INEQUALITIES ON GENERALIZED TRIGONOMETRIC FUNCTIONS

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Abstract. The Sharp Cusa-Huygens inequality involving the generalized trigonometric functions are established.

Introduction
It is well known from basic calculus that
\[ \arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} \, dt, \quad 0 \leq x \leq 1, \]
And \[ \pi/2 = \arcsin(1) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \, dt. \]
For \( 1 < p < \infty \), We can generalize the above function as follows:
\[ \arcsin_p(x) = \int_0^x \frac{1}{(1-t^2)^{1/p}} \, dt, \quad 0 \leq x \leq 1, \]
and \[ \frac{\pi}{2} = \arcsin_p(1) = \int_0^1 \frac{1}{(1-t^2)^{1/p}} \, dt. \]
where \( \pi_p = \frac{2\pi}{p \sin(\pi/p)} \) is decreasing on \((1, \infty)\).

The inverse of \( \arcsin_p \) on \([0, \pi_p/2]\) is called the generalized sine function and denoted by \( \sin_p \).

The generalized cosine function \( \cos_p \) is defined as
\[ \cos_p(x) \equiv \frac{d}{dx} \sin_p(x). \]
It is clear from the definition that
\[ \cos_p(x) = \left(1 - \sin_p(x)^p\right)^{1/p}. \]

The generalized tangent function \( \tan_p \) is defined as
\[ \tan_p(x) \equiv \frac{\sin_p(x)}{\cos_p(x)}. \]
It is easy to see that
\[ \frac{d}{dx} \cos_p(x) = -\cos_p(x)^{2-p} \sin_p(x)^{p-1}, \quad \frac{d}{dx} \tan_p(x) = 1 + \tan_p(x)^p, \]
when \( p = 2 \), the \( p \) functions \( \sin_p, \cos_p, \tan_p \) become our familiar trigonometric functions.

Recently, the generalized trigonometric functions have been studied by many mathematicians from different viewpoints (see [2,4,5,6,7]). In [5,9], the authors gave basic properties of the generalized trigonometric functions. In [6], Klén, Vuorinen and Zhang generalized some classical inequalities for trigonometric functions, such as Mitrinović-Adamović’s inequality, Lazarević’s inequality, Huygens-type inequalities, and Wilker-type inequalities, to the case of generalized functions.
The main results of this paper are the following theorems.

Theorem 1 For $1 < p \leq 2$, the function
\[
f(x) = \frac{x(p + \cos_p(x))}{\sin_p(x)}
\]
is strictly increasing from $(0, \pi_p/2)$ onto $(p + 1, p\pi_p/2)$.

Theorem 2 For $1 < p \leq 2$, the function
\[
F(x) = \frac{\sin_p(x) - x\cos_p(x)}{x^2\sin_p(x)^{p-1}\cos_p(x)^{2-p}}
\]
is strictly increasing from $(0, \pi_p/2)$ onto $(\frac{1}{p+1}, b_p)$.

Where
\[
b_p = \begin{cases} 
\infty, & 1 < p < 2, \\
4, & p = 2.
\end{cases}
\]

Theorem 3 For $1 < p \leq 2$, the function
\[
G(x) = \frac{\ln(\sin_p(x)/x)}{\ln[(p + \cos_p(x))/((p+1))}]
\]
is strictly increasing from $(0, \pi_p/2)$ onto $(1, (\log(\pi_p/2))/\log[(p+1)/p])$.

In particular, for all $p \in (1, 2], x \in (0, \pi_p/2)$,
\[
\left(\frac{p + \cos_p(x)}{p+1}\right)^\alpha < \left(\frac{\sin_p x}{x}\right) < \left(\frac{p + \cos_p(x)}{p+1}\right)^\beta,
\]
Where $\alpha = (\log(\pi_p/2))/\log[(p+1)/p]$ and $\beta = 1$ are the best constants.

Remark 4 For $p = 2$, the above inequalities are due to C.-P. Chen and W.-S. Cheung [8].

**Proof of theorems**

In order to establish our main results we need following lemma:

Lemma 5 (L’Hôpital Monotone Rule see [11]) Let $-\infty < a < b < \infty$, and let $f, g : [a,b] \rightarrow \mathbb{R}$ be continuous functions that are differentiable on $(a,b)$, with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a,b)$.

If $f'/g'$ is increasing (decreasing) on $(a,b)$, then so is $f/g$.

Proof of Theorem 1

By differentiation, we have
\[
f'(x) = \frac{1}{\sin_p(x)^2}g(x),
\]
With
\[
g(x) = p\sin_p(x) + \sin_p(x)\cos_p(x) - px\cos_p(x) - x\cos_p(x)^{2-p}.
\]
a simple computation leads to
\[
g'(x) = \cos_p(x)^{2-p}[-2\sin_p(x) + p\sin_p(x)^{p-1} + (2-p)x\cos_p(x)^{1-p}\sin_p(x)^{p-1}]
\]
\[= \cos_p(x)^{2-p}\sin_p(x)^{1-p}[px - 2\sin_p(x) + (2-p)x \cos_p(x)^{1-p}]
\]
$$= \cos_p(x)^{-p} \sin_p(x)^{p+1} h(x),$$

where

$$h(x) = px - 2 \sin_p(x) + (2 - p)x \cos_p(x)^{-p},$$

and

$$h'(x) = p - 2 \cos_p(x) + (2 - p) \cos_p(x)^{-p} + (2 - p)(p - 1)x \cos_p(x)^{2 - p} \sin_p(x)^{p-1} > (2 - p)(\cos_p(x)^{-p} - \cos_p(x)) > 0.$$  

Hence $h(x) > h(0) = 0$, therefore $g'(x) > 0$, $f(x)$ is strictly increasing on $(0, \pi_p / 2)$, $p + 1 = f(0^+) < f(x) < f(\pi_p / 2) = p\pi_p / 2$.  

**Proof of Theorem 2.** Write  

$$F_1(x) \equiv \sin_p(x) - x \cos_p(x), \text{ and } F_2(x) \equiv x^2 \sin_p(x)^{p-1} \cos_p(x)^{-p},$$

then $F_1(0) = 0$, $F_2(0) = 0$, by simple computations,

$$\frac{F_2'(x)}{F_1'(x)} = \frac{2x \cos_p(x)^{2-p} \sin_p(x)^{p-1} + (p - 1)x^2 \cos_p(x)^{3-p} \sin_p(x)^{p-2} + (p - 2)x^2 \cos_p(x)^{3-2p} \sin_p(x)^{2p-2}}{x \cos_p(x)^{2-p} \sin_p(x)^{p-1}} = 2 + (p - 1)x / \tan_p(x) + (p - 2)x \tan_p(x)^{p-1}.$$  

Which is strictly decreasing, by lemma 5, $F_2(x)$ is strictly decreasing on $(0, \pi_p / 2)$, $F(x)$ is strictly increasing on $(0, \pi_p / 2)$, leads to

$$F(0^+) < F(x) < F(\pi_p / 2).$$

But

$$F(0^+) = \lim_{x \to 0^+} \frac{F_1(x)}{F_2'(x)} = \lim_{x \to 0^+} \frac{F_1'(x)}{F_2'(x)}$$

$$= \lim_{x \to 0^+} \frac{1}{2 + (p - 1)x / \tan_p(x) + (p - 2)x \tan_p(x)^{p-1}} = \frac{1}{p + 1}.$$  

Theorem 2 is proved.

**Proof of Theorem 3.** Write

$$G_1(x) \equiv \ln \frac{\sin_p(x)}{x}, \text{ and } G_2(x) \equiv \ln \frac{p + \cos_p(x)}{p + 1},$$

then $G_1(0) = 0$, $G_2(0) = 0$, by simple computations,

$$\frac{G_2'(x)}{G_1'(x)} = \frac{\sin_p(x) - x \cos_p(x)}{x \sin_p(x)} \cdot \frac{p + \cos_p(x)}{\sin_p(x)^{p-1} \cos_p(x)^{-p}}$$

$$= \frac{x(p + \cos_p(x))}{\sin_p(x)} \cdot \frac{\sin_p(x) - x \cos_p(x)}{x^2 \sin_p(x)^{p-1} \cos_p(x)^{2-p}} = f(x)F(x).$$

By theorem 1 and theorem 2, the functions $f(x)$, $F(x)$ are strictly increasing on $(0, \pi_p / 2)$, $f(x) \geq 0$, $F(x) \geq 0$. Thus

$$\frac{G_2'(x)}{G_1'(x)}$$

is strictly increasing on $(0, \pi_p / 2)$, by lemma 5, the function

$$\frac{G_1(x)}{G_2(x)}$$

is strictly increasing on $(0, \pi_p / 2)$, and we have

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$$1 = G(0^+) < G(x) = \frac{\ln(\sin_p(x)/x)}{\ln[(p + \cos_p(x))/(p + 1)]} < G(\pi_p/2) = \frac{\ln(\pi_p/2)}{\ln((p + 1)/p)}.$$ 

Theorem 3 is proved.

References


