Analysis of a Stochastic Lotka-Volterra Competitive System with Distributed Delays

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Abstract—In this paper, we consider a n -species stochastic Lotka-Volterra competitive system with distributed delays. Sufficient conditions for stability in the mean and extinction of each population are established.

Keywords—competitive system; extinction; delays; stochastic perturbations

I. INTRODUCTION

For the last decades, differential equations with delays have received great attention owing to their extensive application as models in a variety of scientific areas [1]. For example-the delay differential equation

$$\frac{dx(t)}{dt} = x(t)[r - ax(t - \tau)].$$

However, in the real world, population systems are often subject to environmental noise, May [2] has pointed out that due to environmental noises the growth rates in population systems should be stochastic[3-7]. In [3], Meng Liu and Ke Wang studied a delay Lotka-Volterra competitive system with random perturbations; as follows:

$$\begin{cases} dx_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_1)]dt + \sigma_1 x_1(t)dB_1(t) \\ dx_2(t) = x_2(t)[r_2 - a_{21}x_1(t - \tau_2) - a_{22}x_2(t)]dt + \sigma_2 x_2(t)dB_2(t) \end{cases} \quad (1)$$

In practice, we usually estimate the growth rate r_i by an error term plus an average value. According to the famous central limit theorem the error term may be approximated by a normal distribution. At the same time, the noise on r_i may or may not correlate to each other. Taking these possible correlations into account, we hence use n independent Wiener processes $(B_1(t),$

$B_2(t), \dots, B_n(t))$ to model the above normal distribution, e.g.

$r_i \rightarrow r_i + \sum_{j=1}^n \sigma_{ij} dB_j(t)$, where σ_{ij}^2 denotes the intensity of the noise, and $B_j(t)$ is a standard Brownian motion defined on a complete probability space (Ω, F, P) . In this paper, we shall consider a stochastic competitive system with continuous delays:

$$\begin{aligned} dx_i(t) = & x_i(t)[r_i - \sum_{j=1}^n a_{ij}x_j(t) - \sum_{j=1}^n b_{ik} \int_{-\tau}^0 x_j(t+s)du_{ij}(s)]dt \\ & + \sum_{j=1}^n \sigma_{ij}x_i(t)dB_j(t) \end{aligned} \quad (2)$$

where r_i , τ , a_{ik} , b_{ik} are positive constants, σ_{ij} are non-negative constants, the functions $u_{ij}(s)$ is a probability measure on $[-\tau, 0]$, have $\int_{-\tau}^0 du_{ij}(s) = 1$, $i, j = 1, 2, \dots, n$.

The initial conditions of system (2)

$x_i(\theta) = \phi_i(\theta) > 0$, $\theta \in [-\tau, 0]$, $i = 1, 2, \dots, n$. where $\phi_i(\theta)$ is a continuous function on $[-\tau, 0]$.

For simplicity, here we introduce the following notations: Let set I , J , I_m are nonempty set, where $I = \{1, 2, 3, \dots, n\}$, $I_m = \{l_1, l_2, \dots, l_m\}$ and satisfies

$$J \cap I_m = \emptyset, I = I_m \cup J \text{ and } l_1 < l_2 < \dots < l_m.$$

$$f^* = \limsup_{t \rightarrow +\infty} f(t), \quad f_* = \liminf_{t \rightarrow +\infty} f(t),$$

$$\langle f(t) \rangle = \frac{1}{t} \int_0^t f(s)ds,$$

$$R_+^n = \{(x_1, x_2, \dots, x_n) \mid x_i > 0, i = 1, 2, \dots, n\}.$$

We can easy to see that system (2) has no positive equilibrium. Hence, it is a interesting and important to study that of whether system (2) still has some stability around some positive point. In this paper, we shall show it.

II. MAIN RESULTS

First, we introduce some lemmas which are useful and important.

Lemma 2.1 ([4]). Suppose that $z(t) \in C(\Omega \times R_+)$,

(i) If there exist two positive constants T , and u such that $\ln z(t) \leq \lambda t - u \int_0^t z(s)ds + \sum_{i=1}^n \alpha_i B_i(t)$ for all $t \geq T$, where α_i is a constant, then

$$\begin{cases} \langle z(t) \rangle^* \leq \frac{\lambda}{u}, & a.s. \text{ if } \lambda \geq 0 \\ \lim_{t \rightarrow +\infty} z(t) = 0, & a.s. \text{ if } \lambda < 0. \end{cases}$$

(ii) If there exist three positive constants λ , T and u such that $\ln z(t) \geq \lambda t - u \int_0^t z(s) ds + \sum_{i=1}^n \alpha_i B_i(t)$, for all $t \geq T$, have

$$\langle z(t) \rangle_* \geq \frac{\lambda}{u}, \quad a.s.$$

Lemma 2.2 For any given initial value $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \dots, \phi_n(\theta)) \in C([- \tau, 0], R_+^n)$, system (2) have a unique global positive solution $(x_1(t), x_2(t), \dots, x_n(t))$ on $t \geq -\tau$ and the solution will remain in R_+^n with probability 1, and the solution satisfies

$$\left[\frac{\ln x_i(t)}{t} \right]^* \leq 0 \quad a.s. \quad i=1, 2, \dots, n.$$

Proof: Our proof is motivated by the paper of [5], Since the coefficient of system (2) are locally Lipschitz continuous, for any given initial value $\phi(\theta) \in C([- \tau, 0], R_+^n)$, there is a unique maximal local solution $x(t)$ on $t \in [- \tau, \tau_e)$, where τ_e is the explosion time. So, we only need to show that $\tau_e = \infty$. The proof is similar to [6] by defining

$$V(x) = \sum_{i=1}^n (x_i - 1 - \ln x_i)$$

Applied itô's formula to $V(x)$ have

$$\begin{aligned} dV(x) &= \sum_{i=1}^n \left[(x_i - 1) \left[r_i - \sum_{j=1}^n a_{ij} x_j - \sum_{j=1}^n b_{ij} \int_{-\tau}^0 x_j(t+s) du_{ij}(s) \right] \right. \\ &\quad \left. + \sum_{j=1}^n \sigma_{ij} dB_j(t) + 0.5 \sum_{j=1}^n \sigma_{ij}^2 \right] \\ &\leq \sum_{i=1}^n \left[r_i + \sum_{j=1}^n a_{ij} x_j - a_{ii} x_i^2 + \sum_{j=1}^n b_{ij} \int_{-\tau}^0 x_j(t+s) du_{ij}(s) \right. \\ &\quad \left. + \sum_{j=1}^n \sigma_{ij} dB_j(t) + 0.5 \sum_{j=1}^n \sigma_{ij}^2 \right] \\ &\leq K dt + \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} dB_j(t), \end{aligned}$$

where K is a positive constant. By the similar proof of [6], we can obtain system (2) have a unique globally positive solution $(x_1, x_2, \dots, x_N)$ on $t \geq -\tau$, we omitted it in here.

For $k \in I$, consider the following question:

$$dy_i(t) = y_i(t) [r_i - a_{ii} y_i(t)] dt + \sum_{j=1}^n \sigma_{ij} y_i(t) dB_j(t) \quad (3)$$

$$y_i(\theta) = x_i(\theta), \quad \theta \in [-\tau, 0].$$

By virtue of the famous stochastic comparison theorem, we can obtain that $y_i(t) \geq x_i(t)$, then $y_i(t)$ have the following explicit representations respectively

$$y_i(t) = \frac{\exp \{ r_i t - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2 t + \sigma_i \sum_{j=1}^n B_{ij}(t) \}}{\frac{1}{x_i(0)} + a_{ii} \int_0^t \exp \{ r_i s - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2 s + \sigma_i \sum_{j=1}^n B_{ij}(s) \} ds},$$

The proof of $\left[\frac{\ln x_i(t)}{t} \right]^* \leq 0$ a.s. $i=1, 2, \dots, n$ is similar to the method of lemma 3 in [5] $\square$.

If $r_i - 0.5 \sum_{j=1}^n \sigma_{ij}^2 > 0$, by (i) of lemma 2.1 and (3), we have

$$\lim_{t \rightarrow +\infty} \langle y_i(t) \rangle = \frac{r_i - 0.5 \sum_{j=1}^n \sigma_{ij}^2}{a_{ii}}, \quad a.s. \quad i=1, 2, \dots, n. \quad (4)$$

Therefore:

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{t} \int_{-\tau}^0 \int_t^{t+s} y_i(p) dp du_{ij}(s) &= \int_{-\tau}^0 \lim_{t \rightarrow +\infty} \left[\frac{1}{t} \int_0^{t+s} y_i(p) dp - \frac{1}{t} \int_0^t y_i(p) dp \right] du_{ij}(s) \\ &= 0 \quad a.s., \quad i, j=1, 2, \dots, n. \end{aligned}$$

then, together with (3), have

$$\lim_{t \rightarrow +\infty} t^{-1} \int_{-\tau}^0 \int_t^{t+s} x_i(p) dp du_{ij}(s) = 0, \quad a.s., \quad i, j=1, 2, \dots, n. \quad (5)$$

For convenience,

$$\text{Let } R_i = r_i - 0.5 \sum_{j=1}^n \sigma_{ij}^2, \quad A = (a_{ij} + b_{ij})|_{n \times n}, \quad B = (a_{li} + b_{li})|_{m \times m}.$$

$$M_i = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1(i-1)} + b_{1(i-1)} & R_i & a_{1(i+1)} + b_{1(i+1)} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \dots & a_{2(i-1)} + b_{2(i-1)} & R_i & a_{2(i+1)} + b_{2(i+1)} & \dots & a_{2n} + b_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1} & \dots & a_{n(i-1)} + b_{n(i-1)} & R_n & a_{n(i+1)} + b_{n(i+1)} & \dots & a_{nn} + b_{nn} \end{bmatrix},$$

$i=1, 2, \dots, n.$

$$A_k = (-1)^{i+k} \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1(i-1)} + b_{1(i-1)} & a_{1(i+1)} + b_{1(i+1)} & \dots & a_{1n} + b_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{(k-1)1} + b_{(k-1)1} & \dots & a_{(k-1)(i-1)} + b_{(k-1)(i-1)} & a_{(k-1)(i+1)} + b_{(k-1)(i+1)} & \dots & a_{(k-1)n} + b_{(k-1)n} \\ a_{(k+1)1} + b_{(k+1)1} & \dots & a_{(k+1)(i-1)} + b_{(k+1)(i-1)} & a_{(k+1)(i+1)} + b_{(k+1)(i+1)} & \dots & a_{(k+1)n} + b_{(k+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1} & \dots & a_{n(i-1)} + b_{n(i-1)} & a_{n(i+1)} + b_{n(i+1)} & \dots & a_{nn} + b_{nn} \end{bmatrix}$$

$i, k=1, 2, \dots, n.$

$$N_i = \begin{vmatrix} a_{i1} + b_{i1} & \dots & a_{i1} + b_{i1} & R_i & a_{i1} + b_{i1} & \dots & a_{i1} + b_{i1} \\ a_{i2} + b_{i2} & \dots & a_{i2} + b_{i2} & R_i & a_{i2} + b_{i2} & \dots & a_{i2} + b_{i2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{im} + b_{im} & \dots & a_{im} + b_{im} & R_i & a_{im} + b_{im} & \dots & a_{im} + b_{im} \end{vmatrix}$$

$i = 1, 2, \dots, m.$

$$N_{ki} = (-1)^{k+i} \begin{vmatrix} a_{i1} + b_{i1} & \dots & a_{i1} + b_{i1} & a_{i1} + b_{i1} & \dots & a_{i1} + b_{i1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} + b_{k1} & \dots & a_{k1} + b_{k1} & a_{k1} + b_{k1} & \dots & a_{k1} + b_{k1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{im} + b_{im} & \dots & a_{im} + b_{im} & a_{im} + b_{im} & \dots & a_{im} + b_{im} \end{vmatrix},$$

$i, k = 1, 2, \dots, m.$

Lemma 2.3 It is important to point out that if $R_i > 0$ and $A > 0$, then $M_i < 0, i=1, 2, \dots, n$ can't hold simultaneously.

Proof. If this assertion is not true, then $M_i < 0, i=1, 2, \dots, n$, then we easy obtain $M_i = \sum_{k=1}^n R_k A_{ki} < 0$, because of a_{ij}, b_{ij} are positive constants, then $(a_{li} + b_{li})M_i = \sum_{k=1}^n R_k (a_{li} + b_{li}) A_{ki} < 0$, therefore, we easy see that

$$\begin{aligned} \sum_{i=1}^n (a_{li} + b_{li}) M_i &= \sum_{i=1}^n \sum_{k=1}^n R_k (a_{li} + b_{li}) A_{ki} \\ &= \sum_{k=1}^n R_k \sum_{i=1}^n (a_{li} + b_{li}) A_{ki} = R_i A < 0, \end{aligned}$$

so $A < 0$, the contradiction arises.

Therefore, if $R_i > 0$ and $A > 0$, then $M_i < 0 (i=1, 2, \dots, n)$ can't hold simultaneously.

Theorem 2.1 (i) Suppose $R_i > 0$ and $A > 0, i=1, 2, \dots, n$

(a) If $M_i > 0, A_{ii} > 0$ and $A_{ki} \leq 0, (k \neq i, k, i \in 1, 2, \dots, n)$, then

x_i is stable in time average a.s.,

$$\lim_{t \rightarrow +\infty} \langle x_i(t) \rangle = M_i / A, \text{ a.s. } i = 1, 2, \dots, n.$$

(b) Suppose $M_{l_i} > 0$ and $M_k < 0, l_i \in I_m, k \in J$, if $B > 0$,

$N_i > 0, N_{ii} > 0$ and $N_{ki} < 0, (k \neq i, k, i = 1, 2, \dots, m)$, then x_k goes to extinction almost surely and x_{l_i} is stable in time average a.s.,

$$\lim_{t \rightarrow +\infty} \langle x_{l_i}(t) \rangle = N_i / B, \text{ a.s. } l_i \in I_m, i = 1, 2, \dots, m$$

(ii) Suppose $R_i < 0$, then x_i goes to extinction almost surely,

$$\lim_{t \rightarrow +\infty} x_i(t) = 0 \text{ a.s. } i = 1, 2, \dots, n.$$

(iii) Suppose $R_{l_i} > 0$ and $R_k < 0, l_i \in I_m, k \in J$, if $B > 0$,

$N_i > 0, N_{ii} > 0$ and $N_{ki} < 0, (k \neq i, k, i = 1, 2, \dots, m)$, then x_{l_i} is stable in time average a.s. and x_k goes to extinction almost surely,

$$\lim_{t \rightarrow +\infty} \langle x_{l_i}(t) \rangle = N_i / B, \text{ a.s.}, l_i \in I_m, i = 1, 2, \dots, m$$

Proof: Applying Itô's formula to (2), have

$$\begin{aligned} \frac{\ln x_i / x_i(0)}{t} &= R_i + \sum_{j=1}^n b_{ij} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_j(p) dp - \int_0^s x_j(p) dp \right] du_j(s) \\ &\quad - \sum_{j=1}^n b_{ij} \langle x_j(t) \rangle - \sum_{j=1}^n a_{ij} \langle x_j(t) \rangle + \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t}. \end{aligned} \quad (6)$$

So

$$\begin{aligned} A_{ii} \frac{\ln x_i(t) / x_i(0)}{t} &+ \sum_{k \neq i, k=1}^n A_{ki} \frac{\ln x_k(t) / x_k(0)}{t} = \sum_{k=1}^n R_k A_{ki} \\ &+ A_{ii} \sum_{j=1}^n b_{ij} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_i(p) dp - \int_0^s x_i(p) dp \right] du_j(s) \\ &+ \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n b_{kj} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_k(p) dp - \int_0^s x_k(p) dp \right] du_j(s) \\ &- \left(\sum_{j=1}^n (a_{ij} + b_{ij}) A_{ii} + \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n (a_{kj} + b_{kj}) \langle x_j(t) \rangle \right) \\ &+ A_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} + \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &= M_i - \left(\sum_{j=1}^n \sum_{k=1}^n A_{ki} (a_{kj} + b_{kj}) \langle x_j(t) \rangle \right) \\ &+ A_{ii} \sum_{j=1}^n b_{ij} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_i(p) dp - \int_0^s x_i(p) dp \right] du_j(s) \\ &+ \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n b_{kj} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_k(p) dp - \int_0^s x_k(p) dp \right] du_j(s) \\ &+ A_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} + \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &= M_i - A \langle x_i(t) \rangle + A_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} + \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &+ A_{ii} \sum_{j=1}^n b_{ij} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_i(p) dp - \int_0^s x_i(p) dp \right] du_j(s) \\ &+ \sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n b_{kj} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_k(p) dp - \int_0^s x_k(p) dp \right] du_j(s). \end{aligned} \quad (7)$$

(a) By Lemma 2.1 and (5), for arbitrary $\varepsilon > 0$ there is a $T > 0$ such that $t \geq T$ have

$$- \sum_{k \neq i, k=1}^n A_{ki} \frac{\ln x_k(t) / x_k(0)}{t} < \varepsilon / 3,$$

$$A_{ii} \sum_{j=1}^n b_{ij} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_i(p) dp - \int_0^s x_i(p) dp \right] du_{ij}(s) < \varepsilon / 3,$$

$$\sum_{k \neq i, k=1}^n A_{ki} \sum_{j=1}^n b_{kj} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_k(p) dp - \int_0^s x_k(p) dp \right] du_{kj}(s) < \varepsilon / 3.$$

Substituting the above inequalities into (7), we can obtain

$$A_{ii} \frac{\ln x_i(t) / x_i(0)}{t} \leq M_i + \varepsilon - A \langle x_i(t) \rangle + \sum_{k=1}^n A_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \quad (8)$$

Since $M_i > 0$ and $A > 0$, then, by virtue of lemma 2.1 and the arbitrariness of ε , we get

$$\langle x_i(t) \rangle^* \leq M_i / A \quad (9)$$

According to (6) and (9) we easy see that

$$\begin{aligned} \frac{\ln x_i(t) / x_i(0)}{t} &\geq \left[R_i - \sum_{j=1, j \neq i}^n \frac{(a_{ij} + b_{ij}) M_j}{A} - (a_{ii} + b_{ii}) \langle x_i(t) \rangle \right] \\ &\quad + \sum_{j=1}^n \frac{\sigma_{ij} B_j(t)}{t} \\ &= \frac{(a_{ii} + b_{ii}) M_i}{A} - (a_{ii} + b_{ii}) \langle x_i(t) \rangle + \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t}. \end{aligned} \quad (10)$$

By virtue of Lemma 2.1 have $\langle x_i(t) \rangle_* \geq M_i / A$ a.s., together with (9), we can obtain

$$\lim_{t \rightarrow +\infty} \langle x_i(t) \rangle = M_i / A, \quad a.s. \quad i = 1, 2, \dots, n.$$

(b) Since $M_k < 0$, then, we can choose ε sufficiently small, such that $M_k + \varepsilon < 0$, by (8) and Lemma 2.1, we easy get $\lim_{t \rightarrow +\infty} x_k(t) = 0$, $k \in J$. Then, for arbitrary $\varepsilon > 0$ there is a $T > 0$ such that $t \geq T$ have

$$\begin{aligned} -\frac{\varepsilon}{2} &< \sum_{k \in J} b_{ik} t^{-1} \int_0^t \int_{-\tau}^0 x_k(t+s) du_{ik}(s) < \frac{\varepsilon}{2}, \\ -\frac{\varepsilon}{2} &< \sum_{k \in J} t^{-1} \ln x_k(t) < \frac{\varepsilon}{2}, \\ -\frac{\varepsilon}{2} &< \sum_{k=1}^m b_{ik} \int_{-\tau}^0 t^{-1} \left[\int_t^{t+s} x_k(p) dp - \int_0^s x_k(p) dp \right] du_{ik}(s) < \frac{\varepsilon}{2}. \end{aligned}$$

Then, we substituting the above inequalities into (6), we can obtain that $t \geq T$,

$$\begin{aligned} \frac{\ln x_i(t)}{t} &\geq \left[R_i - \varepsilon - (a_{il_i} + b_{il_i}) \langle x_i(t) \rangle - \sum_{k=1, k \neq i}^m (a_{ik} + b_{ik}) \langle x_k(t) \rangle \right] \\ &\quad + \sum_{j=1}^n \frac{\sigma_{ij} B_j(t)}{t}. \end{aligned} \quad (11)$$

Applied the similar way of (7) to (10), we can get

$$\begin{aligned} N_{ii} \frac{\ln x_i(t) / x_i(0)}{t} &+ \sum_{k=1, k \neq i}^m N_{ki} \frac{\ln x_k(t) / x_k(0)}{t} \\ &\leq \sum_{k=1}^m (R_i + \varepsilon) N_{ki} - [(a_{il_i} + b_{il_i}) N_{ii} + \sum_{k=1, k \neq i}^m (a_{ik} + b_{ik}) N_{ki}] \langle x(t) \rangle \\ &\quad + N_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} + \sum_{k=1, k \neq i}^m N_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &\leq N_i + \sum_{k=1}^m \varepsilon N_{ki} - B \langle x_i(t) \rangle + N_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &\quad + \sum_{k \neq i, k=1}^m N_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t}. \end{aligned} \quad (12)$$

In the same way, by (11) we have

$$\begin{aligned} N_{ii} \frac{\ln x_i(t) / x_i(0)}{t} &+ \sum_{k=1, k \neq i}^m N_{ki} \frac{\ln x_k(t) / x_k(0)}{t} \\ &\geq \sum_{k=1}^m (R_i - \varepsilon) N_{ki} + N_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &\quad - ((a_{il_i} + b_{il_i}) N_{ii} + \sum_{k=1, k \neq i}^m (a_{ik} + b_{ik}) N_{ki}) \langle x_i(t) \rangle \\ &\quad + \sum_{k=1, k \neq i}^m N_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &\geq N_i - \sum_{k=1}^m \varepsilon N_{ki} - B \langle x_i(t) \rangle + N_{ii} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t} \\ &\quad + \sum_{k \neq i, k=1}^m N_{ki} \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t}. \end{aligned} \quad (13)$$

By virtue of lemma 2.1 and the arbitrariness of ε , we can obtain from (12) and (13)

$$\lim_{t \rightarrow +\infty} \langle x_i(t) \rangle = N_i / B \quad a.s., \quad l_i \in I_m, i = 1, 2, \dots, m.$$

(ii) Suppose $R_i < 0$, $i = 1, 2, \dots, n$, by (6), we have

$$\frac{\ln x_i(t) / x_i(0)}{t} \leq R_i + \sum_{j=1}^n \sigma_{ij} \frac{B_j(t)}{t}.$$

It follows from $\lim_{t \rightarrow +\infty} t^{-1} B_j(t) = 0$, a.s. $j = 1, 2, \dots, n$, and $R_i < 0$

that $\limsup_{t \rightarrow +\infty} t^{-1} \ln x_i(t) \leq R_i < 0$, a.s. $i = 1, 2, \dots, n$, Therefore,

$$\lim_{t \rightarrow +\infty} x_i(t) = 0, \quad a.s. \quad i = 1, 2, \dots, n.$$

(iii) Suppose $R_{l_i} > 0$ and $R_k < 0$, $l_i \in I_m$, $k \in J$, $i = 1, 2, \dots, m$, by the proof of (ii), we can show that if $R_k < 0$, then $\lim_{t \rightarrow +\infty} x_k(t) = 0$, a.s. $k \in J$. next, the proof is similar to that of (i)(b), hence we omitted it in here. $\square$

III. CONCLUSIONS

This paper is devoted to study the dynamics of a stochastic competitive system with distributed delays. Sufficient conditions for stability in the mean and extinction of each population are established. These results are useful and

important for study species coexist. Some interesting equations deserve further to study.

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