Combination Forecasting Model Using Grey Verhulst Models Coupling to Regression Analysis

Qing Shu and Xinping Xiao*

College of Science, Wuhan University of Technology, Wuhan 430070, China

*Corresponding author

Abstract—Firstly, grey verhulst model based on fractional order accumulate is deduced. Then, take advantage of differential and difference to find the corresponding relationship between the grey verhulst model of each order and the regression equation. Giving the level of significance α , carry out the significance test for the regression equations. Picking up the orders of grey verhulst models corresponding to the first two regression equations which are best-fit for the original time series, then establish the grey verhulst models which have the orders picked up before. Lastly, using these two grey verhulst models to establish the weight arithmetic average combination forecasting model based on grey associate analysis. The practical example indicates that the fitting precision of our model is improved significantly comparing with single forecasting models, logistic model and fractional order accumulate grey verhulst model. It is proved to be an available and practical method for forecasting.

Keywords- grey verhulst model; fractional order accumulation; significance level; combination forecasting; regression analysis

I. INTRODUCTION

Fractional order contains a thought of ‘in between’. In these years, it gets a wide range of recognition. More and more scholars use the thought of fractional order accumulation into grey model. The fractional order accumulate discrete grey model has defined in order to reduce perturbation bound and the question why grey model is suitable for small sample has been explained^[1]. Wu Li-feng^[2] has studied GM (1, 1) model based on fractional order accumulating method and its stability. He has expanded integer order accumulating to fractional order accumulating. Fractional order accumulating has reduced the perturbation bound and relieved the contradictory between the less impact on solution by the new information and the principle of new information priority. Mao Shu-hua^[3] has transformed traditional time-lag GM(1,N, τ) model into fractional order accumulation time-lag GM(1,N, τ) after introducing fractional order accumulation generation operator. The example indicated that the accuracy of fractional order accumulation time-lag GM(1,N, τ) is more satisfactory. Yang Bao-hua^[4] has transformed grey discrete power GM (1, 1) into fractional order grey discrete power GM (1, 1), and application examples showed the higher precision accuracy of the new model. Chen Ye^[5] has extended the original r-AGO technique from integer domain to the real number field, and put forward the corresponding GM (1, 1) model with its solving process.

When evaluate a model, fitting and forecasting effect and the stability of the model are used^[1]. Fractional order

accumulate grey model remits the shortage on the stability of traditional grey model, and reduces the perturbation bound. Application examples also show that compared to the traditional grey model, fitting accuracy of fractional order accumulate grey model is improved to some extent. In this paper, fractional order accumulate grey verhulst model is deduced at first. Then, a new combinational forecasting method based on that model is proposed. Comparing to the fractional order accumulate grey verhulst model, this combination forecasting model can improve the fitting accuracy significantly.

II. COMBINATION FORECAST MODEL USING GREY VERHULST MODELS COUPLING TO REGRESSION ANALYSIS

A. Fractional Order Accumulate Grey Verhulst Model

Definition 2.1. Suppose

$y^{(0)} = (y^{(0)}(1), y^{(0)}(2), \dots, y^{(0)}(n))^T$ is original sequence, d is transform operator of original sequence, $x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n))^T$ is transform sequence. $y^{(0)}(k)d = x^{(0)}(k), k = 1, 2, \dots, n$; A^r is accumulative generator matrix^[5] when the order is r . Then, the sequence $x^{(r)}$ is called r-order accumulative generator sequence which is generated from $x^{(r)} = A^r x^{(0)}$.

$$x^{(r)} = \begin{bmatrix} x^{(r)}(1) \\ x^{(r)}(2) \\ \vdots \\ x^{(r)}(n) \end{bmatrix} = \begin{bmatrix} 1 & 0 & L & 0 \\ r & 1 & L & 0 \\ C_{r+1}^2 & r & L & 0 \\ M & M & O & M \\ C_{r+n-2}^{n-1} & C_{r+n-3}^{n-2} & L & 1 \end{bmatrix} g \quad (1)$$

$$\begin{bmatrix} x^{(0)}(1) \\ x^{(0)}(2) \\ \vdots \\ x^{(0)}(n) \end{bmatrix} = A^r x^{(0)}$$

Definition 2.2. Grey verhulst model based on fractional order accumulate:

$$x^{(r)}(k) - x^{(r)}(k-1) + ax^{(r)}(k) = b(z^{(r)}(k))^2 \quad (2)$$

There is $x^{(r)} = A^r x^{(0)}$, $z^{(r)}(k) = 0.5x^{(r)}(k) + 0.5x^{(r)}(k-1)$.

Theory 2.1. (1) The lever 1 parameters pack of fractional order accumulate grey verhulst model is $P = (a, b)^T$, there is matrix formula under the least squares principle:

$$P = \begin{bmatrix} a \\ b \end{bmatrix} = (B^T B)^{-1} B^T X \quad (3)$$

And

$$B = \begin{bmatrix} -z^{(r)}(2) & (z^{(r)}(2))^2 \\ -z^{(r)}(3) & (z^{(r)}(3))^2 \\ \vdots & \vdots \\ -z^{(r)}(n) & (z^{(r)}(n))^2 \end{bmatrix}, X = \begin{bmatrix} x^{(r)}(2) - x^{(r)}(1) \\ x^{(r)}(3) - x^{(r)}(2) \\ \vdots \\ x^{(r)}(n) - x^{(r)}(n-1) \end{bmatrix}.$$

(2) The level 2 parameters pack of fractional order accumulate grey verhulst model is $(s(i), D, E)$ with $s(i) = \sum_{k=2}^n (z^{(r)}(k))^{(i)}$, $D = \sum_{k=2}^n x^{(r-1)}(k)z^{(r)}(k)$, $E = \sum_{k=2}^n x^{(r-1)}(k)(z^{(r)}(k))^2$ so

$$P = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{s(2)s(2) - s^2(3)} \begin{bmatrix} s(3)E - s(4)D \\ s(2)E - s(3)D \end{bmatrix} \quad (4)$$

Theory 2.2. Under the boundary condition $\hat{x}^{(r)}(1) = x^{(r)}(1)$, the white response of fractional order accumulate grey verhulst model is

$$\hat{x}^{(r)}(k+1) = \frac{ax^{(r)}(1)}{bx^{(r)}(1) + (a - bx^{(r)}(1))e^{ak}} \quad (5)$$

With

$$\hat{x}^{(0)} = (A^r)^{-1} \hat{x}^{(r)}, \hat{y}^{(0)} = \hat{x}^{(0)} d^{-1}$$

Intelligent algorithms such as genetic algorithm are used to search order r .

B. Choose the Optimal Grey Verhulst Model Based on Regression Equation

Grey Verhulst model usually uses 1-AGO on the original sequence $x^{(0)}(t)$ to generate accumulative sequence $x^{(1)}(t)$, and then we can get the mean value generating sequence $z^{(1)}(t)$.

Lastly, $x^{(0)}(t)$, $x^{(1)}(t)$ and $z^{(1)}(t)$ are used to set up grey verhulst model:

$$x^{(0)}(k) + az^{(1)}(k) = b(z^{(1)}(k))^2 \quad (6)$$

And its whitenization equation is

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b(x^{(1)}(t))^2 \quad (7)$$

Define $x^{(1)}(t) = \int_0^t x^{(0)}(s)ds \approx \sum x^{(0)}(t)\Delta t$. When Δt is small, the derivative of $x^{(1)}(t)$ to the time t is $x^{(1)'}(t) = x^{(0)}(t) \approx \frac{x^{(1)}(t+\Delta t) - x^{(1)}(t)}{\Delta t}$. Especially take $\Delta t = 1$, there is $x^{(1)'}(t) \approx x^{(1)}(t+1) - x^{(1)}(t)$, so $x^{(1)}(t) \approx \sum x^{(0)}(x)$, $x^{(1)'}(t) = x^{(0)}(t) \approx x^{(1)}(t+1) - x^{(1)}(t)$, then the formula (7) can be turned to:

$$x^{(1)}(k) - x^{(1)}(k-1) + ax^{(1)}(k) = b(x^{(1)}(k))^2 \quad (8)$$

And

$$\frac{x^{(1)}(k) - x^{(1)}(k-1)}{x^{(1)}(k)} + a = b(x^{(1)}(k)) \quad (9)$$

It equivalent to

$$\frac{x^{(0)}(k)}{x^{(1)}(k)} = b x^{(1)}(k) - a \quad (10)$$

Set $\frac{x^{(0)}(k)}{x^{(1)}(k)} = Y(k)$ as dependent variable and $x^{(1)}(k)$ as independent variable. The linear relationship between $Y(k)$ and $x^{(1)}(k)$ is represented as $Y(k) = b x^{(1)}(k) - a$. If the regression equation could fit the original sequence $x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n))^T$ well, then the corresponding grey verhulst model (6) has a better fitting effect.

Significance test is given to the linear regression equation $Y(k) = b x^{(1)}(k) - a$ while significance level is α . If

the test is passed, the traditional grey verhulst model $x^{(1)}(k) - x^{(1)}(k-1) + az^{(1)}(k) = b(z^{(1)}(k))^2$ can be set up.

$$Y(k) = \frac{x^{(1)}(k)}{x^{(2)}(k)}$$

Similarly, if $Y(k)$ is dependent variable and $x^{(2)}(k)$ is independent variable, the grey verhulst model generated from 2-AGO: $x^2(k) - x^{(2)}(k-1) + az^{(2)}(k) = b(z^{(2)}(k))^2$ can be set up as the same way.

The order of the accumulative sequence within the independent corresponds to the order of grey verhulst model. Giving the significance lever α and repeating these significance tests until the regression equation can't pass through.

The value of coefficient of determination R^2 reflects to the fitting degree of regression equation. Comparing all R^2 of the regression equations which are pass the significance test and choosing the best two regression equations, the corresponding orders of the accumulative sequence within the independent are orders of the most optimal single grey verhulst models.

C. Combination Forecasting Model Based on Grey Correlation Analysis

Choose the best two optimal single grey verhulst models to set up weighted arithmetic means combination forecasting model based on grey correlation analysis.

Wang Yingming^[6] put forward that forecasting results can be evaluated comprehensively from five aspects at least. They are Sum Squared Error (SSE), Mean Absolute Error (MAE), Mean Squared Error (MSE), Mean Absolute Percentage Error (MAPE) and Mean Squared Percentage Error (MSPE). If just choose one of these error indicators to evaluate the forecasting result, it is not that comprehensive. In addition to these five cost indices, this paper intend to add a benefit index—grey absolute correlation degree—to evaluate the forecasting result of the combination model. Tang Shaoliang^[7] gives the definition of grey absolute correlation degree between single forecast sequence, combination forecast sequence with original sequence.

Take the method of reference [8], standardize the decision matrix and get the weight vector of indicators, then the positive and negative ideal solutions can be confirmed. Calculate the optimum subordinate degrees and normalize them, and then we can get the weight of solutions.

III. EXAMPLE ANALYSIS

Choose data of carbon emissions^[9] (Table 1) of Shanxi province from 2002 to 2010 to set up combination forecasting model this paper proposed upon.

A. Choose the Optimal Grey Verhulst Model Based on Regression Equation

Take $X^{(0)}$ as the original sequence, suppose dependent variable is $Y(t) = \frac{X^{(-1)}(t)}{X^{(0)}(t)}$, and independent variable is $X^{(0)}(t)$, then we can get $a = 0.9185$, $b = -0.0001$ and the regression equation $y = -0.0001x + 0.9185$. Giving significance level $\alpha = 0.05$, we can get $R^2 = 0.6299$, obviously pass through the significance test. Then we can set up straight grey verhulst model^[10] from the regression equation upon.

Similarly, suppose dependent variable is $Y(t) = \frac{X^{(0)}(t)}{X^{(1)}(t)}$, then we can get the regression equation $y = -0.00002x + 0.6795$. Giving significance level $\alpha = 0.05$, we can get $R^2 = 0.6126$, obviously pass through the significance test. This regression equation corresponds to traditional grey verhulst model.

Set up corresponding regression equation successively. When the significance level is $\alpha = 0.05$, give significance test to all these regression equation. Then transfer all the regression equation which pass through the significance test into grey verhulst model which has the corresponding order, such as 2-AGO grey verhulst model, 3-AGO grey verhulst model and so on.

In all the regression equations which have passed through significance test, the largest two R^2 which show best fitness to original sequence are correspond to straight grey verhulst model and traditional grey verhulst model. Set up corresponding single grey verhulst model M_1 and M_2 separately.

As to formula (2), substitute into $r=0$, straight grey verhulst model can be get according to theory 2.1.1:

$$x^{(0)}(k) - x^{(0)}(k-1) - 0.2374z^{(0)}(k) = -0.00003397(z^{(0)}(k))^2 \text{ . Its solution is } \hat{x}^{(0)}(k+1) = \frac{473.661}{0.0678+0.1696e^{-0.2374k}}, k = 0, 1, \dots \text{ . We}$$

can get the fitting data of carbon emission in Shanxi province and all indicators we will take to evaluate the model. The results will be shown in Table 1. And substitute $r=1$ to formula (2) in the same way.

B. Combination Forecasting Model Based on Grey Correlation Analysis

Choose the best two optimal single grey verhulst models upon to set up weighted arithmetic means combination forecasting model based on grey correlation analysis. The standardized decision matrix can be gotten. And the weight of indicators

is $\omega = [0.235552, 0.183359, 0.185701, 0.186147, 0.189086, 0.020154]$. Then, we can get the positive and negative ideal solutions

Calculate the grey correlation degree and subordinate degree between solutions M_i with M^+ and M^- . Then we can

get the weight of solutions $k_1 = 0.871367, k_2 = 0.128633$, and the results of combination forecasting model are shown in Table I.

C. Comparing Fitting Effect Between Different Models

The fractional order grey verhulst model is set up using

genetic algorithm which base on the object that minimum MAPE to search the optimal order which is $r = 0.3740$, and parameters are $a = -0.3590, b = -0.00003$. The fitting values and corresponding indicators are also shown in Table I.

TABLE I. FITTING VALUES AND VALUE OF EVALUATE INDICATORS

Year	Data of Carbon Emission	M_1	M_2	Combination Forecast Model	Logistic Model	Fractional Order Grey Verhulst Model
2002	1995.5	1995.5	1995.5	1995.5	2233.8	1995.5
2003	2446.6	2350.283	965.9958	2185.95	2494.6	1942.993
2004	2950.8	2733.602	1388.821	2573.959	2786	2312.352
2005	3366.7	3137.144	1947.822	2995.956	3111.4	2807.031
2006	3715.2	3550.535	2638.909	3442.313	3474.7	3350.943
2007	3970	3962.341	3412.418	3897.058	3880.6	3881.852
2008	4373.5	4361.3	4156.285	4336.962	4333.8	4342.443
2009	4580.6	4737.52	4708.558	4734.081	4839.9	4690.671
2010	5152.5	5083.372	4915.821	5063.482	5405.2	4908.514
SSE		460002.082	8234054.384	165871.833	349930.5	1187528
MAE		181.455692	741.909567	105.960077	176.444444	282.1381
MSE		678.234533	2869.504205	407.273659	591.549237	1089.737
MAPE		0.055036	0.234404	0.030875	0.052522	0.087464
MSPE		0.216817	0.965947	0.122255	0.184133	0.360269
Absolute of Grey Incidence		0.934421	0.833432	0.950512	0.940991	0.905239

The fitting values of logistic model come out of reference [9].

From Table 1, it shows that comparing to single grey verhulst models, fractional order accumulate grey verhulst model and logistic model, combination forecasting model using grey verhulst models with different orders coupling to regression analysis proposed in this paper has smallest value in five error indicators and largest value in grey absolute correlation degree. It has the largest value of grey absolute correlation degree which is 0.9505, and has the smallest value of MAPE which is 3.09%. Comparing to MAPE of fractional order grey verhulst model, it reduces nearly 6%. It indicates that the method proposed in this paper improves the fitting precision, and has a better fitting effect than fractional order accumulate grey verhulst model.

IV. CONCLUSIONS

Traditional combination forecasting model combines more than two different kinds of models, but in this paper, the same kind of models with different orders are combined. Because

bases on the grey model's character of AGO (Accumulated Generating Operation), this method can excavate the information from original sequence fully, and provide new idea for combination forecasting model. This paper proposes using the significance test of regression equation to choose appropriate single grey verhulst models with different orders and combining to forecast. Owing to the corresponding relations between regression equations and grey verhulst models with each order, the regression equation which has the best fitting effect corresponds to the grey verhulst model with best fitting effect. This can be very persuasive. The practical example proves that the method this paper proposes improves the fitting precision and provides a new idea for combination forecasting model.

ACKNOWLEDGMENT

We are very grateful to the research group of Grey System in Wuhan University of Technology for providing meaningful suggestions while this paper is being written. This paper is supported by National Natural Science Foundation of China (Grant No. 71540027, Grant No. 51479151), and Research

Fund for the Doctoral Program of Higher Education of China
(Grant No. 20120143110001).

REFERENCES

- [1] Wu Li-feng, Liu Si-feng and Yao Li-gen, "Discrete grey model based on fractional order accumulate," *Systems Engineering—Theory &Practice*, Vol.34, No.7, July 2014, pp. 1822-1827.
- [2] Wu Li-feng, Liu Si-feng and Liu Jian, "GM(1,1) model based on fractional order accumulating method and its stability," *Control and Decision*, Vol.29, No.7, May 2014, pp. 919-924.
- [3] Mao Shu-hua, Gao Ming-yun, Xiao Xin-ping, "Fractional order accumulation time-lag GM(1,N, τ) model and its application," *Systems Engineering—Theory &Practice*, Vol.35, No.2, Feb 2015, pp. 430-436.
- [4] Yang Bao-hua, Zhao Jin-shuai, "Fractional order discrete grey GM(1,1) power model and its application," *Control and Decision*, Vol.30, No.7, Jul 2015, pp. 1264-1268.
- [5] Chen Ye, *Modeling Research on Fractional Order Accumulation in Short-Term Traffic Flow Forecasting*, Wuhan: Wuhan University of Technology, 2013.
- [6] Wang Ying-ming, "Research on the methods of combining forecasts based on correlativity," *Forecasting*, Vol.21, No.2, Feb 2002, pp. 58-62.
- [7] Tang Shao-liang, Li Nan, Gong Zai-wu, "Research on properties of combination forecasting model based on absolute of grey incidence," *System Engineering and Electronic*, Vol.30, No.1, Jan 2008, pp. 89-92.
- [8] Yan Shu-li, *Research on Methods of Multiple Attribute Decision-Making and Integration*, Wuhan: Wuhan University of Technology, 2005.
- [9] Du Qiang, Chen Qiao, Yang Rui, "Forecast caron emissions of provinces in China based on logistic model", Vol.22, No.22, Feb 2013, pp. 143-151.
- [10] Feng Zheng-yuan, "Direct grey model," *Acta Mathematicae Applicatae Sinica*, Vol.15, No.3, July 1992, pp. 345-354.