Mixed finite volume method for elliptic equations problems

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Abstract. This paper briefly describes the relevant conclusions of the mixed finite volume method is mainly based on a set and two mesh mixed finite volume method on triangular mesh, also lists the convergence results corresponding format.

1 Introduction

A mesh of elliptic equations with mixed finite volume method problem

1.1 Mixed finite volume method triangular grid

Consider the second-order elliptic equations:

$$\begin{cases} -\nabla\cdot\kappa\nabla p = f & in\Omega \\ p = 0 & on\partial\Omega \end{cases} \qquad (1)$$

Introducing a new flow variable：$u := -\kappa\nabla p$, Then the equation can be rewritten as a first-order partial differential equations below

$$\begin{cases} \nabla\cdot u - f = 0 & in\Omega \\ u + \kappa\nabla p = 0 & in\Omega \\ p = 0 & on\partial\Omega \end{cases} \qquad (2)$$

Mixed finite volume method：Find $(u_h, p_h)\in V_h\times Y_{h,0}$ making

$$\begin{cases} (\nabla\cdot u_h - f, \chi_K) = 0 \\ (u_h + \kappa\nabla p_h, \chi_K) = 0 \end{cases} \qquad (3)$$

Error estimates：

If the data of the problem is smooth enough data，Making $p\in H^2\cap H_0{}^1$, and $u(x) = --\kappa\nabla p(x)\in H^1(\Omega)^2$,

and We assume $\kappa\in W^{1,\infty}$, then there exists a plus constant C independent of h making

$\|p-p_h\|_0 \le Ch^2(|f|_h + \|f\|_0),\ f|_K \in H^1(K), \forall K\in\Gamma_h$

$|p-p_h|_h \le Ch\|f\|_0$

$\|u-u_h\|_0 \le Ch(\|f\|_0 + \|u\|_0)$

$|u-u_h|_{H(div;\Omega)} \le Ch|f|_h,\ f|_K \in H^1(K), \forall K\in\Gamma_h$

1.2 Online quadrilateral mixed finite volume method

Consider a common second order elliptic problem with boundary condition:

$$-\nabla(\kappa\nabla p) = f(x), \forall x\in\Omega \qquad (4)$$

Weak form：

$$\begin{cases} (\kappa^{-1}\underline{u}, v) - \sum_{Q\in Q_h}(\int_q \nabla\underline{v}p - \int_{\partial Q} p\underline{v}\cdot n) = 0, & \forall v\in H_{loc}(div); \\ \sum_{T\in\Gamma_h}(\int_T -\nabla q\cdot\underline{u} + \int_{\partial T}\underline{u}\cdot nq) = (f,q), & \forall q\in H^1{}_{loc}(\Gamma_h) \end{cases} \qquad (5)$$

We depart from the weak form derived finite volume method, we establish two control volume method: a symmetrical and the other asymmetrical.Defind a bilinear form defined

$\mathrm{A}(v,p;w,q)$ on $(V_h',W_h)\times(\overline{V}_h,W_h)$

$$\mathrm{A}(v,p;w,q)=a(v,w)+b(w,p)-b(v,q) \qquad (6)$$

Then a non-symmetric control volume method：

Order $V_h'=\widehat{V}_h,\overline{V}_h=\gamma_h\widehat{V}_h$, That find

$\widehat{u}_h\in\widehat{V}_h,\ p_h\in W_h$,

$$\text{Making } \mathrm{A}(\widehat{u}_h,p_h;\gamma_h\widehat{w},q)=(f,q) \qquad (7)$$

Another symmetrical control volume method：Order

$$V_h'=\overline{V}_h=\gamma_h\widehat{V}_h,$$

that find $\widehat{u}_h\in\widehat{V}_h,\ p_h\in W_h$, Making

$$\mathrm{A}(\gamma_h\widehat{u}_h,p_h;\gamma_h\widehat{w},q)=(f,q) \qquad (8)$$

Error estimates：

Theorem：Assuming u is a solution of the weak form factor (5)，The u_h is the solution of symmetrical method (8) or asymmetric methods (7),then there exits a number $C_1>0$ independent of h,making

$$\|u-u_h\|_0+\|p-p_h\|_0\le C_1h(\|u\|_1+\|p\|_1) \qquad (9)$$

2 Two meshing problems elliptic equations mixed finite volume method

2.1 Mixed finite volume method triangular grid

We still consider the second-order elliptic equations on a plane convex domains:

$$\begin{cases}-\nabla\cdot\kappa\nabla p=f & in\Omega\\ \quad p=0 & on\partial\Omega\end{cases} \qquad (10)$$

Which is symmetric positive definite matrix functions and meet the conditions, we introduce a new variable flow， The problem can be transformed into a mixed question follows:

$$\begin{cases}\nabla\cdot u-f=0 & in\Omega\\ u+\kappa\nabla p=0 & in\Omega\\ \quad p=0 & on\partial\Omega\end{cases} \qquad (11)$$

Solving space：

$V=H(div,\Omega)=\{v\in L^2(\Omega):divv\in L^2(\Omega)\}$,

$H^s(div,\Omega)=\{v\in L^2(\Omega):divv\in H^s(\Omega)\}$

In the definition of the lowest order spatial Raviart-Thomas, we use the reference cell on the local space, which is defined as:

$V_h(\widehat{Q})=\{\widehat{v}:\widehat{v}=(a+b\widehat{x},c+d\widehat{x}),a,b,c,d\in R\}$

Then each quadrilateral Q on local spatial unit is defined as:

$V_h(Q)=\{v|_Q=\mathrm{P}_Q\widehat{v}:\widehat{v}\in V_h(\widehat{Q})\}$

The whole space:

$V_h=\{v\in V:v|_Q\in V_h(Q),\forall Q\in Q_h\}$

If n_i is a unit normal vector outside edge of the unit Q, then right $\widehat{v}\in V_h(\widehat{Q})$ there,

$|e_i|v\cdot n_i=\widehat{v}\cdot\widehat{n}_i,i=1,2,3,4$

Among， $\widehat{n}_i$ as a unit outside the normal vector on the edge e_i， The arbitrary $v \in V_h$ In any one unit Q edge of the component to a conventional method.

2.2 Online quadrilateral mixed finite volume method

We still consider the problem of second order elliptic equations with Neumann boundary condition on the plane convex domains Ω

$$\begin{cases} -\nabla \cdot \kappa \nabla p = f & in\Omega \\ \kappa \nabla p \cdot n = 0 & on\partial\Omega \end{cases} \tag{12}$$

Corresponding to the problem (7) standard mixed finite element method is:find $(\tilde{u}_h, \tilde{p}_h) \in H_h \times L_h$ making

$$\begin{cases} (\kappa^{-1}\tilde{u}_h, v_h) - (divv_h, \tilde{p}_h) = 0 & \forall v_h \in H_h \\ (div\tilde{u}_h, q_h) = (f, q_h) & \forall q_h \in L_h \end{cases} \tag{13}$$

Solving space： Trial function space H_h is called the lowest order Raviart-Thomas space， Then define the trial function space Y_h Next, define the operator

$$\gamma_h : H_h \to Y_h : \gamma_h w_h := (\sum_{i,j} u_h(c_{i+1/2,j})\chi_{i+1/2,j}, \sum_{i,j} v_h(c_{i,j+1/2})\chi_{i,j+1/2})$$

$$w_h = (u_h, v_h)$$

Corresponding variation in the form of volume control： find $u_h, p_h \in H_h \times L_h$ making

$$\begin{cases} a(u_h, \gamma_h w_h) - b(\gamma_h w_h, p_h) = 0 & \forall w_h \in H_h \\ -c(u_h, q_h) = (f, q_h) & \forall q_h \in L_h \end{cases} \tag{14}$$

Error estimates：

Theorem： For Ω regular mesh $Q_{i,j}$， Solution Let the problem (9) is u_h, p_h， And u, p is the problem (7) solution， There is nothing to do with an h， However, the constant C depends on $\|\kappa^{-1}\|_\infty, \|u\|_1, \|divu\|_1, \|p\|_1$ such

$$\|u - u_h\|_{H(div)} + \|p - p_h\|_0 \le Ch \tag{15}$$

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