Research on the Problem of Maximum-Minimum Related to Probability of n Events

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Abstract—The problem of maximum and minimum on the difference between the product of two events' probability and the probability of product of two events has been discussed, in this paper, two events are generalized to n events and the problem is solved by using the method of induction and construction.

Keywords—probability; maximum; minimum; inductive method; constructive method

I. INTRODUCTION

The maximum- minimum problem of the probability of two events is discussed in [1-3], Proposition 1 discussed the maximum and Proposition 2 discussed the minimum problem.

Proposition 1.

$$P(AB) - P(A)P(B) \leq \frac{1}{4} \quad (1)$$

Proposition 2.

$$-\frac{1}{4} \leq P(AB) - P(A)P(B) \quad (2)$$

In the paper, we generalized Proposition 1 and Proposition 2 from two events to three events and n events, used inductive method and constructive method to solve the problem, the maximum value of three events is discussed in Proposition 3, the maximum value of n events is discussed in Proposition 4, the minimum value of three events is discussed in Proposition 5, the minimum value of n events is discussed in Proposition 6.

II. THE MAXIMUM VALUE

Proposition 3.

$$P(ABC) - P(A)P(B)P(C) \leq \frac{2}{9}\sqrt{3} \quad (3)$$

Proof.

Assume that

$$P(A) \geq P(B) \geq P(C),$$

$$P(ABC) - P(A)P(B)P(C) \leq P(C)(1 - (P(C))^2)$$

Define f(x) as follows:

$$f(x) = x - x^3, x \in [0, 1], f(x) \leq \frac{2}{9}\sqrt{3}$$

$$P(ABC) - P(A)P(B)P(C) \leq \frac{2}{9}\sqrt{3}$$

$$\text{if } P(A) = P(B) = P(C) = \frac{\sqrt{3}}{3}, A = B = C,$$

the equality is true.

The proof is complete.

Proposition 4.

$$P\left(\prod_{i=1}^n A_i\right) - \prod_{i=1}^n P(A_i) \leq \left(\frac{1}{n}\right)^{\frac{1}{n-1}} \left(1 - \frac{1}{n}\right) \quad (4)$$

Proof.

Assume that

$$P(A_1) \geq P(A_2) \geq \dots \geq P(A_n),$$

$$P\left(\prod_{i=1}^n A_i\right) - \prod_{i=1}^n P(A_i) \leq P(A_n) - P^n(A_n)$$

Define f(x) as follows:

$$f(x) = x - x^n, x \in [0, 1].$$

the maximum value of f(x) is

$$\left(\frac{1}{n}\right)^{\frac{1}{n-1}} \left(1 - \frac{1}{n}\right),$$

$$P\left(\prod_{i=1}^n A_i\right) - \prod_{i=1}^n P(A_i) \leq \left(\frac{1}{n}\right)^{\frac{1}{n-1}} \left(1 - \frac{1}{n}\right),$$

$$\text{if } P(A_1) = P(A_2) = \dots = P(A_n) = \left(\frac{1}{n}\right)^{\frac{1}{n-1}},$$

$$A_1 = A_2 = \dots = A_n,$$

the equality is true.

The proof is complete.

III. MINIMUM VALUE

Proposition 5.

$$-\frac{8}{27} \leq P(ABC) - P(A)P(B)P(C) \quad (5)$$

Proof.

$$\begin{aligned} & P(A)P(B)P(C) - P(ABC) \\ &= P(A)P(B)[P(ABC) + P(\overline{ABC})] - P(ABC) \\ &= P(A)P(B)P(\overline{ABC}) - P(ABC)(1 - P(A)P(B)) \\ &\leq P(A)P(B)P(\overline{ABC}) \\ &\leq P(A)P(B)P(\overline{AB}) = P(A)P(B)(1 - P(AB)) \\ &P(A \cup B) = P(A) + P(B) - P(AB) \leq 1, \\ &P(A) + P(B) + 1 - P(AB) \leq 2 \\ &P(A)P(B)(1 - P(AB)) \\ &\leq \left(\frac{P(A) + P(B) + 1 - P(AB)}{3}\right)^3 \leq \left(\frac{2}{3}\right)^3 = \frac{8}{27}, \end{aligned}$$

$$\text{if } P(A) = P(B) = P(C) = \frac{2}{3},$$

$$P(AB) = P(AC) = P(BC) = \frac{1}{3}, P(ABC) = 0,$$

the equality is true.

The proof is complete.

Proposition 6.

$$-\left(\frac{n-1}{n}\right)^n \leq P\left(\bigcap_{i=1}^n A_i\right) - \prod_{i=1}^n P(A_i) \quad (n \geq 3) \quad (6)$$

Proof.

$$\begin{aligned} & \prod_{i=1}^n P(A_i) - P\left(\bigcap_{i=1}^n A_i\right) \\ &= \left(\prod_{i=1}^{n-1} P(A_i)\right) \left(P\left(\bigcap_{i=1}^n A_i\right) + P\left(\bigcap_{i=1}^{n-1} A_i \overline{A_n}\right)\right) - P\left(\bigcap_{i=1}^n A_i\right) \\ &= P\left(\bigcap_{i=1}^{n-1} A_i \overline{A_n}\right) \left(\prod_{i=1}^{n-1} P(A_i)\right) - P\left(\bigcap_{i=1}^n A_i\right) \left(1 - \prod_{i=1}^{n-1} P(A_i)\right) \\ &\leq P\left(\bigcap_{i=1}^{n-1} A_i \overline{A_n}\right) \left(\prod_{i=1}^{n-1} P(A_i)\right) \\ &\leq P\left(\bigcap_{i=1}^{n-1} A_i\right) \left(\prod_{i=1}^{n-1} P(A_i)\right) \\ &= \left(1 - P\left(\bigcap_{i=1}^{n-1} A_i\right)\right) \left(\prod_{i=1}^{n-1} P(A_i)\right) \end{aligned}$$

We use inductive method to prove the conclusion as follows:

$$\sum_{i=1}^{n-1} P(A_i) - P\left(\bigcap_{i=1}^{n-1} A_i\right) \leq n-2 \quad (7)$$

$$n=3, P(A_1 \cup A_2) = \sum_{i=1}^2 P(A_i) - P\left(\bigcap_{i=1}^2 A_i\right) \leq 1$$

the inequality is true.

We assume that if $n=k$, then

$$\sum_{i=1}^{k-1} P(A_i) - P\left(\bigcap_{i=1}^{k-1} A_i\right) \leq k-2, \quad (8)$$

if $n=k+1$,

$$\begin{aligned} & P\left(\bigcap_{i=1}^k A_i\right) = P\left(\left(\bigcap_{i=1}^{k-1} A_i\right) \cap A_k\right) \\ &= P\left(\bigcap_{i=1}^{k-1} A_i\right) + P(A_k) - P\left(\left(\bigcap_{i=1}^{k-1} A_i\right) \cup A_k\right) \\ &\geq \sum_{i=1}^{k-1} P(A_i) - (k-2) + P(A_k) - P\left(\left(\bigcap_{i=1}^{k-1} A_i\right) \cup A_k\right) \\ &\geq \sum_{i=1}^k P(A_i) - (k-1) \\ &\quad \left(P(A_k) \geq 0, P\left(\left(\bigcap_{i=1}^{k-1} A_i\right) \cup A_k\right) \leq 1 \right) \end{aligned}$$

$$\sum_{i=1}^k P(A_i) - P(\bigcap_{i=1}^k A_i) \leq k-1$$

So,(7) is proved.

$$\begin{aligned} & (1 - P(\bigcap_{i=1}^{n-1} A_i)) \prod_{i=1}^{n-1} P(A_i) \\ & 1 - P(\bigcap_{i=1}^{n-1} A_i) + \sum_{i=1}^{n-1} P(A_i) \\ & \leq \left(\frac{1 - P(\bigcap_{i=1}^{n-1} A_i) + \sum_{i=1}^{n-1} P(A_i)}{n} \right)^n \leq \left(\frac{n-1}{n} \right)^n, \end{aligned}$$

$$-\left(\frac{n-1}{n}\right)^n \leq P(\bigcap_{i=1}^n A_i) - \prod_{i=1}^n P(A_i)$$

$$\text{if } P(A_1) = P(A_2) = \dots = P(A_n) = \frac{n-1}{n},$$

$$P(\bigcap_{i=1}^{n-1} A_i) = P(\bigcap_{\substack{i=1 \\ i \neq n-1}}^n A_i) = \dots = P(\bigcap_{\substack{i=2 \\ i \neq n-1}}^n A_i) = \frac{1}{n},$$

$$\bigcap_{i=1}^n A_i = \phi$$

the equality is true.

We define n events as follows:

$$\{A_i, i = 1, 2, 3, \dots, n\},$$

$$I = \{1, 2, 3, \dots, n\}$$

$$A_1 = \{2, 3, \dots, n\}, A_2 = \{1, 3, \dots, n\}, \dots, A_n = \{1, 2, 3, \dots, n-1\}$$

$$P(A_1) = P(A_2) = \dots = P(A_n) = \frac{n-1}{n},$$

$$P(\bigcap_{i=1}^{n-1} A_i) = P(\bigcap_{\substack{i=1 \\ i \neq n-1}}^n A_i) = \dots = P(\bigcap_{\substack{i=2 \\ i \neq n-1}}^n A_i) = \frac{1}{n}, \quad \bigcap_{i=1}^n A_i = \phi,$$

$$-\left(\frac{n-1}{n}\right)^n \leq P(\bigcap_{i=1}^n A_i) - \prod_{i=1}^n P(A_i).$$

The proof is complete.

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