

New Remarks on Oscillation of Second-Order Linear Difference Equation

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Abstract—In this paper, we first point out in the literature [6] the symbol definition of the theorem B is wrong. Secondly, I want to say in the literature [7] the proof process of lemma 1 is wrong. Thirdly, we point out the proof process of lemma 2.2 in the literature [8] is incorrect. Finally, given the correct proof and draw new key results.

Keywords—second-order linear difference equation; oscillation; non oscillation

I. INTRODUCTION

Consider the second-order linear difference equation

$$\Delta^2 x_{n-1} + p_n x_n = 0, \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where Δ is the forward difference operator, $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ is a sequence of non negative real numbers. Equations (1.1) of the oscillatory and non oscillatory behavior have been studied by several authors, such as literature [1, 2, 3, 4, 5]. In this paper, firstly, we will point out in [6] the main theorem B and lemma is wrong. Secondly, we will be pointed out in [7] the main lemma that lemma 1 is wrong. Lastly, we will point out in [8] the main lemma that lemma 2.2 is wrong, resulting in [8] theorem 2.1, corollary 2.1 and theorem 2.3 are also incorrect.

The main errors in the literature [6] are as follows:

(1) In theorem B, definite $u_n(\alpha) = n^{\alpha-1} \sum_{k=n+1}^{\infty} k^{\alpha} p_k$, $\alpha > 1$

is wrong, which should be corrected $u_n(\alpha) = n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} p_k$.

(2) Using the given symbol, the proof of lemma 1 can only

prove that $\sum_{k=n_1}^{n+1} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}} \leq \frac{\alpha^2}{\alpha-1} (n+1)^{\alpha-1}$, this formula cannot prove the final result, therefore the latter conclusion is wrong.

The main error in the literature [7] is as follows:

(1)The lemma formula derivation is not correct, for example

$$\begin{aligned} & \frac{\alpha^2}{1-\alpha} \sum_{k=n_1}^n \frac{\Delta(k-1)^{1-\alpha}}{[(k-1)k]^{1-\alpha}} (k+1)^{2\alpha-2} [(k-1)k]^{1-\alpha} \\ & \leq \frac{\alpha^2}{1-\alpha} \sum_{k=n_1}^n \left(\frac{k-1}{k}\right)^{2\alpha-2} \frac{\Delta(k-1)^{1-\alpha}}{[(k-1)k]^{1-\alpha}} \end{aligned}$$

The main error in the literature [8] is as follows:

(1)There is no proof of lemma 2.2 $q \leq R - R^2$. This conclusion can be proved by $q \leq R(1-r)$, the conclusion can be a new theorem.

Theorem A [6] .If

$$\liminf_{n \rightarrow \infty} n \sum_{k=n+1}^{\infty} p_k > \frac{1}{4} \quad (1.2)$$

Then every solution of the equation (1.1) is oscillatory.

Theorem B I

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n k^2 p_k > 1 \quad (1.3)$$

Then every solution of the equation (1.1) is oscillatory.

Our purpose in this paper is to further study the oscillation of all solutions of equation(1.1) neither (1.2) nor (1.3) does not hold.

We can easily show that if there exists $\alpha < 1$ such that

$$\sum_{k=1}^{\infty} k^{\alpha} p_k = \infty \quad (1.4)$$

Then every solution of the equation (1.1) is oscillatory. So we can assume that

$$\sum_{k=1}^{\infty} k^{\alpha} p_k < \infty, \quad \alpha < 1 \quad (1.5)$$

For the sake of convenience, throughout this paper, we use the following notation:

$$u_n(\alpha) = n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} p_k \quad \alpha < 1,$$

$$p_*(\alpha) = \liminf_{n \rightarrow \infty} u_n(\alpha), \quad p^*(\alpha) = \limsup_{n \rightarrow \infty} u_n(\alpha),$$

$$q = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n k^2 p_k$$

II. SOME LEMMAS

Lemma 1.1 Assume that $\alpha \in [0, 1)$, then there exists

$$\sum_{k=n+1}^{\infty} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}} \leq \frac{\alpha^2}{1-\alpha} n^{\alpha-1}, \quad (1.6)$$

$$\frac{(n+1)^{\alpha-1}}{1-\alpha} < \sum_{k=n+1}^{\infty} k^{\alpha-2} < \frac{n^{\alpha-1}}{1-\alpha} \quad (1.7)$$

Proof By the mean value theorem, the existence of $\xi_k \in (k, k+1), \eta_k \in (k-1, k)$ bring

$$\frac{(\Delta k^{\alpha})^2}{k^{\alpha}} = \frac{\alpha^2 \xi_k^{2\alpha-2}}{k^{\alpha}} \leq \frac{\alpha^2 k^{2\alpha-2}}{k^{\alpha}} \quad (1.8)$$

$$\frac{\Delta(k-1)^{1-\alpha}}{1-\alpha} = \eta_k^{-\alpha} \geq \frac{1}{k^{\alpha}} \quad (1.9)$$

Fro(1.8),(1.9)we have

$$\sum_{k=n+1}^{\infty} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}} \leq \frac{\alpha^2}{1-\alpha} \sum_{k=n+1}^{\infty} \frac{\Delta(k-1)^{1-\alpha}}{k^{2-2\alpha}} \quad (1.10)$$

Present definition

$$r(t) = (k-1)^{1-\alpha} + (t-k+1)\Delta(k-1)^{1-\alpha} \quad k-1 \leq t \leq k$$

Easy to launch

$$r'(t) = \Delta(k-1)^{1-\alpha}, \quad (k-1)^{1-\alpha} \leq r(t) \leq k^{1-\alpha}, \quad k-1 \leq t \leq k,$$

So that we have

$$\frac{\Delta(k-1)^{1-\alpha}}{k^{2-2\alpha}} = \int_{k-1}^k \frac{\Delta(k-1)^{1-\alpha}}{k^{2-2\alpha}} dt \leq \int_{k-1}^k \frac{r'(t)}{r^2(t)} dt = \frac{1}{(k-1)^{1-\alpha}} - \frac{1}{k^{1-\alpha}}$$

$$\text{Due to (1.10) set up, so } \sum_{k=n+1}^{\infty} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}} \leq \frac{\alpha^2}{1-\alpha} n^{\alpha-1}$$

$$\text{Also } \int_{n+1}^{\infty} \frac{1}{t^{2-\alpha}} dt < \sum_{k=n+1}^{\infty} \frac{1}{k^{2-\alpha}} < \int_n^{\infty} \frac{1}{t^{2-\alpha}} dt. \text{ Thus (1.7)}$$

was established.

Lemma 1.2 Assume $\{x_n\}$ is a non oscillatory solution of the equation (1.1) such that $x_{n-1} > 0$ for $n \geq n_0$. In view of the definition of $w_n = \frac{\Delta x_{n-1}}{x_{n-1}}$, we obtain

$$\Delta w_n + w_n w_{n+1} + p_n \leq 0, \quad n \geq n_0 \quad (1.11)$$

$$w_n \geq w_{n+1}, \quad (n-n_0)w_n < 1, \quad n \geq n_0 \quad (1.12)$$

$$p_*(0) \leq r - r^2, \quad q \leq R(1-r) \quad (1.13)$$

where

$$r = \liminf_{n \rightarrow \infty} n w_{n+1}, \quad R = \limsup_{n \rightarrow \infty} n w_{n+1} \quad (1.14)$$

Proof Because $\{x_n\}$ is a non oscillatory solution of equation(1.1) such that $x_{n-1} > 0$ for $n \geq n_0$, we easily following when $n \geq n_0$, we have

$$\Delta x_{n-1} \geq 0, \quad \Delta^2 x_{n-1} \leq 0 \quad (1.15)$$

In view of the definition of w_n , we have

$$\Delta w = \Delta w_n = \frac{\Delta^2 x_{n-1}}{x_n} - \frac{(\Delta x_{n-1})^2}{x_{n-1} x_n}, \quad n \geq n_0 \quad (1.16)$$

Together with(1.1)and(1.15), yields

$$\Delta w_n + w_n w_{n+1} + p_n \leq 0, \quad n \geq n_0 \quad (1.17)$$

For $n \geq n_0$, $w_n > 0$, so we have

$$w_n \geq w_{n+1}, \quad n \geq n_0 \quad (1.18)$$

$$\text{and} \quad \frac{\Delta w_n}{w_n w_{n+1}} < -1, \quad n \geq n_0 \quad (1.19)$$

From n_0 to $n-1$ on (1.19) both sides of sum, we have

$$(n - n_0)w_n < 1, \quad n \geq n_0 \quad (1.20)$$

So we have

$$\lim_{n \rightarrow \infty} w_n = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=n_0}^n w_k = 0 \quad (1.21)$$

By (1.14) and (1.20) we have

$$0 \leq r \leq 1, \quad 0 \leq R \leq 1 \quad (1.22)$$

$$r - r^2 \geq 0, \quad R - R^2 \geq 0 \quad (1.23)$$

We now prove that (1.13) holds. By (1.13), we assume that $p_*(0) \neq 0$, $q \neq 0$.

From (1.17) and (1.18), we easily find that for any $n_1 > n_0$

$$nw_{n+1} \geq n \sum_{k=n+1}^{\infty} p_k + n \sum_{k=n+1}^{\infty} \frac{(k-1)w_k w_{k+1}}{(k-1)k}, \quad n > n_1 \quad (1.24)$$

$$\Delta w_n + w_{n+1}^2 + p_n \leq 0, \quad n \geq n_1 \quad (1.25)$$

Multiplying (1.25) by n^2 . Summing it from n_1 to n , we obtain

$$\begin{aligned} \sum_{k=n_1}^n k^2 p_k &\leq - \sum_{k=n_1}^n k^2 \Delta w_k - \sum_{k=n_1}^n k^2 w_{k+1}^2 \\ &= -(n+1)^2 w_{n+1} + n_1^2 w_{n_1} + \sum_{k=n_1}^n w_{k+1} \Delta k^2 - \sum_{k=n_1}^n k^2 w_{k+1}^2 \\ &= -(n+1)^2 w_{n+1} + n_1^2 w_{n_1} + \sum_{k=n_1}^n w_{k+1} + \sum_{k=n_1}^n k w_{k+1} (2 - k w_{k+1}) \end{aligned}$$

It follows that

$$nw_{n+1} < \frac{n_1^2 w_{n_1} + \sum_{k=n_1}^n w_{k+1}}{n} - \frac{1}{n} \sum_{k=n_1}^n k^2 p_k + \frac{1}{n} \sum_{k=n_1}^n k w_{k+1} (2 - k w_{k+1}), \quad n > n_1 \quad (1.26)$$

Since

$$\lim_{n \rightarrow \infty} \frac{n_1^2 w_{n_1} + \sum_{k=n_1}^n w_{k+1}}{n} = 0, \quad k w_{k+1} (2 - k w_{k+1}) < 1$$

By (1.24) and (1.26) we have $r \geq p_*(0)$ and $R \leq 1 - q$

It is easy to see that for any $\varepsilon : 0 < \varepsilon < \min\{r, 1 - R\}$, there exists $n_2 > n_1$ such that when $n > n_2$, we have

$$r - \varepsilon < nw_{n+1} < R + \varepsilon, \quad n \sum_{k=n+1}^{\infty} p_k > p_*(0) - \varepsilon,$$

$$\text{and} \quad \frac{1}{n} \sum_{k=n_1}^n k^2 p_k > q - \varepsilon$$

Together with (1.24) and (1.26) yields $nw_{n+1} > p_*(0) - \varepsilon + (r - \varepsilon)^2$ for $n > n_2$

$$nw_{n+1} < \frac{1}{n} (n_2^2 w_{n_2} + \sum_{k=n_2}^n w_{k+1}) - q + \varepsilon + (R + \varepsilon)(2 - r - \varepsilon)$$

for $n > n_2$.

This implies that $r \geq p_*(0) + r^2$, $R \leq -q + R(2 - r)$

Thus (1.13) holds and the proof is complete.

III. MAIN RESULTS

Theorem 1.1 If

$$q = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n k^2 p_k > 1 \quad (1.27)$$

Then every solution of equation (1.1) oscillates.

Proof If not, let $\{x_n\}$ be a non oscillatory solution of equation (1.1) such that $x_{n-1} > 0$ for $n \geq n_0$.

Let $R = \limsup_{n \rightarrow \infty} nw_{n+1}$, by lemma 1.2, we have $q \leq R(1 - r) < 1$ which contradicts (1.27). The proof is complete.

Theorem 1.2 Let $q \leq 1$. Assume further that there exists $\alpha \in [0, 1)$ such that

$$p^*(\alpha) \leq \frac{\alpha^2}{4(1-\alpha)} + R \quad (1.28)$$

Then every solution of equation (1.1) oscillates.

Proof If not, let $\{x_n\}$ be a non oscillatory solution of equation (1.1) such that $x_{n-1} > 0$ for $n \geq n_0$. By Lemma 1.2, we have

$$\Delta w_n + w_{n+1}^2 + p_n \leq 0, \quad n \geq n_0 \quad (1.29)$$

$$\text{and} \quad q \leq R(1-r) \leq R \quad (1.30)$$

$$\text{where } w_n = \frac{\Delta x_{n-1}}{x_{n-1}}, \quad R = \limsup_{n \rightarrow \infty} n w_{n+1},$$

by $\Delta w_n + w_{n+1}^2 + p_n \leq 0$ we have

$$\begin{aligned} \sum_{k=n+1}^{\infty} k^{\alpha} p_k &\leq - \sum_{k=n+1}^{\infty} k^{\alpha} \Delta w_k - \sum_{k=n+1}^{\infty} k^{\alpha} w_{k+1}^2 \\ &= (n+1)^{\alpha} w_{n+1} + \sum_{k=n+1}^{\infty} w_{k+1} \Delta k^{\alpha} - \sum_{k=n+1}^{\infty} k^{\alpha} w_{k+1}^2 \\ &= (n+1)^{\alpha} w_{n+1} + \frac{1}{4} \sum_{k=n+1}^{\infty} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}} - \sum_{k=n+1}^{\infty} (k^{\frac{\alpha}{2}} w_{k+1} - \frac{1}{2} k^{\frac{-\alpha}{2}} \Delta k^{\alpha})^2 \\ &< (n+1)^{\alpha} w_{n+1} + \frac{1}{4} \sum_{k=n+1}^{\infty} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}} \end{aligned}$$

Hence,

$$\limsup_{n \rightarrow \infty} n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} p_k < \limsup_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{\alpha} n w_{n+1} + \limsup_{n \rightarrow \infty} \frac{1}{4} n^{1-\alpha} \sum_{k=n+1}^{\infty} \frac{(\Delta k^{\alpha})^2}{k^{\alpha}}$$

$$\text{So } p^*(\alpha) \leq \limsup_{n \rightarrow \infty} n w_{n+1} + \frac{\alpha^2}{4(1-\alpha)}$$

$$\text{That is } p^*(\alpha) \leq R + \frac{\alpha^2}{4(1-\alpha)} \text{ which contradicts (1.28).}$$

The proof is complete.

Theorem 1.3

Let $p_*(0) \leq \frac{1}{4}$, $q \leq 1$, $M = \limsup_{n \rightarrow \infty} n w_{n+1}$, and assume further that $\alpha \in [0, 1)$ such

that, $p_*(0) > \frac{\alpha(2-\alpha)}{4}$, $p^*(\alpha) > M + \frac{M(\alpha-m)}{1-\alpha}$, Then every solution of equation (1.1) oscillates.

Proof If not, let $\{x_n\}$ be a non oscillatory solution of equation (1.1) such that $x_{n-1} > 0$ for $n \geq n_0$. By lemma 1.2, we have

$$\Delta w_n + w_{n+1}^2 + p_n \leq 0, \quad n \geq n_0 \quad (1.31)$$

$$\text{And } p_*(0) \leq r - r^2 \quad q \leq R(1-r) \leq R$$

$$\text{where } r = \liminf_{n \rightarrow \infty} n w_{n+1}, \quad R = \limsup_{n \rightarrow \infty} n w_{n+1}$$

$$\text{Implies that } r \geq m = \frac{1}{2}(1 - \sqrt{1 - 4p_*(0)}), \quad q \leq R$$

$$\text{By } p_*(0) > \frac{\alpha(2-\alpha)}{4} \quad \text{we have} \quad m > \frac{\alpha}{2}$$

$$\text{For any } \forall \varepsilon : 0 < \varepsilon < m - \frac{\alpha}{2}, \exists n_1 > n_0 \text{ such that}$$

$$m - \varepsilon < n w_{n+1} < \left(\frac{n+1}{n} \right)^{\alpha} n w_{n+1}$$

$$\text{Let } \limsup_{n \rightarrow \infty} n w_{n+1} = M \quad \text{then}$$

$$m - \varepsilon < n w_{n+1} < \left(\frac{n+1}{n} \right)^{\alpha} n w_{n+1} < M + \varepsilon$$

$$\text{By } \Delta w_n + w_{n+1}^2 + p_n \leq 0, \text{ we have}$$

$$\begin{aligned} \sum_{k=n+1}^{\infty} k^{\alpha} p_k &\leq - \sum_{k=n+1}^{\infty} k^{\alpha} \Delta w_k - \sum_{k=n+1}^{\infty} k^{\alpha} w_{k+1}^2 \\ &= (n+1)^{\alpha} w_{n+1} + \sum_{k=n+1}^{\infty} w_{k+1} \Delta k^{\alpha} - \sum_{k=n+1}^{\infty} k^{\alpha} w_{k+1}^2 \end{aligned} \quad (1.32)$$

$$\Delta k^{\alpha} = (k+1)^{\alpha} - k^{\alpha} < \alpha k^{\alpha-1}$$

Simultaneous (1.32) we have

$$n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} p_k \leq \left(\frac{n+1}{n} \right)^{\alpha} n w_{n+1} + n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha-2} [k w_{k+1} (\alpha - k w_{k+1})]$$

That

$$\begin{aligned}
n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} p_k &< M + \varepsilon + n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha-2} [(M + \varepsilon)(\alpha + \varepsilon - m)] \\
&< (M + \varepsilon) + n^{1-\alpha} \frac{n^{\alpha-1}}{1-\alpha} [(M + \varepsilon)(\alpha + \varepsilon - m)] \\
&= (M + \varepsilon) + \frac{1}{1-\alpha} [(M + \varepsilon)(\alpha + \varepsilon - m)]
\end{aligned}$$

This contradicts $p^*(\alpha) \leq M + \frac{1}{1-\alpha} [M(\alpha - m)]$.

The proof is complete.

An excellent style manual for science writers is [7].

IV. EXAMPLE

Considering the difference equation

$$\Delta^2 x_{n-1} + p_n x_n = 0, \quad n = 0, 1, 2, \dots, \quad (1.33)$$

$$\text{Here } p_n = \begin{cases} 6^{-k} & n = 6^k \\ 0 & n \neq 6^k \end{cases} \quad k = 0, 1, 2, \dots,$$

to $6^m - 1 < n < 6^{m+1} - 1, m = 0, 1, 2, \dots$, we have

$$u_n(0) = n \sum_{k=n+1}^{\infty} p_k = n \sum_{k=m+1}^{\infty} 6^{-k} = \frac{n}{5 \times 6^m} \in \left[\frac{1}{5}, \frac{6}{5} \right)$$

$$p_*(0) = \liminf_{n \rightarrow \infty} u_n(0) = \frac{1}{5}$$

$$p^*(0) = \limsup_{n \rightarrow \infty} u_n(0) = \frac{6}{5}$$

$$m = \frac{1}{2} (1 - \sqrt{1 - 4p_*(0)}) = \frac{5 - \sqrt{5}}{10}$$

$$p^*(0) = \frac{6}{5} > 1 + \frac{(\alpha - m)}{1 - \alpha} \quad \alpha < \frac{7 - \sqrt{5}}{12} \quad \text{so}$$

In accordance with the theorem 1.3, so the solution of the equation (1.33) is oscillatory.

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