

The Maximal 2-independent Sets in Trees

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Abstract— A 2-independent set in a graph G is a subset I of the vertices such that the distance between any two vertices of I in G is at least three. We say that I is a maximal 2-independent set in G if it is not a proper subset of any other 2-independent set. In this paper, we study the problem of determining the small numbers of maximal 2-independent sets among all trees of order n . Extremal graphs achieving these values are also given.

Keywords: independent set; maximal independent set; 2-independent set; maximal 2-independent set; tree

I INTRODUCTION

Throughout this paper, graphs will be finite, simple and loopless. A subset I of $V(G)$ is said to be a 2-independent set of G such that the distance between any two vertices of I in G is at least three. A maximal 2-independent set is a 2-independent set that is not a proper subset of any other 2-independent set. The set of all maximal 2-independent sets of a graph G is denoted by $MI_2(G)$ and its cardinality by $mi_2(G)$. The study of the number of independent sets in a graph has a rich history. The maximum weight k -independent set problem has applications in many practical problems like k -machines job scheduling problem, k -colourable subgraph problem, VLSI design layout and routing problem [2]. Finding a maximum k -independent set of a graph is NP-hard (see [3], [4]). In this paper, we study the problem of determining the small numbers of maximal 2-independent sets among all trees of order n .

For a graph G , we refer to $V(G)$ and $E(G)$ as the vertex set and the edge set, respectively. The cardinality of $V(G)$ is called the order of G , denoted by $|G|$. The (open) neighborhood $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G , and the close neighborhood $N_G[x]$ is $N_G(x) \cup \{x\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \cup_{x \in A} N_G(x)$ and $N_G[A] = \cup_{x \in A} N_G[x]$. The degree $deg_G(x)$ of a vertex x is the cardinality of $N_G(x)$. A vertex x is said to be a leaf if $|N_G(x)| = 1$. A vertex v of G is a support vertex if it is adjacent to a leaf in G . Let $L(G)$ and $U(G)$ denote the sets of leaves and support vertices, respectively, of G . For

a subset $A \subseteq V(G)$, the deletion of A from G is the graph $G - A$ by removing all vertices in A and all edges incident to these vertices. If $A = \{v\}$, we write $G - v$ instead of $G - \{v\}$. A forest is a graph with no cycles, and a tree is a connected forest. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G . Denote by S_n a star is the graph $K_{1,n-1}$. A double star $S(n, m)$ is the graph consisting of the union of two stars S_n and S_m together with a line joining their centers. For notation and terminology in graphs we follow [1] in general.

II PRELIMINARY

For convenience, we define the following sets. For a subset $A, B \subseteq V(G)$, the set $MI_2(G; +A, -B, \dots)$ is defined as follow.

$$MI_2(G; +A, -B, \dots) = \{I \in MI_2(G); I \cap A \neq \emptyset, I \cap B = \emptyset, \dots\}$$

If $A = \{v\}$, we write $MI_2(G; +v)$ ($MI_2(G; -v)$) instead of $MI_2(G; +\{v\})$ ($MI_2(G; -\{v\})$). Suppose that v is a vertex in a graph G , we define $MI_2(G; -v^\alpha)$ and $MI_2(G; -v^\beta)$ as follows.

$$\begin{aligned} MI_2(G; -v^\alpha) &= \{I \in MI_2(G); v \notin I, I \cap N_G(v) \neq \emptyset\}. \\ MI_2(G; -v^\beta) &= \{I \in MI_2(G); I \cap N_G[v] = \emptyset, I \cap N_G(N_G(v)) \neq \emptyset\}. \end{aligned}$$

Let $I \in MI_2(G)$. If $I \notin MI_2(G; +v)$, then $I \in MI_2(G; -v^\alpha)$ or $I \in MI_2(G; -v^\beta)$. Thus we have

$$\begin{aligned} MI_2(G) &= MI_2(G; +v) \cup MI_2(G; -v) \\ &= MI_2(G; +v) \cup MI_2(G; -v^\alpha) \cup MI_2(G; -v^\beta) \end{aligned}$$

The following useful lemmas which are needed in this paper.

Lemma 2.1 Suppose v is a vertex of a graph G , then

$$\begin{aligned} mi_2(G) &= mi_2(G; +v) + mi_2(G; -v) \\ &= mi_2(G; +v) + mi_2(G; -v^\alpha) + mi_2(G; -v^\beta), \end{aligned}$$

where $mi_2(G; -v^\alpha) = |MI_2(G; -v^\alpha)|$ and $mi_2(G; -v^\beta) = |MI_2(G; -v^\beta)|$.

Lemma 2.2 If $P : x, y, z, \dots$ is a longest path of a tree T and $N_T(y) \cap L(T) = L'$, then $mi_2(T) = mi_2(T; +L') + mi_2(T; +y) + mi_2(T; +z)$.

Proof. Let $L'' = L' - \{x\}$. So $mi_2(T; +L') = mi_2(T; +x) + mi_2(T; +L'')$. Since P is a longest path, this implies that $mi_2(T; -x^\alpha) = mi_2(T; +y)$ and $mi_2(T; -x^\beta) = mi_2(T; +z) + mi_2(T; +L'')$. Thus $mi_2(T) = mi_2(T; +x) + mi_2(T; -x^\alpha) + mi_2(T; -x^\beta) = mi_2(T; +x) + mi_2(T; +y) + [mi_2(T; +z) + mi_2(T; +L'')] = mi_2(T; +L') + mi_2(T; +y) + mi_2(T; +z)$. \square

Lemma 2.3 Suppose that $f(x) = x(n - x)$, where $1 \leq x \leq n - 1$, then $f(x) \geq n - 1$.

Proof. Thus $f'(x) = n - 2x$. So $f(x)$ is absolutely increasing on $(1, \frac{n}{2})$ and absolutely decreasing on $(\frac{n}{2}, n - 1)$. Hence $f(x) \geq \min\{f(1), f(n - 1)\} = n - 1$. \square

III MAIN THEOREMS

In this section, we will show the small numbers of maximal 2-independent sets among all trees of order $n \geq 2k + 5$. Extremal graphs achieving these values are also given.

For $a \geq 0$ and $c \geq b \geq 1$, $T = \tilde{S}(z; a, b, c)$ is a tree with a center z such that the following all hold.

- (i) The distance between any vertex and z is at most two.
- (ii) $A = N(z) \cap L(T)$ with cardinality $|A| = a$.
- (iii) $B = N(z) - A$ with cardinality $|B| = b$.
- (iv) $C = L(T) - A$ with cardinality $|C| = c$.

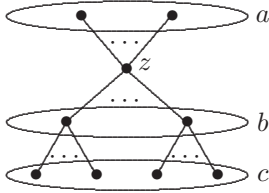


Figure 1: The graph $\tilde{S}(z; a, b, c)$, where $a \geq 0$ and $c \geq b \geq 1$.

For $0 \leq i \leq n - 3$, we define

$$T^{(k)}(n) = \begin{cases} \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2}), & \text{if } 0 \leq \lceil \frac{i}{2} \rceil \leq k = 1; \\ \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2}), & \text{if } \lceil \frac{i}{2} \rceil = k \geq 2. \end{cases}$$

For $k \geq 1$, let $t^{(k)}(n) = mi_2(T^{(k)}(n)) = \lceil \frac{n}{2} \rceil + k$.

The following theorems are the main results.

Theorem 3.1 If T is a tree of order $n \geq 4$, then $mi_2(T) \geq t^{(1)}(n)$, and the equality holding if and only if $T = T^{(1)}(n)$.

Theorem 3.2 For $k \geq 2$, suppose that T is a tree of order $n \geq 2k + 5$ different from $T^{(1)}(n), \dots, T^{(k-1)}(n)$, then $mi_2(T) \geq t^{(k)}(n)$, and the equality holding if and only if $T = T^{(k)}(n)$.

First we need the following lemmas.

Lemma 3.3 If T is a star or a double star of order $n \geq 2k + 5$, where $k \geq 1$, then $mi_2(T) > t^{(k)}(n)$.

Proof. Then $T = S_n$ or $\tilde{S}(z; i, 1, n - i - 2)$, where $1 \leq i \leq n - 3$. By Lemma 2.3, $mi_2(T) = \min\{mi_2(S_n), mi_2(\tilde{S}(z; i, 1, n - i - 2))\} = \min\{n, i(n - i - 2) + 2\} = n - 1 = \frac{n+n-2}{2} \geq \frac{n+(2k+5)-2}{2} > \frac{n+2k+1}{2} \geq t^{(k)}(n)$. \square

Lemma 3.4 Suppose T is a tree of order $n \geq 5$ and $P : x, y, z, \dots$ is a longest path of T , where $|P| \geq 5$. If x and x' are two distinct leaves adjacent to y , then $T - x'$ is a tree of order $n - 1$ and $mi_2(T - x') \leq mi_2(T) - 2$.

Proof. We can see that $T - x'$ is a tree of order $n - 1$. Since $|P| \geq 5$, this implies that $mi_2(T; +x') \geq mi_2(T - N_T[y]) \geq mi_2(P_2) = 2$. By Lemma 2.2, we have

$$\begin{aligned} mi_2(T - x') &= mi_2(T - x'; +x) + mi_2(T - x'; -x) \\ &= mi_2(T; +x) + mi_2(T; -x, -x') \\ &= [mi_2(T; +x) + mi_2(T; +x') + mi_2(T; -x, -x')] - mi_2(T; +x') \\ &= mi_2(T) - mi_2(T; +x') \\ &\geq mi_2(T) - 2. \end{aligned}$$

This completes the proof. \square

Lemma 3.5 Suppose T is a tree of order $n \geq 5$ and $P : x, y, z, \dots$ is a longest path of T , where $|P| \geq 5$. If $\deg(y) = 2$, then $T - N_T[x]$ is a tree of order $n - 2$ and $mi_2(T - N_T[x]) \leq mi_2(T) - 1$. The equality holds if and only if $T = \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})$ for some $i \geq 0$.

Proof. Since $\deg(y) = 2$, this implies that $T^* = T - N_T[x]$ is a tree of order $n - 2$. Since $|P| \geq 5$, this implies that $mi_2(T; +x, -z^\beta) \geq 1$. So $mi_2(T; +x) \geq mi_2(T; +x, -z^\alpha) + mi_2(T; +x, -z^\beta) \geq mi_2(T^*; -z^\alpha) + 1$. The equalities hold if and only if $|P| = 5$ and $\deg(v) \leq 2$ for every vertex $v \neq z$. That is $T = \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})$ for some $i \geq 0$.

Note that $mi_2(T; +z) = mi_2(T^*; +z)$. We also know that $mi_2(T; +y) \geq mi_2(T; +y, -z^\beta) = mi_2(T^*; -z^\beta)$, and the equality holding if and only if $|P| = 5$. Thus

$$\begin{aligned} mi_2(T) &= mi_2(T; +z) + mi_2(T; +y) + mi_2(T; +x) \\ &\geq mi_2(T^*; +z) + mi_2(T^*; -z^\beta) + [mi_2(T^*; -z^\alpha) + 1] \\ &= [mi_2(T^*; +z) + mi_2(T^*; -z^\beta) + mi_2(T^*; -z^\alpha)] + 1 \\ &= mi_2(T^*) + 1, \end{aligned}$$

and the equalities holds if and only if $T = \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})$ for some $i \geq 0$. \square

Now we prove the Theorem 3.1.

Proof. We prove this theorem by induction on $n \geq 4$. It's true for $n = 4, 5$ and 6 . Assume that it is true for all $n' < n$. Let T be a tree of order $n \geq 7$ such that $mi_2(T)$ is as small as possible. Then $mi_2(T) \leq t^{(1)}(n)$. Let $P : x, y, z, \dots$ be a longest path of T . If $|P| \leq 4$, then T is a star or a double star. By Lemma 3.3, $t^{(1)}(n) \geq mi_2(T) > t^{(1)}(n)$. This is a contradiction, hence $|P| \geq 5$. **Claim.** $deg(y) = 2$.

If x and x' are two distinct leaves adjacent to y , then $T - x'$ is a tree of order $n - 1$. By induction hypothesis and Lemma 3.4, $\frac{(n-1)+2}{2} \leq t^{(1)}(n-1) \leq mi_2(T - x') \leq mi_2(T) - 2 \leq t^{(1)}(n) - 2 \leq \frac{n+3}{2} - 2 = \frac{n-1}{2}$. This is a contradiction, so y is adjacent to only one leaf x . Hence $deg(y) = 2$.

By claim, the subgraph $T^* = T - N_T[x]$ is a tree of order $n - 2$. By induction hypothesis and Lemma 3.5, $t^{(1)}(n-2) \leq mi_2(T^*) \leq mi_2(T) - 1 \leq t^{(1)}(n) - 1 = t^{(1)}(n-2)$. So $mi_2(T^*) = mi_2(T) - 1 = t^{(1)}(n-2)$, by Lemma 3.5, $T = \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})$ for some $i \geq 0$. Thus $\frac{n+3}{2} \geq t^{(1)}(n) \geq mi_2(T) = mi_2(\tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})) = \frac{n+i+1}{2}$. Thus $i \leq 2$. That is $T = T^{(1)}(n)$. \square

Theorem 3.6 If T is a tree of order $n \geq 9$ having $T \neq T^{(1)}(n)$, then $mi_2(T) \geq t^{(2)}(n)$, and the equality holding if and only if $T = T^{(2)}(n)$.

Proof. Let T be a tree of order $n \geq 9$ having $T \neq T^{(1)}(n)$ such that $mi_2(T)$ is as small as possible. By Theorem 3.1 and the hypothesis, $t^{(1)}(n) + 1 \leq mi_2(T) \leq t^{(2)}(n) = t^{(1)}(n) + 1$. Hence $mi_2(T) = t^{(2)}(n)$. Let $P : x, y, z, \dots$ be a longest path of T . If $|P| \leq 4$, then T is a star or a double star. By Lemma 3.3, $t^{(2)}(n) = mi_2(T) > t^{(2)}(n)$. This is a contradiction, so $|P| \geq 5$.

Claim. $deg(y) = 2$.

If x and x' are two distinct leaves adjacent to y in T , then $T - x'$ is a tree of order $n - 1$ and, by Theorem 3.1 and Lemma 3.4, $\frac{(n-1)+2}{2} \leq t^{(1)}(n-1) \leq mi_2(T - x') \leq mi_2(T) - 2 = t^{(2)}(n) - 2 \leq \frac{n+5}{2} - 2 = \frac{n+1}{2}$. The equalities hold and n is odd. So $T - x' = T^{(1)}(n-1)$ and $T = \tilde{S}(z; 1, \frac{n-3}{2}, \frac{n-1}{2})$. So $\frac{n+5}{2} = t^{(2)}(n) = mi_2(T) = mi_2(\tilde{S}(z; 1, \frac{n-3}{2}, \frac{n-1}{2})) = n - 1 = \frac{n+n-2}{2} \geq \frac{n+9-2}{2} > \frac{n+5}{2}$, where $n \geq 9$. This is a contradiction, so y is adjacent to only one leaf x . That is $deg(y) = 2$.

By claim, $T^* = T - N[x]$ is a tree of order $n - 2$. By Theorem 3.1 and Lemma 3.5, $t^{(1)}(n-2) \leq mi_2(T^*) \leq mi_2(T) - 1 = t^{(2)}(n) - 1 = t^{(1)}(n-2)$. So $mi_2(T^*) = mi_2(T) - 1 = t^{(1)}(n-2)$, by Lemma 3.5, $T = \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})$ for some $i \geq 3$. Since $mi_2(T) = \lceil \frac{n}{2} \rceil + 2$, this implies that $i = 3$ or 4 . That is $T = T^{(2)}(n)$. \square

We prove another result, Theorem 3.2.

Proof. We prove this theorem by induction on $k \geq 2$. By Theorem 3.6, it's true for $k = 2$. Assume that it's

true for all $k' < k$, where $k \geq 3$. Let T be a tree of order $n \geq 2k + 5$ different from $T^{(1)}(n), \dots, T^{(k-1)}(n)$ such that $mi_2(T)$ is as small as possible. By induction hypothesis, $t^{(k-1)}(n) + 1 \leq mi_2(T) \leq t^{(k)}(n) = t^{(k-1)}(n) + 1$. So $mi_2(T) = t^{(k)}(n)$. Let $P : x, y, z, \dots$ be a longest path of T . If $|P| \leq 4$, then T is a star or a double star. By Lemma 3.3, $t^{(k)}(n) = mi_2(T) > t^{(k)}(n)$. This is a contradiction, so $|P| \geq 5$.

Claim 1. $deg(y) = 2$.

If x and x' are two distinct leaves adjacent to y in T , then $T - x'$ is a tree of order of order $n - 1$ and, by Lemma 3.4, $mi_2(T - x') \leq mi_2(T) - 2 = t^{(k)}(n) - 2 \leq \frac{n+2k+1}{2} - 2 = \frac{(n-1)+2(k-1)}{2} \leq t^{(k-1)}(n-1)$. By induction hypothesis, $T - x' = T^{(m)}(n-1)$ for some $m \leq k-1$. That is $T = \tilde{S}(z; i, \frac{n-i-2}{2}, \frac{n-i}{2})$, where $0 \leq \lceil \frac{i}{2} \rceil \leq k-1$. So $\frac{n+2k+1}{2} \geq t^{(k)}(n) = mi_2(T) \geq \min\{mi_2(\tilde{S}(z; 0, \frac{n-2}{2}, \frac{n}{2})), mi_2(\tilde{S}(z; i, \frac{n-i-2}{2}, \frac{n-i}{2}))\} = n-1 = \frac{n+n-2}{2} \geq \frac{n+(2k+5)-2}{2} > \frac{n+2k+1}{2}$. This is a contradiction, so y is adjacent to only one leaf x . That is $deg(y) = 2$.

By claim 1, $T^* = T - N[x]$ is a tree of order $n - 2$. By Lemma 3.5, $mi_2(T^*) \leq mi_2(T) - 1 = t^{(k)}(n) - 1 = t^{(k)}(n-2)$.

Claim 2. $mi_2(T^*) = t^{(k)}(n-2)$.

Suppose that $mi_2(T^*) < t^{(k)}(n-2)$, by induction hypothesis, $T^* = \tilde{S}(z'; i, \frac{n-i-3}{2}, \frac{n-i-3}{2})$, where $0 \leq \lceil \frac{i}{2} \rceil \leq k-1$. Since T is different from $T^{(1)}(n), \dots, T^{(k-1)}(n)$, this implies that $z' \neq z$. We consider two cases.

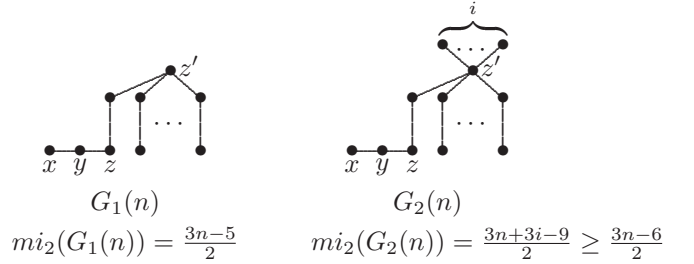


Figure 2: The graphs $G_1(n)$ and $G_2(n)$

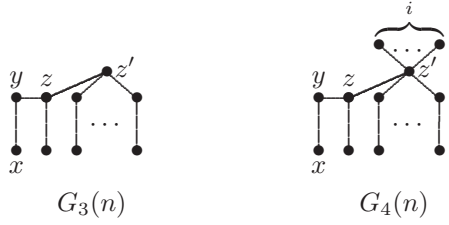
Case 1. $z \notin N(z')$.

Then $T = G_1(n)$ or $G_2(n)$. So

$$\begin{aligned}
 \frac{n+2k+1}{2} &\geq mi_2(T) \\
 &\geq \min\{mi_2(G_1(n)), mi_2(G_2(n))\} \\
 &= \frac{3n-6}{2} \\
 &= \frac{n+2n-6}{2} \\
 &\geq \frac{n+2(2k+5)-6}{2} \\
 &> \frac{n+2k+1}{2}.
 \end{aligned}$$

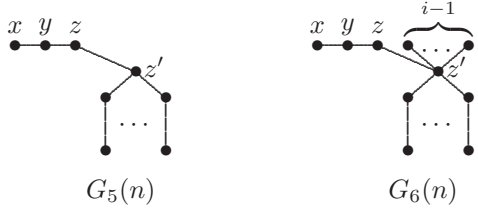
This is a contradiction.

Case 2. $z \in N(z')$.



$$mi_2(G_3(n)) = n - 1 \quad \quad \quad \begin{aligned} mi_2(G_4(n)) \\ = n + i - 3 \geq n - 2 \end{aligned}$$

Figure 3: The graphs $G_3(n)$ and $G_4(n)$



$$mi_2(G_5(n)) = n \quad \quad \quad \begin{aligned} mi_2(G_6(n)) \\ = n + i - 3 \geq n - 2 \end{aligned}$$

Figure 4: The graphs $G_5(n)$ and $G_6(n)$

Then $T = G_3(n), G_4(n), G_5(n)$ or $G_6(n)$. So

$$\begin{aligned} & \frac{n + 2k + 1}{2} \\ & \geq mi_2(T) \\ & \geq \min\{mi_2(G_3(n)), mi_2(G_4(n)), mi_2(G_5(n)), mi_2(G_6(n))\} \\ & = n - 2 \\ & = \frac{2n - 4}{2} \\ & = \frac{n + n - 4}{2} \\ & \geq \frac{n + (2k + 5) - 4}{2} \\ & = \frac{n + 2k + 1}{2}. \end{aligned}$$

Thus $mi_2(T) = \frac{n+2k+1}{2}$, this implies that n is odd and $i = 1$. This is a contradiction.

By Case 1 and 2, we complete the Claim 2.

We obtain that $mi_2(T^*) = t^{(k)}(n - 2) = mi_2(T) - 1$. By Lemma 3.5, $T = \tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})$ for some i , where $i > 2(k - 1)$. Thus $\frac{n+2k+1}{2} \geq mi_2(T) = mi_2(\tilde{S}(z; i, \frac{n-i-1}{2}, \frac{n-i-1}{2})) = \frac{n+i+1}{2}$. So $2k - 2 < i \leq 2k$. That is $T = T^{(k)}(n)$. We complete this proof. \square

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