho}^{\partial\varphi_2}(Ay^n - \rho_2 f_2(Ay^n)),$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)]$$

For all $n = 0, 1, 2, \dots, \rho_1, \rho_2, \gamma > 0$.

If $\varphi_i: H \rightarrow R \cup \{+\infty\}$ is the indicator function of closed convex set $C_i \subset H_i$, then Algorithm 1 is reduced to the following iterative algorithm for SpGVIP (6)-(7):

Algorithm 3([2]). Given $x_1^0 \in C_1$, compute the iterative sequences $\{x_1^n\}$ defined by the following of iterative schemes:

$$g_1(y^n) = P_{C_1}(g_1(x_1^n) - \rho_1 f_1(x_1^n)),$$

$$g_2(z^n) = P_{C_2}(g_2(Ay^n) - \rho_2 f_2(Ay^n)),$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)]$$

For all $n = 0, 1, 2, \dots, \rho_1, \rho_2, \gamma > 0$.

If $\varphi_i: H \rightarrow R \cup \{+\infty\}$ is the indicator function of closed convex set $C_i \subset H_i$, and $g_i = I_i$, where I_i is an identity operator on H_i , then Iterative Algorithm 1 is reduced to the following iterative algorithm for SpVIP (8)-(9):

Algorithm 4([10]). Given $x_1^0 \in C_1$, compute the iterative sequences $\{x_1^n\}$ defined by the following of iterative schemes:

$$g_1(y^n) = P_{C_1}(x_1^n - \rho_1 f_1(x_1^n)),$$

$$g_2(z^n) = P_{C_2}(Ay^n - \rho_2 f_2(Ay^n)),$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)]$$

For all $n = 0, 1, 2, \dots, \rho_1, \rho_2, \gamma > 0$.

If we set $H_1 = H_2, f_1 = f_2, g_i = I_i$, for all $x_i, i \in \{1, 2\}$, then Iterative Algorithm 1 is reduced to the following iterative algorithm for MVIP (1.1):

Algorithm 5. Given $x_1^0 \in H_1$, compute the iterative sequences $\{x_1^n\}$ defined by the following of iterative schemes:

$$y^n = J_{\rho}^{\partial\varphi_1}(x_1^n - \rho_1 f_1(x_1^n)),$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n y^n.$$

Definition 1. A nonlinear mapping $f_1: H_1 \rightarrow H_1$ is said to be

(i) α_1 -strongly monotone if there exists a constant $\alpha_1 > 0$ such that

$$\langle f_1(x) - f_1(\bar{x}), x - \bar{x} \rangle \geq \alpha_1 \|x - \bar{x}\|^2 \quad \text{for all } x, \bar{x} \in H_1,$$

(ii) β_1 -Lipschitz continuous if there exists a constant $\beta_1 > 0$ such that

$$\|f_1(x) - f_1(\bar{x})\| \leq \beta_1 \|x_1 - \bar{x}\| \quad \text{for all } x, \bar{x} \in H_1.$$

Lemma1([9]). Let $\varphi: H \rightarrow R \cup \{+\infty\}$ be a proper convex lower semi-continuous function. Then for a constant $\rho > 0$, the resolvent operator of its subdifferential mapping $J_{\rho}^{\partial\varphi} = (I - \rho\partial\varphi)^{-1}$ is nonexpansive, that is

$$\|J_{\rho}^{\partial\varphi}(x) - J_{\rho}^{\partial\varphi}(\bar{x})\| \leq \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in H.$$

Main Results

Theorem1. For each $i \in \{1, 2\}$, let $g_i: H_i \rightarrow H_i$ be τ_i -Lipschitz continuous such that $(g_i - I_i)$ is σ_i -strongly monotone, where I_i is the identity operator on H_i . Let $f_i: H_i \rightarrow H_i$ be α_i -strongly monotone with respect to g_i and β_i -Lipschitz continuous. Let $A: H_1 \rightarrow H_2$ be a bounded linear operator and let A^* be its adjoint operator. Suppose $x_1^* \in H_1$ is a solution to SpGMVIP (2)-(3). Then the sequence $\{x_1^n\}$ generated by Algorithm1 converges strongly to x_1^* provided that the constant ρ_i, γ satisfy the following conditions:

$$\left| \rho_1 - \frac{\alpha_1}{\beta_1^2} \right| < \frac{\sqrt{\alpha_1^2 - \beta_1^2(1 - k_1^2)}}{\beta_1^2}, \alpha_1 > \beta_1 \sqrt{\tau_1^2 - k_1^2}, k_1 = \frac{\sqrt{2\sigma_1 + 1}}{1 + 2\theta_2}, k_1 < \tau_1,$$

$$\theta_2^2 = \frac{\tau_2^2 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}{2\sigma_2 + 1}, \rho_2 > 0, \gamma \in \left(0, \frac{2}{\|A\|^2}\right).$$

Proof. Since $x_1^* \in H_1$ is a solution to SpGMVIP (2)-(3), then we have

$$g_1(x_1^*) = J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)), \quad (15)$$

$$g_1(Ax_1^*) = J_{\rho_2}^{\partial\varphi_2}(g_2(Ax_1^*) - \rho_2 f_2(Ax_1^*)). \quad (16)$$

For $\rho_i > 0$. From Algorithm 1 (12), (15) and Lemma 1, we have

$$\begin{aligned} \|g_1(y^n) - g_1(x_1^*)\| &= \|J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^n) - \rho_1 f_1(x_1^n)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*))\| \\ &\leq \|g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*))\|. \end{aligned}$$

Next, since f_1 is α_1 -strongly monotone with respect to g_1 and β_1 -Lipschitz continuous, and g_1 be τ_1 -Lipschitz continuous, we have

$$\begin{aligned} &\|g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*))\|^2 \\ &= \|g_1(x_1^n) - g_1(x_1^*)\|^2 - 2\rho_1 \langle f_1(x_1^n) - f_1(x_1^*), g_1(x_1^n) - g_1(x_1^*) \rangle + \rho_1^2 \|f_1(x_1^n) - f_1(x_1^*)\|^2 \\ &\leq (\tau_1^2 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2) \|x_1^n - x_1^*\|^2. \end{aligned}$$

As a result we obtain

$$\|g_1(y^n) - g_1(x_1^*)\| \leq \sqrt{\tau_1^2 - 2\rho_1\alpha_1 + \beta_1^2} \|x_1^n - x_1^*\|. \quad (17)$$

Since $(g_1 - I_1)$ is σ_1 -strongly monotone, we have

$$\begin{aligned} \|y^n - x_1^*\|^2 &\leq \|g_1(y^n) - g_1(x_1^*)\|^2 - 2\langle (g_1 - I_1)y^n - (g_1 - I_1)x_1^*, y^n - x_1^* \rangle \\ &\leq \|g_1(y^n) - g_1(x_1^*)\|^2 - 2\sigma_1 \|y^n - x_1^*\|^2. \end{aligned}$$

Which implies

$$\|y^n - x_1^*\| \leq \frac{1}{\sqrt{2\sigma_1 + 1}} \|g_1(y^n) - g_1(x_1^*)\|. \quad (18)$$

From (17) and (18), we have

$$\|y^n - x_1^*\| \leq \theta_1 \|x_1^n - x_1^*\|, \quad (19)$$

where $\theta_1 = \sqrt{\frac{\tau_1^2 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2}{2\sigma_1 + 1}}$. Similarly, from Algorithm 1 (13), (16), Lemma 1 and using the

facts that f_2 is α_2 -strongly monotone with respect to g_2 and β_2 -Lipschitz continuous, $(g_1 - I_1)$ is σ_1 -strongly monotone, and g_2 is τ_2 -Lipschitz continuous, we have

$$\|g_1(z^n) - g_1(Ax_1^*)\| \leq \sqrt{\tau_2^2 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} \|Ay^n - Ax_1^*\|. \quad (20)$$

$$\|z^n - Ax_1^*\| \leq \theta_2 \|Ay^n - Ax_1^*\|, \quad (21)$$

where $\theta_2 = \sqrt{\frac{\tau_2^2 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}{2\sigma_2 + 1}}$. Next, from Algorithm 1 (14), we have

$$\|x_1^{n+1} - x_1^*\| \leq (1 - \alpha^n) \|x_1^n - x_1^*\| + \alpha^n [\|y^n - x_1^* - \gamma A^*(Ay^n - Ax_1^*)\| + \gamma \|A^*(z^n - Ax_1^*)\|] \quad (22)$$

Further, using the definition of A^* , the fact that A^* is a bounded linear operator with $\|A^*\| = \|A\|$, and given condition on γ , we have

$$\begin{aligned} & \|y^n - x_1^* - \gamma A^*(Ay^n - Ax_1^*)\|^2 \\ &= \|y^n - x_1^*\|^2 - 2\gamma \langle y^n - x_1^*, A^*(Ay^n - Ax_1^*) \rangle + \gamma^2 \|A^*(Ay^n - Ax_1^*)\|^2 \\ &\leq \|y^n - x_1^*\|^2 - \gamma(2 - \gamma\|A\|^2) \|Ay^n - Ax_1^*\|^2 \\ &\leq \|y^n - x_1^*\|^2. \end{aligned} \quad (23)$$

From (21), we have

$$\|A^*(z^n - Ax_1^*)\| \leq \|A\| \|z^n - Ax_1^*\| \leq \theta_2 \|A\| \|Ay^n - Ax_1^*\| \leq \theta_2 \|A\|^2 \|y^n - x_1^*\|. \quad (24)$$

Combining (23) and (24) with inequality (3.8), we have

$$\|x_1^{n+1} - x_1^*\| \leq [1 - \alpha^n(1 - \theta)] \|x_1^n - x_1^*\|,$$

where $\theta = \theta_1(1 + \gamma\|A\|^2\theta_2)$. Hence, after n iterations, we obtain

$$\|x_{n+1} - x^*\| \leq \prod_{j=1}^n [1 - \alpha_j(1 - \theta)] \|x_0 - x^*\|. \quad (25)$$

It follows from the conditions on ρ_1 and ρ_2 that $\theta \in (0, 1)$. Since $\sum_{n=1}^{\infty} \alpha^n = +\infty$ and $\theta \in (0, 1)$, this implies in the light of [13] that

$$\lim_{n \rightarrow \infty} \prod_{j=1}^n [1 - \alpha_j(1 - \theta)] = 0.$$

Thus it follows from (3.11) that $\{x_n\}$ converges strongly to x^* as $n \rightarrow +\infty$. Since A is continuous, it follows from (17) and (19)-(21) that $y^n \rightarrow x_1^*$, $g_1(y^n) \rightarrow g_1(x_1^*)$, $Ay^n \rightarrow Ax_1^*$, $z^n \rightarrow Ax_1^*$ and $g_2(z^n) \rightarrow g_2(Ax_1^*)$ as $n \rightarrow +\infty$. This completes the proof.

If we set $g_i = I_i$, then Theorem 1 reduces to the following result for the convergence of Algorithm 2 for SpMVIP (4) -(5).

Corollary 2. For each $i \in \{1, 2\}$, let $f_i : H_i \rightarrow H_i$ be α_i -strongly monotone and β_i -Lipschitz continuous. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and let A^* be its adjoint operator. Suppose $x_1^* \in H_1$ is a solution to SpMVIP (4)-(5). Then the sequence $\{x_1^n\}$ generated by Algorithm 2 converges strongly to x_1^* provided that the constants ρ_i and γ satisfy the following conditions:

$$\left| \rho_1 - \frac{\alpha_1}{\beta_1^2} \right| < \frac{\sqrt{\alpha_1^2 - \beta_1^2(1 - k_1^2)}}{\beta_1^2}, \alpha_1 > \beta_1 \sqrt{1 - k_1^2}, k_1 = \frac{1}{1 + 2\theta_2},$$

$$\theta_2^2 = 1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2, \rho_2 > 0, \gamma \in \left(0, \frac{2}{\|A\|^2}\right).$$

If we set $H_1 = H_2, f_1 = f_2, g_i = I_i$, for all $x_i, i \in \{1, 2\}$, then Theorem 1 is reduced to the following result for the convergence of Algorithm 5 for MVIP (1).

Corollary 3. For each $i \in \{1, 2\}$, let $f_i : H_i \rightarrow H_i$ be α_i -strongly monotone and β_i -Lipschitz continuous. Suppose $x_1^* \in H_1$ is a solution to MVIP (1). Then the sequence $\{x_1^n\}$ generated by Algorithm 5 converges strongly to x_1^* provided that the constants ρ_1 satisfies $0 < \rho_1 < \frac{2\alpha_1}{\beta_1^2}$.

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