The Multiplicity of Solutions for a Certain Class of Fractional Elliptic Equation

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***Abstract*—This paper explores the multiplicity of solutions for a Certain class of fractional elliptic equation under the Dirichlet boundary conditions. By using the asymptotic property of the nonlinear term f(x,u) at zero and at infinite point, the mountain pass theorem and proper truncation methods can be applied to get both a positive and a negative solution for all parameters under the condition of not satisfying the Ambrosetti-Rabinowitz.**

Keywords-fractional laplace's equation; dirichlet boundary value condition; mountain pass theorem; positive solution; negative solution.

I. INTRODUCTION

In this paper, we study the multiplicity of solutions for the following fractional elliptic:

$$\begin{cases}(-\Delta)^s u = \lambda a(x)|u|^{r-2}u + \lambda f(x,u); x \in \Omega \\ u = 0; x \in \partial\Omega\end{cases} \tag{1}$$

Where $\Omega \in R^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. $(-\Delta)^s$ is the Fractional Laplace operator, $s \in (0,1)$, $f \in C(\bar{\Omega} \times \mathrm{R}, \mathrm{R})$.

Fractional Laplace operator $(-\Delta)^s$ is an infinitesimal generator of *Le'vy* during its steady state diffusion process, which has been widely used in many fields. In order to study the nontrivial solution of nonlinear equations, AR condition is often used to ensure the compactness of corresponding energy. In recent years, many researchers have confronted with the multiplicity of various nonlinear solutions when it fails to satisfy the AR condition, but researches of multiplicity the fractional order elliptic equations has gradually got more attention in recent years . In this paper, we mainly explore the multiplicity of solutions of fractional order elliptic equations with parameters. With mountain pass theorem, we get both a positive and a negative solution for all parameters while not satisfying the Ambrosetti-Rabinowitz.

Note:

$H(x,\mathrm{u}) = \mathrm{u}\, f(x,u) - 2F(x,u)$ and $F(x,u) = \int_0^u f(x,t)dt$.

Suppose the function *f*(*x*,*u*) on (1) meet the following conditions:

(*H*₁) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0$ uniformly for almost all $x \in \Omega$;

(*H*₂) there exists positive C_1, C_2, p, and:

$$|f(x,u)| \leq C_1 + C_2 |t|^p$$
$$1 < p < \frac{N+2s}{N-2s}, \forall t \in R, a.e. x \in \Omega;$$

(*H*₃) $\lim_{|t|\to +\infty} \frac{f(x,t)}{t^2} = +\infty$ uniformly for a.a $x \in \Omega$;

(*H*₄) there exists $\mathrm{C}^* > 0$ such that $\mathrm{H}(x,t_1) \leq \mathrm{H}(x,t_2) + \mathrm{C}^*$ for all $0 < t_1 < t_2$ or $t_2 < t_1 < 0$ and uniformly for a.e. $x \in \Omega$.

II. MAIN RESULTS

Theorem 1. If hypotheses (H_1)-(H_4) is available, then Problem (1) have at least a positive solution and a negative answer when $\lambda > 0$.

Notes 1: in [1], the multiplicity of solutions of $(-\Delta)^s u = f(x,u)$ is obtained under the condition of satisfying the Ambrosetti-Rabinowitz.

It is the multiplicity of solutions of $-\Delta_p(b(x)u) = a(x)|u|^{r-2}u + f(x,u)$ in [2]. When the (AR) condition is not satisfied, [3] discussed the multiplicity of solutions for problem (1) when *s*=1. In this paper, under the influence of the parameters λ, we discussed the multiplicity of solutions for fractional Laplacian equations (1) when the (AR) condition is not satisfied.

III. BACKGROUND KNOWLEDGE

We can denote $\{\varphi_k\}$ as A set of orthogonal basis of $L_2(\Omega)$, with $\|\varphi_k\|_2 = 1$, and make up the spectrum decomposition of $(-\nabla)^s$ under the Dirichlet boundary condition. λ_k is the corresponding characteristic value .

Define:

$$H_0^s(\Omega) = \left\{ u = \sum_{k=0}^{\infty} a_k \varphi_k \in L^2(\Omega) : \|u\|_{H_0^s(\Omega)} = (\sum_{k=0}^{\infty} a_k{}^2 \varphi_k{}^s)^{\frac{1}{2}} < \infty \right\}.$$

Denote $H^{-s}(\Omega)$ as dual space of $H_0^s(\Omega)$, $u \in H_0^s(\Omega)$, $u=\sum_{k=0}^{\infty} a_k \varphi_k$ with $a_k=\int_\Omega u\varphi_k dx$ .We define $(-\nabla)^s$ as: $(-\nabla)^s u=\sum_{k=0}^{\infty} a_k \lambda_k{}^s \varphi_k \in H^{-s}(\Omega)$.

$\{(\varphi_k,\lambda_k{}^s)\}$ is the characteristic function and the characteristic value of $(-\nabla)^s$.

Define inner product in $H_0^s(\Omega)$ as:

$$(u,v)_{H_0^s(\Omega)}=\int_\Omega (-\nabla)^{\frac{s}{2}} u (-\nabla)^{\frac{s}{2}} v dx.$$

It is easy to find that $H_0^s(\Omega)$ is a Hilbert space.

Define $u \in H_0^s(\Omega)$ is a weak solution on (1), if $\int_\Omega (-\nabla)^{\frac{s}{2}} u(-\nabla)^{\frac{s}{2}} v dx=\lambda\int_\Omega a(x)|u|^{r-2} u dx+\lambda\int_\Omega f(x,u) v dx$ are all available to $\forall v \in H_0^s(\Omega)$. First, let's consider the positive solution of the following questions (1):

$$f_+(x,u)=\begin{cases} f(x,u); \mathrm{u}>0 \\ 0: u \le 0 \end{cases}$$

$F_+(x,u)=\int_0^u f_+(x,t)dt$, then taking the following problems into account:

$$\begin{cases} (-\Delta)^s u=\lambda a(x)|u|^{r-2} u+\lambda f^+(x,u); x\in\Omega \\ u=0; x\in\partial\Omega \end{cases} \tag{2}$$

The corresponding energy functional $I_\lambda^+(\mathrm{u}): \mathrm{H}_0^s(\Omega)\to \mathrm{R}$ is as follow:

$$I_\lambda^+(u)=\frac{1}{2}\int_\Omega \left|(-\nabla)^{\frac{s}{2}}\right| \mathrm{dx}-\frac{\lambda}{r}\int_\Omega a(x)(u^+)^r dx-\lambda\int_\Omega F_+(x,u)dx.$$

Where $u \in H_0^s(\Omega)$.

Apparently, $I_\lambda^+ \in C^1(\mathrm{H}_0^s(\Omega),\mathrm{R})$, u is the critical point of I_λ^+, and u is the solution of (2). Furthermore, according to the principle of the maximum value, u>0 in Ω. Therefore, u is also the solution of (1).

Similarly, definite:

$$f_-(x,u)=\begin{cases} f(x,u); \mathrm{u}\le 0 \\ 0: u>0 \end{cases} \tag{3}$$

$$F_-(x,u)=\int_0^u f_-(x,t)dt,$$

$$I_\lambda^-(u)=\frac{1}{2}\int_\Omega \left|(-\nabla)^{\frac{s}{2}} u\right| \mathrm{dx}-\frac{\lambda}{r}\int_\Omega a(x)(u^-)^r dx-\lambda\int_\Omega F_-(x,u)dx.$$

$I_\lambda^- \in C^1(\mathrm{H}_0^s(\Omega),\mathrm{R})$, u is the critical point of I_λ^-. According to the principle of the maximum value, u<0 in Ω. So, u is also the solution of (1).

IV. Proof of Main Results

First, from (H_1)-(H_3), I_λ^+ and I_λ^- are mountain road geometry.

Lemma 1. If hypotheses (H_1)-(H_2) is available, for $\forall\lambda>0$, there exists $\rho, R>0$. Thus, if $\|u\|=\rho, I_\lambda^\pm \ge R$.

Proof. By only proving the situation of I_λ^+, we can prove the situation of I_λ^- by the same way. Get $\alpha \in (2,\frac{2N}{N-2s})$, from ($H_1$) and ($H_2$) ,which implies there is $C_\varepsilon>0$ for $\forall\varepsilon>0$,

$$F_+(x,u) \le \frac{\varepsilon}{2}u^2+C_\varepsilon u^\alpha \quad \forall x\in\Omega, \forall u>0 \tag{4}$$

According to (4), *Poincare'* in-equation, *Sobolve* in-equation and *Holder* in-equation, we have

$$\begin{aligned} I_\lambda^+(u) &\ge \frac{1}{2}\|u\|^2-\frac{\lambda}{r}\|a(x)\|_{L^2}\|u\|_{L^{2r}}^r-\frac{\lambda\varepsilon}{2}\int_\Omega |u|^2 dx-\lambda C_\varepsilon\int_\Omega |u|^\alpha dx \\ &\ge \frac{1}{2}\|u\|^2-C_r\|a(x)\|\|u\|^2-\frac{\lambda\varepsilon}{2}\|u\|^2-C_\varepsilon\|u\|^2 \\ &=(\frac{1}{2}-\frac{\lambda}{2}\varepsilon-C_r)\|u\|^2-C_\varepsilon\|u\|^2 \end{aligned}$$

Where $C_s>0$, taking any small value of $\varepsilon>0$, make sure $\frac{1}{2}-\frac{\lambda}{2}\varepsilon-C_r \ge \frac{1}{4}$. Select enough small $\|u\|=\rho>0$, we can find R>0 , so if $\|u\|=\rho>0$, we have $I_\lambda^+(u)\ge R$.

Lemma 2. If hypotheses (H_3) is available, I_λ^- and I_λ^+ are no lower bound for $\forall\lambda>0$.

Proof. From (H_3) , $\liminf_{t\to+\infty}\frac{1}{t^2}\int_\Omega F_+(x,ty(x)dx \ge \frac{1}{2\varepsilon}$ for $\varepsilon>0$.

Since ε is arbitrary, so $\lim_{t\to+\infty}\frac{1}{t^2}\int_\Omega F_+(x,ty(x)dx \to +\infty$.

We have

$$\frac{I_\lambda^+(ty)}{t^2}=\frac{1}{2}\|y\|^2-\frac{\lambda t^{r-2}}{r}\int_\Omega a(x)y^r dx-\frac{\lambda}{t^2}\int_\Omega F_+(x,ty)dx.$$

Because of r>2, so we have

$$\lim_{t\to+\infty} t^{r-2}=+\infty, \lim_{t\to+\infty}\frac{1}{t^2}\int_\Omega F_+(x,ty)dx=+\infty \ .$$

So $\lim_{t\to+\infty}\frac{I_\lambda^+(ty)}{t^2}=-\infty$ · $I_\lambda^+(ty)\to-\infty$ as $t\to+\infty.$

The following is to prove that each of the palais-Smale sequences is relatively tight.

Lemma 3. If hypotheses (H_2)-(H_4) is available, every palais-Smale sequence of $I_\lambda^\pm$ contains a convergent subsequence for all $\lambda>0$.

Proof. It is just to prove the situation of I_λ^+ is by the same way of I_λ^-. Since Ω is bounded and (H_2) is available, so if $\{u_n\}$ is bounded in $H_0^s(\Omega)$, we can get strong convergence sequence by Sobolev embedded theorem So it is just to proof that $\{u_n\}$ is bounded in $H_0^s(\Omega)$.

Supposing $\{u_n\}\subset H_0^s(\Omega)$ is the palais-Smale sequence of I_λ^+, i.e. $I_\lambda^+(u_n)\to C_\lambda$, $(I_\lambda^+)'(u_n)\to 0$, at $n\to+\infty$, it is suggested to adopt the proof by contradiction. We suppose that $\|u_n\|\to+\infty$ as $n\to+\infty$. If setting $w_n=\frac{u_n}{\|u_n\|}$ and $\|w_n\|=1$. We may assume that for some $w\in H_0^s(\Omega)$, it makes w_n weak convergence of w in $H_0^s(\Omega)$, $w_n\to w$ and in $L^2(\Omega)$, furthermore $w_n(x)\to w(x)$, a.e. $x\in\Omega$. From (H_3), by combining *Fatou lemma*, we have $w(x)\equiv 0$, a.e. $x\in\Omega$. Set $t_n\in[0,1]$ such that $I_\lambda^+(t_n u_n)=\max_{t\in[0,1]}I_\lambda^+(t u_n)$, which implies that:

$$\frac{d}{dt}I_\lambda^+(tu_n)\Big|_{t=t_n}=t_n\|u_n\|^2-\lambda t_n^{r-1}\int_\Omega a(x)u_n^r dx-\lambda\int_\Omega f_+(x,t_nu_n)dx=0\cdot$$

Since

$$(I_\lambda^+)'(t_n u_n)\cdot(t_n u_n)=t_n^2\|u_n\|^2-\lambda t_n^r\int_\Omega a(x)u_n^r dx-\lambda\int_\Omega f_+(x,t_nu_n)t_nu_n dx\cdot$$

We have $(I_\lambda^+)'(t_n u_n)\cdot(t_n u_n)=t_n\frac{d}{dt}I_\lambda^+(tu_n)\Big|_{t=t_n}=0\cdot$

From (H_4), we have

$$\begin{aligned}
&2I_\lambda^+(tu_n)\le 2I_\lambda^+(t_nu_n)-(I_\lambda^+)'(t_nu_n)(t_nu_n)\\
&=\lambda\int_\Omega(t_nu_nf_+(x,t_nu_n)-2F_+(x,t_nu_n))\,dx+\lambda(1-\frac{2}{r})t_n^r\int_\Omega a(x)u_n^r dx\\
&\le\lambda\int_\Omega(u_nf_+(x,u_n)-2F_+(x,u_n)+C_*)\,dx+\lambda(1-\frac{2}{r})\int_\Omega a(x)u_n^r dx\\
&=2I_\lambda^+(u_n)-(I_\lambda^+)'(u_n)(u_n)+\lambda|\Omega|C_*\\
&=2C_\lambda+\lambda|\Omega|C_*
\end{aligned}$$

O n the other side, $\forall R_0>0$

$$\begin{aligned}
2I_\lambda^+(R_0 w_n)&=R_0^2-\frac{2\lambda}{r}\int_\Omega a(x)(R_0 w_n)^r dx-2\lambda\int_\Omega F_+(x,R_0 w_n)dx\\
&=R_0^2+o(1)
\end{aligned}$$

The proof of the theorem 1.

From (H_1) we know $I_\lambda^\pm(0)=0$. By lemma 2, there is $e\in H_0^s(\Omega),\|e\|>\rho$ satisfies $I_\lambda^\pm(e)<0$.Besides, there are ρ and $R>0$,so we have $I_\lambda^\pm(u)\big|_{\partial B_\rho}\ge R$·

Define

$$c_\lambda^\pm=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda^\pm(\gamma(t)),$$

Where

$$\Gamma=\left\{\gamma:[0,1]\to H_0^s(\Omega):\gamma continuous,\gamma(0)=0,\gamma(1)=e\right\}.$$

By lemma 3, we have $I_\lambda^\pm$ satisfying the *Palais-Smale* condition. By mountain pass theorem, we know that c_λ^+ is the critical value of I_λ^+ and there are at least a nontrivial critical point $u_\lambda^+\in H_0^s(\Omega)$ satisfying $I_\lambda^+(u_\lambda^+)=c_\lambda^+.Clearly,u_\lambda^+\ge 0.$ By the strong maximum principle, we have $u_\lambda^+>0$, so u_λ^+ is the positive solution of (1). In the say way, there are at least a negative solution $u_\lambda^-\in H_0^s(\Omega)$. As a result, there are at least a positive solution and a negative solution on problem (1).

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