Maximal Element Theorems In GFC-Spaces With The Application To Systems of General Quasiequilibrium Problems

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Abstract. G_{KS} -mappings and G_{KS}-majorized mappings are introduced, and maximal element theorems for G_{KS} -mappings and G_{KS}-majorized mappings are established in GFC-spaces. As application, a new equilibrium existence theorem for systems of general quasiequilibrium problems is obtained.

1. Introduction

In 2009, Khanh et al.[1, 2] introduced GFC-spaces. In 2010, Khanh et al.[3] established coincidence theorems, maximal elements and nonempty intersections in GFC-spaces. In 2011, Wen[4,5] established intersection theorems, fixed point theorems, variational inequalities and existence theorems for solutions of generalized equilibrium problems with lower and upper bounds in GFC-spaces. In 2012, Wen[6] obtained equilibrium existence theorems for systems of quasiequilibrium problems and systems of general quasiequilibrium problems in GFC-spaces. In 2013, Wen et al.[7-9] obtained matching theorems, saddle point theorems, section theorems and existence theorems of solutions for systems of generalized mixed vector quasiequilibrium problems in GFC-spaces. In 2014, Wen et al.[10-11] established equilibrium existence theorems for systems of general quasiequilibrium problems and abstract economies in GFC-spaces. In 2015, Wen et al.[12-15] studied equilibrium existence for constrained multiobjective games, abstract generalized vector equilibrium problems, coincidence problems and GFC-KKM theorem in GFC-spaces.

In this paper, G_{KS}-mappings and G_{KS}-majorized mappings are introduced, and maximal element theorems for G_{KS}-mappings and G_{KS}-majorized mappings are established in GFC-spaces. As application, a new equilibrium existence theorem for systems of general quasiequilibrium problems is obtained.

2. Preliminaries

Let X be a nonempty set. We denote by $\langle X\rangle$ and 2^X the family of all nonempty finite subsets of X and the family of all subsets of X, respectively, by $\triangle_n$ the standard n-dimensional simplex with vertices $e_0,\dots, e_n$. Let X and Y be topological spaces. We denote by $C(X, Y)$ the class of single-valued continuous maps of X into Y. Following Khanh et al.[1-15], let X be a topological space, Y be a nonempty set and Φ a family of continuous mappings $\varphi_N:\Delta_n\to X$. (X,Y,Φ) is said to be a GFC-space if for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle$, there exists $\varphi_N:\Delta_n\to X$ of the family Φ. Let (X,Y,Φ) be a GFC-space, Z a nonempty set, $S:X\to Z$ a map. $T:Y\to 2^Z$ a mapping. T is said to be a GFS-KKM mapping if for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we have $S(\varphi_N(\Delta_k))\subset\bigcup_{j=0}^{k}T(y_{i_j})$.

Now, we introduce the following definitions.

Definition 2.1

Let (X,Y,Φ) be a GFC-space, Z a topological space, K a nonempty compact subset of Z. $S:X\to Z$ a map. $T:Z\to 2^Y$ is said to be a G_{KS}-mapping if T has weakly compactly local intersection property relatively to K and for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we have $S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k} c\mathrm{int}_Z T^{-1}(y_{i_j})=\varnothing$.

Definition 2.2

Let (X,Y,Φ) be a GFC-space, Z a topological space, K a nonempty compact subset of Z. $S:X\to Z$ a map. $T:Z\to 2^Y$ is said to be a G_{KS}-majorized mapping if for each $z\in Z$ satisfying $T(z)\neq\varnothing$, there exists a mapping $T_z:Z\to 2^Y$ and an open neighborhood $N(z)\subset Z$ of z in Z such that

(1) T_z^{-1} is weakly transfer compactly open valued relatively to K;

(2) $T(x)\subset T_z(x)$ for each $x\in N(z)$;

(3) for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we have

$$N(z)\bigcap S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k} c\mathrm{int}_Z T_z^{-1}(y_{i_j})=\varnothing.$$

Remark 2.1

Definition 2.1 and 2.2 unify and generalize correlated definitions in Wu[16], Wen[17], Wen[18], Yang[19] and Ding[20].

The following lemma is Theorem 3.1 of Wen[15].

Lemma 2.1

Let (X,Y,Φ) be a GFC-space, Z a topological space, $S\in C(X,Z)$ a continuous map, $F:Y\to 2^Z$ a GFS-KKM mapping with compactly closed values. Then

(1) $\{F(y)\}_{y\in Y}$ has the finite intersection property;

(2) $\bigcap_{y\in Y}F(y)\neq\varnothing$ if there exists a nonempty compact subset $K\subset Z$ and $M\in\langle Y\rangle$ such that $\bigcap_{y\in M}F(y)\subset K$.

3. Main Results

Theorem 3.1

Let (X,Y,Φ) be a GFC-space, Z a topological space, K a nonempty compact subset of Z. $S\in C(X,Z)$ a continuous map, $T:Z\to 2^Y$ a G_{KS} -mapping. Suppose that there exists $M\in\langle Y\rangle$ such that $Z\setminus K\subset\bigcup_{y\in M} c\mathrm{int}_Z T^{-1}(y)$. Then there exists $\hat{z}\in K$ such that $T(\hat{z})=\varnothing$.

Proof Define $F:Y\to 2^Z$ by $F(y):=Z\setminus c\mathrm{int}_Z T^{-1}(y)$ for each $y\in Y$. Since $c\mathrm{int}_Z T^{-1}(y)$ is compactly open for each $y\in Y$, then F is compactly closed valued. Note that T is a G_{KS} -mapping. Then for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we have $S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k} c\mathrm{int}_Z T^{-1}(y_{i_j})=\varnothing$, i.e.,

$$S(\varphi_N(\Delta_k))\subset Z\setminus\bigcap_{j=0}^{k} c\mathrm{int}_Z T^{-1}(y_{i_j})=\bigcup_{j=0}^{k}(Z\setminus c\mathrm{int}_Z T^{-1}(y_{i_j}))=\bigcup_{j=0}^{k}F(y_{i_j}),$$

which means that F is a GFS-KKM mapping with compactly closed values. Since there exists $M\in\langle Y\rangle$ such that $Z\setminus K\subset\bigcup_{y\in M} c\mathrm{int}_Z T^{-1}(y)$, then

$$\bigcap_{y\in M}F(y)=\bigcap_{y\in M}(Z\setminus c\mathrm{int}_Z T^{-1}(y_{i_j}))=Z\setminus\bigcup_{y\in M} c\mathrm{int}_Z T^{-1}(y_{i_j})\subset K.$$

By lemma 2.1, we have $\bigcap_{y\in Y}F(y)\neq\varnothing$. Take $\hat{z}\in\bigcap_{y\in Y}F(y)\subset\bigcap_{y\in M}F(y)\subset K$, we claim that $T(\hat{z})=\varnothing$. Otherwise, since T is G_{KS} –mapping, then T has weakly compactly local intersection property relatively to K. Define $T_K:K\to 2^Y$ by $T_K(z)=T(z)$ for each $z\in K$. Since K is compact, than T_K has local intersection property. Note that $\hat{z}\in K$ and $T_K(\hat{z})=T(\hat{z})\neq\varnothing$. Hence, there exists an open neighborhood $N(\hat{z})\subset Z$ such that $\bigcap_{z\in N(\hat{z})}T_K(z)\neq\varnothing$. Take $\hat{y}\in\bigcap_{z\in N(\hat{z})}T_K(z)$. Then $N(\hat{z})\subset T_K^{-1}(\hat{y})$, so that $\hat{z}\in \operatorname{int}_K T_K^{-1}(\hat{y}) = c\operatorname{int}_K T_K^{-1}(\hat{y}) \subset c\operatorname{int}_Z T_K^{-1}(\hat{y})$, and hence $\hat{z}\notin Z\setminus c\operatorname{int}_K T_K^{-1}(\hat{y}) = F(\hat{y})$, a contradiction to $\hat{z}\in\bigcap_{y\in Y}F(y)$.

Remark 3.1

Theorem 3.1 unifies, improves and generalizes Theorem 1 of Wu[16], Theorem 1 of Wen[17], Theorem 2.1 of Wen[18].

Theorem 3.2

Let (X,Y,Φ) be a GFC-space, Z a Hausdorff topological space, K a compact subspace of Z. $S\in C(X,Z)$ a continuous map, $F:Z\to 2^Y$ a G_{KS}-majorized mapping. Then there exists $\hat{z}\in K$ such that $F(\hat{z})=\varnothing$.

Proof Suppose that the conclusion is false, i.e.,

(1) $F(z)\neq\varnothing$ for each $z\in K$.

Since F is a G_{KS}-majorized mapping, then for each $z\in K$, there exists a mapping $F_Z:Z\to 2^Y$ and an open neighborhood $N(z)\subset Z$ of z in Z such that

(2) F_z^{-1} is weakly transfer compactly open valued relatively to K;

(3) $F(x)\subset F_Z(x)$ for each $x\in N(z)$;

(4) for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we have

$$N(z)\cap S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}c\operatorname{int}_Z F_z^{-1}(y_{i_j})=\varnothing.$$

Define $T:K\to 2^Y$ by $T(z)=F(z)$ and $T_Z:K\to 2^Y$ by $T_Z(z)=F_Z(z)$ for each $z\in K$. Since K is compact, then in virtue of (2), we have

(5) T_z^{-1} is transfer open valued for each $z\in K$.

For each $z\in K$, we may suppose that open neighborhood $N(z)$ is also an open neighborhood of z in K. Then by (3), we have

(6) $T(x)\subset T_Z(x)$ for each $x\in N(z)$

Note that K is compact. Hence for each $y\in Y$, we have

$$\operatorname{int}_K T_Z^{-1}(y)=c\operatorname{int}_K T_Z^{-1}(y)\subset c\operatorname{int}_Z T_z^{-1}(y)\subset c\operatorname{int}_Z F_z^{-1}(y).$$

Hence, (4) implies that

(7) for each $z\in K, N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we have

$$N(z)\cap S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}c\operatorname{int}_Z T_z^{-1}(y_{i_j})=\varnothing.$$

Since Z is a Hausdorff topological space and K is a compact subspace of Z, then K is a compact Hausdorff topological space, and hence K is a regular topological space, so that for each $z\in K$ and the open neighborhood $N(z)$ of z in K, there exists an open neighborhood $U(z)$ of z in K such that $U(z)\subset cl_K U(z)\subset N(z)$. By the compactness of K, we have

(8) there exists $\{z_0,\cdots,z_m\}\in\langle K\rangle$ such that $K\subset\bigcup_{i=0}^{m}U(z_i)$.

For each $i\in\{0,\cdots,m\}$, define $T_i:K\to 2^Y$ by

$$T_i(z):=\begin{cases}T_{z_i}(z), & if \quad z\in cl_K U(z_i),\\ Y, & if \quad z\notin cl_K U(z_i).\end{cases}$$

Then $T_i^{-1}(y)=T_{z_i}^{-1}(y)\bigcup(K\setminus cl_K U(z_i))$ for each $y\in Y$. By (5), for each $i\in\{0,\cdots,m\}$, $T_{z_i}^{-1}$ is transfer open valued. Note that $K\setminus cl_K U(z_i)$ is open. Hence,

(9) for each $i\in\{0,\cdots,m\}$, T_i^{-1} is transfer open valued.

Define $\tilde{T}:K\to 2^Y$ by $\tilde{T}(z):=\bigcap_{i=0}^{m}T_i(z)$ for each $z\in Z$. We claim that $\tilde{T}$ is a G_{KS}-mapping. First of all, since $\tilde{T}^{-1}(y):=\bigcap_{i=0}^{m}T_i^{-1}(y)$ for each $y\in Y$, then $\tilde{T}^{-1}$ is transfer open valued by (9). In virtue of Lemma 1.3 of Wen[21], $\tilde{T}$ has local intersection property, so that $\tilde{T}$ has weakly compactly local intersection property relatively to K. Secondly, for each $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$, we claim that $S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}c\text{int}_K\tilde{T}^{-1}(y_{i_j})=S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}\text{int}_K\tilde{T}^{-1}(y_{i_j})=\varnothing$. Otherwise, there exist $N:=\{y_0,\cdots,y_n\}\in\langle Y\rangle,\{y_{i_0},\cdots,y_{i_k}\}\in\langle N\rangle$ such that $S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}\text{int}_K\tilde{T}^{-1}(y_{i_j})\neq\varnothing$. Take $z^*\in S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}\text{int}_K\tilde{T}^{-1}(y_{i_j})\subset K$. Then by (8), there exists $p\in\{0,\cdots,m\}$ such that $z^*\in U(z_p)\subset N(z_p)$. Note that

$$z^*\in\bigcap_{j=0}^{k}\text{int}_K\tilde{T}^{-1}(y_{i_j})=\bigcap_{j=0}^{k}\text{int}_K(\cap_{i=0}^{m}T_i^{-1}(y_{i_j}))\subset\bigcap_{j=0}^{k}\text{int}_K T_p^{-1}(y_{i_j}).$$

Hence, there exists an open neighborhood $O(z^*)$ of z^* in K such that $O(z^*)\subset U(z_p)\cap\bigcap_{j=0}^{k}T_p^{-1}(y_{i_j})$. For each $z\in O(z^*)\subset U(z_p)\cap\bigcap_{j=0}^{k}T_p^{-1}(y_{i_j})$, by the definition of T_i, we have $\{y_{i_0},\cdots,y_{i_k}\}\subset T_p(z)=T_{z_p}(z)$, which implies that $z\in\bigcap_{j=0}^{k}T_{z_p}^{-1}(y_{i_j})$, and then $O(z^*)\subset\bigcap_{j=0}^{k}T_{z_p}^{-1}(y_{i_j})$, i.e., $z^*\in\bigcap_{j=0}^{k}\text{int}_K T_{z_p}^{-1}(y_{i_j})$. So we have $z^*\in N(z_p)\cap S(\varphi_N(\Delta_k))\cap\bigcap_{j=0}^{k}\text{int}_K T_{z_p}^{-1}(y_{i_j})$, which is a contradiction to (7). Finally, since $\tilde{T}$ is a G_{KS}-mapping and K is compact, then for each $M\in\langle Y\rangle$, we have $K\setminus K\subset\bigcup_{y\in M}c\,\text{int}_Z\tilde{T}^{-1}(y)$. In virtue of Theorem 3.1, there exists $\hat{z}\in K$ such that $\tilde{T}(\hat{z})=\varnothing$. By (6) and the condition of T_i, for each $z\in K,i\in\{0,\cdots,m\},F(z)=T(z)\subset T_i(z)$, hence, $F(z)\subset\cap_{i=0}^{m}T_i(z)=\tilde{T}(z)$, especially, $F(\hat{z})\subset\tilde{T}(\hat{z})=\varnothing$, which is a contradiction to (1).

Remark 3.1

Theorem 3.2 unifies, improves and generalizes Theorem 2 of Wu[16], Theorem 2 of Wen[17], Theorem 2.2 of Wen[18].

Theorem 3.3

Let I be a finite or infinite index set, for each $i\in I$, (X_i,Y_i,Φ_i) be a GFC-space, Z_i a Hausdorff topological space, $Y_i\subset Z_i$, $T_i:Z:=\prod_{j\in I}Z_j\to Y_i$ a map, $A_i:Z\to 2^{Y_i}\setminus\{\varnothing\}$ a nonempty valued mapping, $\psi_i:Z\times Y_i\times Z\to\bar{R}$ a function, K a compact subspace of Z, $X:=\prod_{j\in I}X_j$, $Y:=\prod_{j\in I}Y_j$, $S\in C(X,Z)$ a continuous map. Suppose that $A:Z\to 2^Y$ defined by $A(z):=\prod_{i\in I}A_i(z)$ for each $z\in Z$ is a G_{KS}-majorized mapping. Then the system of general quasiequilibrium problems SGQEP$(T_i,A_i,\psi_i)_{i\in I}$ has an equilibrium in Z, i.e., there exists $\hat{z}\in Z$ such that

$$\begin{cases}\hat{z}_i:=\pi_i(\hat{z})\in A_i(\hat{z}), & \forall i\in I,\\ \psi_i(\hat{z},T_i(\hat{z}),y)\le 0, \quad \forall y\in A(\hat{z}), & \forall i\in I.\end{cases}$$

Proof For each $i\in I$, $N_i\in\langle Y_i\rangle,\varphi_{N_i}\in\Phi_i$, let $\Phi=\{\varphi_N:\varphi_N:=\prod_{i\in I}\varphi_{N_i}\}$. Then (X,Y,Φ) is a GFC-space. Since Z_i is a Hausdorff topological space for each $i\in I$, then $Z:=\prod_{j\in I}Z_i$ is a Hausdorff topological space. For each $i\in I$, define $P_i:Z\to 2^Y$ by $P_i(z):=\{y\in Y:\psi_i(z,T_i(z),y)>0\}$ for each $z\in Z$. Let $D:=\{z\in Z:\ z\in A(z)\}$. Define $F:Z\to 2^Y$ by

$$F(z):=\begin{cases}A(z), & if\quad z\in Z\setminus D,\\ A(z)\cap\bigcup_{i\in I}P_i(z), & if\quad z\in D.\end{cases}$$

We claim that there exists $\hat{z}\in Z$ such that $F(\hat{z})=\varnothing$. Otherwise, since A is a G_{KS}-majorized mapping, then for each $z\in Z$, there exists $A_z:Z\to 2^Y$ and an open neighborhood $N(z)\subset Z$ such that

(1) A_z^{-1} is weakly transfer compactly open valued relatively to K;

(2) $A(x) \subset A_Z(x)$ for each $x \in N(z)$;

(3) for each $N := \{y_0, \cdots, y_n\} \in \langle Y \rangle, \{y_{i_0}, \cdots, y_{i_k}\} \in \langle N \rangle$, we have

$$N(z) \bigcap S(\varphi_N(\Delta_k)) \cap \bigcap_{j=0}^{k} c\mathrm{int}_Z A_z^{-1}(y_{i_j}) = \varnothing.$$

By the definition of F, $F(z) \subset A(z)$ for each $z \in Z$, hence by (2) we have

(4) $F(x) \subset A_Z(x)$ for each $x \in N(z)$.

By (1), (3) and (4), F is a G_{KS}-majorized mapping. In virtue of Theorem 3.2, there exists $\tilde{z} \in K \subset Z$ such that $F(\tilde{z}) = \varnothing$, which is a contradiction. Finally, since A_i is nonempty valued, then A is nonempty valued. Note that $F(\hat{z}) = \varnothing$. By the definition of F, we have $\hat{z} \in D$ and $A(\hat{z}) \bigcap \bigcup_{i \in I} P_i(\hat{z}) = \varnothing$. Since $\hat{z} \in D$, hence $\hat{z} \in A(\hat{z})$, and then $\hat{z}_i := \pi_i(\hat{z}) \in A_i(\hat{z})$ for each $i \in I$. Since $A(\hat{z}) \bigcap \bigcup_{i \in I} P_i(\hat{z}) = \varnothing$, which implies that for each $y \in A(\hat{z}), y \notin \bigcup_{i \in I} P_i(\hat{z})$, and then for each $i \in I, y \notin P_i(\hat{z})$, i.e., for each $i \in I$, $y \in A(\hat{z}), \psi_i(\hat{z}, T_i\hat{z}, y) \le 0$. Therefore, $\hat{z} \in Z$ is an equilibrium of the system of general quasiequilibrium problems SGQEP $(T_i, A_i, \psi_i)_{i \in I}$ in Z.

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