Hybrid Synchronization of Fractional Order Chaotic Systems

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Abstract. This paper studies the modified hybrid projective synchronization of fractional order chaotic systems, which generalizes many various synchronization forms. We firstly present this kind synchronization that the response and drive systems have scaling factors at the same time. In addition, based on the stability analysis of fractional order systems and adaptive control technique, a suitable controller and parameter update law can be designed to achieve the modified hybrid projective synchronization for uncertain fractional-order chaotic systems.

Introduction

Recently, synchronization of chaotic fractional-order differential systems have received a significant attention among scientists from various different fields [1]. On one hand, inspired by the pioneering work in 1990 [2], synchronization has attracted increasing attention due to its potential applications in secure communication and signal processing etc, on the other hand, fractional order derivatives provide an excellent instrument for the description of memory properties of various materials and processes. At the same time, it was proved that many fractional-order systems behave chaotically, such as fractional-order Lü system [3], fractional-order Chua’s circuit [4] and fractional-order unified system [5]. Among all kinds of chaos synchronization, projective synchronization is the most noticeable one because of its proportional feature [6]. How to effectively realize the modified hybrid projective of two fractional-order chaotic systems with unknown parameters is an important problem for both the theoretical research and practical applications.

Adaptive modified hybrid projective synchronization

Fractional-order Chen system is described as follows:

\[
\begin{align*}
D^q x_1 &= a(x_2 - x_1) \\
D^q x_2 &= (c - a)x_1 - x_1x_3 + cx_2 \\
D^q x_3 &= x_1x_2 - bx_3,
\end{align*}
\]  

(1)

When parameters are set by default as \(a = 35, b = 3, c = 28\), and \(q = 0.9\), system (1) exhibits chaotic behaviors as shown in Fig.1(a). The fractional-order Lü system is described by the following state equation:

\[
\begin{align*}
D^q y_1 &= d(y_2 - y_1) \\
D^q y_2 &= -y_1y_3 + hy_2 \\
D^q y_3 &= y_1y_2 - fy_3,
\end{align*}
\]  

(2)

Fig.1(b) displays the chaotic attractor of the fractional-order Lü chaotic system.

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We assume that fractional-order chaotic Chen system with unknown parameters is the drive system and the response Lu system is given by:

\[
\begin{align*}
D^\alpha y_1(t) &= d(y_2 - y_1) + u_1 \\
D^\alpha y_2(t) &= -y_1y_3 + hy_2 + u_2 \\
D^\alpha y_3(t) &= y_1y_2 - fy_3 + u_3,
\end{align*}
\]  

(3)

Denote the state errors as \( e_1 = \alpha_i y_1 - \beta_i x_1, e_2 = \alpha_i y_2 - \beta_i x_2, e_3 = \alpha_i y_3 - \beta_i x_3 \).

It follows from (1)-(3), Then we have the following error dynamical system

\[
\begin{align*}
D^\alpha e_1(t) &= \alpha_i [d(y_2 - y_1) + u_1] - \beta_i [a(x_2 - x_1)] \\
D^\alpha e_2(t) &= \alpha_i [-y_1y_3 + hy_2 + u_2] - \beta_i [(c - a)x_1 - x_2x_3 + cx_3] \\
D^\alpha e_3(t) &= \alpha_i [y_1y_2 - fy_3 + u_3] - \beta_i [x_1x_2 - bx_3].
\end{align*}
\]  

(4)

**Theorem** For given constant scaling matrices \( \alpha_i (i = 1, 2, 3), \beta_i (i = 1, 2, 3) \), modified hybrid projective synchronization between system (1) and system (3) will occur by following controller:

\[
\begin{align*}
u_1 &= 1/\alpha_i (\beta_i x_1 - \alpha_i y_1 + \beta_i (\tilde{a}(x_2 - x_1)) - \tilde{d}(y_2 - y_1)) \\
u_2 &= 1/\alpha_i (\beta_i x_2 - \alpha_i y_2 + \beta_i ((\tilde{c} - \tilde{a})x_1 - x_2x_3 + \tilde{c}x_3) + y_1y_3 - \tilde{h}y_2) \\
u_3 &= 1/\alpha_i (\beta_i x_3 - \alpha_i y_3 + \beta_i (x_1x_2 - \tilde{b}x_3) + \tilde{f}y_3 - y_1y_2).
\end{align*}
\]  

(5)

and all the parameter update rule for unknown parameters \( a, b, c, d, h, f \)

\[
\begin{align*}
D^\alpha \tilde{a} &= \beta_i(x_2 - x_1)e_1 + \beta_i x_1e_2 \\
D^\alpha \tilde{b} &= \beta_i x_1e_1 \\
D^\alpha \tilde{c} &= \beta_i(x_1 + x_2)e_2 \\
D^\alpha \tilde{d} &= \alpha_i (y_1 - y_2)e_1 \\
D^\alpha \tilde{h} &= -\alpha_i y_3e_2 \\
D^\alpha \tilde{f} &= \alpha_i y_3e_3,
\end{align*}
\]  

(6)

**Proof:** From Eqs.(5)-(7), we can obtain the error dynamical system as below

\[
\begin{align*}
D^\alpha e_1 &= \alpha_i (y_2 - y_1)e_2 - \beta_i(x_2 - x_1)e_3 - e_1 \\
D^\alpha e_2 &= \alpha_i y_2e_3 - e_2 + \beta_i e_2(x_2 - x_1) - \beta_2 x_2e_3 \\
D^\alpha e_3 &= \alpha_i y_3e_4 - \beta_i x_3e_3 - e_3.
\end{align*}
\]  

(7)

Combining (6) with (7), one has

\[
J = e_1D^\alpha e_1 + e_2D^\alpha e_2 + e_3D^\alpha e_3 + e_4D^\alpha e_4 + e_5D^\alpha e_5 + e_6D^\alpha e_6 + e_yD^\alpha e_y + e_fD^\alpha e_f.
\]  

(8)
From Eq.(5-8), we can get that

\[ e_t^1 e_1^1 + e_t^2 e_2^1 + e_t^3 e_3^1 + e_t^4 e_4^1 + e_t^5 e_5^1 + e_t^6 e_6^1 + e_t^7 e_7^1 + e_t^8 e_8^1 + e_t^9 e_9^1 + e_t^10 e_{10}^1 + e_t^11 e_{11}^1 = e_t^1 [\alpha_1 (y_1 - y_t) e_1^1 + \beta_1 (x_1 - x_t) e_1^1] + e_t^2 [\alpha_2 y_1 e_2^1 - e_2^1 + \beta_2 e_1^1 (x_1 + x_t) - \beta_2 x_1 e_2^1] + e_t^3 [\alpha_3 y_1 e_3^1] + e_t^4 [\alpha_4 y_1 e_4^1] + e_t^5 [\alpha_5 y_1 e_5^1] + e_t^6 [\alpha_6 y_1 e_6^1] + e_t^7 [\alpha_7 y_1 e_7^1] + e_t^8 [\alpha_8 y_1 e_8^1] + e_t^9 [\alpha_9 y_1 e_9^1] + e_t^{10} [\alpha_{10} y_1 e_{10}^1] + e_t^{11} [\alpha_{11} y_1 e_{11}^1] = -e_1^1 - e_2^1 - e_3^1 \leq 0. \]

Then the response system (3) can synchronize the drive system (1) globally and asymptotically.

**Numerical simulations**

We choose the scaling matrices \( \alpha = \text{diag}(1,1,1), \beta = \text{diag}(-2,-2,-2) \). The values of unknown parameters converge to \( \tilde{a} = 35, \tilde{b} = 3, \tilde{c} = 28, \tilde{d} = 36, \tilde{h} = 20, \tilde{f} = 3 \) is shown in Fig. 4.

![Fig.2. Errors of drive and response system with \( \alpha = \text{diag}(1,1,1), \beta = \text{diag}(-2,-2,-2) \)](image)

![Fig.3. Identification curves of the unknown parameters](image)

**Conclusions**

Based on the stability theory of fractional-order system, adaptive controller and parameter update law are designed to ensure Chen system synchronize with Lu system up to double scaling matrices. Numeric results show that the proposed scheme is analytically rigorous and practically feasible.
References


