Application of the Regularization Strategy in Solving Numerical Differentiation for Function with Error

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Abstract. In many practical problems, it is sometimes necessary to evaluate the derivative of function whose values are given approximately. Firstly, the problem of estimating the derivative of a function observed with error is studied. It presents a proper regularization strategy and explain how to choose regularization parameter. Secondly, the regularization strategy above to the numerical differentiation is applied and discussed in the implementation of the numerical method and the tests which it has performed in order to investigate the accuracy and stability of the numerical differentiation procedure. Finally, some numerical examples will further illustrate that this method is reasonable, effective and reasonable.

Introduction

In recent years, inverse problems in mathematical physics have been one of the fastest growing areas. However, inverse problems are closely linked to the ill-posed, and due to a great deal of difficulty to numerical solution. At the beginning of 1960s, the regularization strategy is proposed by Tikhonov creatively[1]. From then on, Dinh Nho Hao studied the mollification method for ill-posed problems[2]. Y F Wang, W Q Liang, J T Zhao conducted a lot of research[3,4,5]. As it is known, the choice of the regularization parameter is a key matter for ensuring proper regularization. There are many problems that can be described by numerical differentiation in the natural science and engineering technology field. It is easy to imagine many different situations--- mostly involving ordinary and partial differential equations—related with the question of numerical differentiation of noisy (measured) data. So it is sometimes necessary to evaluate the derivative of function whose values are given approximately. How to get the numerical solution of these functions with noisy data has become a special course. In this paper, the operator with Gaussian kernel is studied and its reasonable regularization parameters in an efficient manner is introduced.

Therefore, a lot of numerical examples show that the process has good stability and high accuracy. The general method to solve the ill-posed problem is to approximate the solution of the original problem with a set of well-posed problems. How to establish an effective regularization method is an important part of the research on the problem of ill-posed problems in the field of inverse problem. J J Cao, Y F Wang and B F Wang[6,7] conducted a lot of research.

Regularization Strategy

Theorem 1: Let $g_\alpha \in L^1(R^n)$ and $\int_{R^n} g_\alpha(x)dx = 1$,

(1) if $f \in L^p(R^n)$, where $1 \le p < +\infty$, then $\lim_{\alpha\to 0} \| g_\alpha * f - f\|_p = 0$;

(2) if $f \in L^\infty(R^n)$, then $\lim_{\alpha\to 0}(g_\alpha * f)(x) = f(x)$, where x is the continuous point of f.

Proof: Set $g_\alpha * f = f_\alpha$, $f_\alpha(x) - f(x) = \int_{R^n} [f(x-y) - f(x)]g_\alpha(y)dy$.

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By the generalized Minkowski inequality,

$$\| f_\alpha - f \|_p = \left\{ \int_{R^n} \left| \int_{R^n} [f(x-y) - f(x)] g_\alpha(y) dy \right|^p dx \right\}^{1/p} \le \int_{R^n} [\int_{R^n} | f(x-\alpha t) - f(x) |^p dx]^{1/p} |g(t)| dt .$$

To every $t \in R^n$,

$$\lim_{\alpha \to 0} \left[\int_{R^n} | f(x-\alpha t) - f(x) |^p dx \right]^{1/p} = 0$$

and

$$\left[\int_{R^n} | f(x-\alpha t) - f(x) |^p dx \right]^{1/p} \le 2 \| f \|_p .$$

By dominated convergence theorem,

$$\int_{R^n} [\int_{R^n} | f(x-\alpha t) - f(x) |^p dx]^{1/p} |g(t)| dt \to 0 .$$

When $\alpha \to 0$. (1) has already proved.

Because $f \in L(R^n)$ and x is a continuous point of f, then

$$| f_\alpha(x) - f(x) | \le \int_{R^n} |(f(x-y) - f(x)) g_\alpha(y)| dy = \int_{R^n} | f(x-\alpha t) - f(x) || g(t) | dt .$$

To every $t \in R^n$,

$$\lim_{\alpha \to 0} | f(x-\alpha t) - f(x) | = 0$$

and

$$| f(x-\alpha t) - f(x) | \le 2 \| f \| .$$

By dominated convergence theorem,

$$\int_{R^n} | f(x-\alpha t) - f(x) || g(t) | dt \to 0, \text{ when } \alpha \to 0 .$$

(2) has already proved.

By Theorem 1, g_α is needed to choose properly, then $g_\alpha * f \to f$ when $\alpha \to 0$.

The Gaussian kernel $g_\alpha(t)$ is defined by $g_\alpha(t) = \frac{1}{\alpha\sqrt{\pi}} \exp(-t^2/\alpha^2)$, $t \in R$,

where $\alpha > 0$ denotes a parameter. Then $\int_{-\infty}^{+\infty} g_\alpha(t) dt = 1$ and the convolution

$$T_\alpha f(t) = (g_\alpha * f)(t) = \int_{-\infty}^{+\infty} g_\alpha(t-s) f(s) ds = \int_{-\infty}^{+\infty} g_\alpha(s) f(t-s) ds,\ t \in R$$

exists and is an L^2 -function for every $f(t) \in L^2(R)$. Furthermore, by Young's inequality,

$\| T_\alpha f \|_{L^2} = \| g_\alpha * f \|_{L^2} \le \| g_\alpha \|_{L^1} \| f \|_{L^2} = \| f \|_{L^2}, \forall f \in L^2(R)$ is obtained.

Therefore, the operator $f \to g_\alpha * f$ is uniformly bounded in $L^2(R)$ with respect to α. So regard $T_\alpha f(t)$ as regularization operator.

However, in practical application, $f(t)$ cannot be accurately given. The noisy (measured) function $f_\delta(t)$ which satisfied the error bound is obtained:

$$\| f(t) - f_\delta(t) \|_{\infty, I} \le \delta$$

in the data interval $I = [0,1]$.

Theorem 2: (Error estimation):

$$\begin{aligned} |T_\alpha f_\delta(t) - f(t)| &= |T_\alpha f_\delta(t) - T_\alpha f(t) + T_\alpha f(t) - f(t)| \\ &\le |T_\alpha f_\delta(t) - T_\alpha f(t)| + |T_\alpha f(t) - f(t)| \le \delta + C\alpha \end{aligned}$$

where α is a constant. So $T_\alpha f_\delta(t)$ is used to compute the approximate derivative of the exact $f(t)$.

The original ill-posed problem of finding f is replaced by new problem of finding $T_\alpha f_\delta$. The new problem is well-posed, and depends on a parameter $\delta > 0$.

For given $T_\alpha f_\delta$, regularization parameter α is choosed by Morozov deviation principle or by GCV[8,9].

In the numerical computation, Newton method is used as follows:

Step1: Set initial value α_0, then we compute $f_0 = f(\alpha_0)$, $f_0' = f'(\alpha_0)$,
where $f(\alpha) = \| T_\alpha f_\delta - f_\delta \|^2 - \delta^2$.

Step2: Compute $\alpha_1 = \alpha_0 - \dfrac{f_0}{f_0'}$, $f_1 = f(\alpha_1)$, $f'_1 = f'(\alpha_1)$.

Step3: If α_1 satisfied $|\eta| < \varepsilon_1$ or $|\eta| < \varepsilon_2$, then $\alpha = \alpha_1$;
else go to step 4.

Where $\eta = \begin{cases} |\alpha_1 - \alpha_0|; & |\alpha_1| < C \\ \dfrac{|\alpha_1 - \alpha_0|}{\alpha_1}; & |\alpha_1| \ge C \end{cases}$, C is a control constant.

Step4: if $f_1' = 0$, then over,
otherwise alternate (α_0, f_0, f_0') by (α_1, f_1, f_1'), then go to Step1.

Numerical Result

In this section , the regularization strategy is applied above to the numerical differentiation, and discussed the implementation of the numerical method and the tests which it performed in order to investigate the accuracy and stability of the numerical differentiation procedure[10,11].

In the examples, the exact data function is denoted by $f(t)$ and the noisy function $f_\delta(t)$. $f_\delta(t)$ is obtained by adding an random error or an high frequency disturbance error to $f(t)$.

That is $f_\delta(t_i) = f(t_i)(1+\delta\sigma_i)$, where $t_i = ih; i = 0,1,2\cdots\cdots n; nh = 1$, and σ_i is a uniform random variable with values in [-1,1], such that $\max_{0\le i\le n}\left|f_\delta(t_i) - f(t_i)\right| \le \delta$ or $f_\delta(t_i) = f(t_i)(1+\delta\sin(\omega t_i))$, where $t_i = ih; i = 0,1,2\cdots\cdots n; nh = 1$, and ω is a perturbation frequency.

Example 1: The first example is rather oscillatory on [0,1]. We choose $f(t) = \sin(10\pi t)$ and the exact derivative $f^{/}(t) = 10\pi\cos(10\pi t)$. In table1, we give the error between the exact derivative of $f(t) = \sin(10\pi t)$ and the solution obtained with the regularization strategy. In order to illustrate the numerical approximation to the derivative $f^{/}(x)$ denoted $f_\delta^{/}(x)$, $error_{2,h} = (\frac{1}{n}\sum_{i=1}^{n}\left|f_\delta^{/}(t_i) - f^{/}(t_i)\right|^2)^{\frac{1}{2}}$ is applied.

Table 1. This is the the error between the exact derivative and regularization derivative

δ	*error* $n=100$, $\omega=100$	$n=100$, $\omega=200$	$n=200$, $\omega=100$	$n=200$, $\omega=200$
0.001	0.49067	0.49201	0.11878	0.12708
0.010	0.58587	0.55242	0.45855	0.42768
0.050	1.70032	1.32859	2.24404	3.25952
0.100	3.36879	2.78627	4.52232	3.27416

Example 2: The exact function $f(t) = normcdf(t,2,3)$, solve $\rho(x)$, such that $\int_{-\infty}^{+\infty}\rho(t)dt = f(t)$.

The complete algorithm for the regularization strategy is as follows:

Step1: Provide the exact discrete data $f_i(t)$ of $f(t)$ by MATLAB.

Step 2: Obtain the noisy function $f_\delta(t_i)$ by adding a δ high frequency disturbance error to $f(t_i)$.

Step 3: Compute the approximate solution $\rho_\delta(x)$ of $\rho(x)$.

Step 4: Compare $\rho_\delta(x)$ with $\rho(x)$.

In experience the exact $\rho(x)$ and error function *error* are given as follow:

$\rho(x) = \frac{1}{2\sqrt{2\pi}}\exp(-\frac{2}{2\times 2^2}(t-3)^2)$, $error = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\rho(t_i) - \rho_\delta(t_i))^2}$.

Table 2. This is the the error between the exact derivative and regularization derivative(ω=217)

	error (ω=217) δ=0.1	δ=0.01	δ=0.001
regularization stratery	0.1231	0.0123	0.0012
bisection method	0.1428	0.0143	0.0014
interpolation	2.5203	0.2520	0.0252
spline	0.3270	0.0327	0.0033
extrapolation	0.2025	0.0202	0.0020
discrete regularization(α unknown)	0.0143	0.0022	7.2694e-004
discrete regularization(α known)	0.0133	0.0021	7.2634e-004

Table3. This is the the error between the exact derivative and regularization derivative(ω =100)

	error (ω =100)		
	δ =0.1	δ =0.01	δ =0.001
regularization stratery	0.3215	0.0322	0.0032
bisection method	0.2261	0.0226	0.0023
interpolation	2.2087	0.2209	0.0221
spline	0.5400	0.0540	0.0054
extrapolation	0.3434	0.0343	0.0034
discrete regularization(α unknown)	0.0126	0.0020	6.1855e-004
discrete regularization(α known)	0.0130	0.0023	6.1835e-004

Conclusions

In many practical problems, it is sometimes necessary to evaluate the derivative of function whose values are given approximately. Firstly, the problem of estimating the derivative of a function observed with error is studied. It presents a proper regularization strategy and explain how to choose regularization parameter. Secondly, the regularization strategy above to the numerical differentiation is applied and discussed in the implementation of the numerical method and the tests which it has performed in order to investigate the accuracy and stability of the numerical differentiation procedure. Finally, some numerical examples will further illustrate that this method is reasonable, effective and reasonable[12].

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