A Numerical Method for Time Fractional Diffusion Equation

Guangzhen Song^{1,a}, Weijia Zhao^{2,b} and Jianfei Huang^{3,b}

¹School of Mathematics, Qingdao University, Qingdao, 266071, China

² School of Mathematics, Qingdao University, Qingdao, 266071, China

³ School of Mathematics, Qingdao University, Qingdao, 266071, China

^a847139747@qq.com, ^bzhaoweijia@qdu.edu.cn

Keywords: time fractional diffusion equation, block-by-block method, Caputo derivative.

Abstract. Use block-by-block method and classic second-order central difference approximation to discretize its temporal direction and spatial direction, respectively. This new scheme can effectively solve a class of time fractional diffusion equations with initial boundary value conditions.

1. Introduction

It is usually difficult to obtain the analytical solution for a fractional differential equation (FDE). So there has been a growing interest to develop numerical approaches in solving the FDE. For example, Lin and Liu^[1] developed a kind of linear multistep methods for fractional initial value problems based on Lubich's high-order approximations^[2] to fractional derivatives and integrals. In 2006, Kumar and Agrawal^[3] also utilized the equivalent Volterra integral equation in^[4] and extended the block-by-block method proposed by Linz^[5] to some kind of FDEs. In this paper, block-by-block method is extended to the solution of FDEs. The new numerical format has three advantages: spent little time in computing the starting weights and has the advantage of efficient computing time; in the direction of time and space can achieve high accuracy; the scheme is numerically stable.

2. Block-by-block method

Consider the following nonlinear FDE with the initial conditions:

$${}_0^C D_t^\alpha y(t) = f(t, y(t)), \quad 0 \leq t \leq T, \quad n-1 < \alpha \leq n \quad (1)$$

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \dots, n-1. \quad (2)$$

In Eq. (1), ${}_0^C D_t^\alpha$ denotes the Caputo derivative of order α , defined by

$${}_0^C D_t^\alpha y(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n y(\tau)}{d\tau^n} d\tau$$

If $f(t, y)$ is continuous, the initial value problem (IVP) Eq.(1-2) is equivalent to the following Volterra integral equation of the second kind[6]

$$y(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \quad (3)$$

where

$$g(t) := \sum_{k=0}^{n-1} c_k \frac{t^k}{k!}.$$

The specific format structure for solving initial value problems Eq. (1) and Eq. (2) is as follows. First divide the interval $[0, T]$ into $2N$ parts with stepsize $\tau = T/(2N)$, and set $t_j = j\tau, (j = 0, 1, \dots, 2N)$. The numerical solution of Eq. (3) at the point t_j is denoted by y_j , let $g_j = g(t_j)$ and $f_j = f(t_j, y_j)$.

Now assume that $y_j (j = 0, 1, \dots, 2m)$, the approximations of $y(t_j) (j = 0, 1, \dots, 2m)$ are obtained, and the numerical solutions y_{2m+1} and y_{2m+1} can be obtained by the following formulas

$$y_{2m+2} = g_{2m+2} + \sum_{k=0}^{m-1} [W_{k,0}^{[2m+2]} f_{2k} + W_{k,1}^{[2m+2]} f_{2k+1} + W_{k,2}^{[2m+2]} f_{2k+2}] \\ + W_{m,0}^{[2m+2]} f_{2m} + W_{m,1}^{[2m+2]} f_{2m+1} + W_{m,2}^{[2m+2]} f_{2m+2}, \quad (4)$$

$$y_{2m+1} = g_{2m+1} + \sum_{k=0}^{m-1} [W_{k,0}^{[2m+1]} f_{2k} + W_{k,1}^{[2m+1]} f_{2k+1} + W_{k,2}^{[2m+1]} f_{2k+2}] \\ + W_{m,0}^{[2m+1]} f_{2m} + W_{m,1}^{[2m+1]} f_{2m+1} + W_{m,2}^{[2m+1]} f_{2m+2}, \quad (5)$$

where for $i = 0, 1, 2$,

$$W_{k,i}^{[2m+2]} := \frac{1}{\Gamma(\alpha)} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+2} - \tau)^{\alpha-1} \phi_{k,i}(\tau) d\tau, \quad k = 0, 1, \dots, m, \quad (6)$$

$$W_{k,i}^{[2m+1]} := \frac{1}{\Gamma(\alpha)} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+1} - \tau)^{\alpha-1} \phi_{k,i}(\tau) d\tau, \quad k = 0, 1, \dots, m-1, \quad (7)$$

In Eq. (5), if $k = m$, $W_{m,i}^{[2m+1]} (i = 0, 1, 2)$ are defined as

$$W_{m,0}^{[2m+1]} := d_{2m} + \frac{3}{8} d_{2m+1/2}, \quad W_{m,1}^{[2m+1]} := \frac{3}{4} d_{2m+1/2} + d_{2m+1}, \quad W_{m,2}^{[2m+1]} := -\frac{1}{8} d_{2m+1/2} \quad (8)$$

With $d_{2m} := \frac{1}{\Gamma(\alpha)} \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1} - \tau)^{\alpha-1} \psi_{m,0}(\tau) d\tau, \quad d_{2m+\frac{1}{2}} := \frac{1}{\Gamma(\alpha)} \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1} - \tau)^{\alpha-1} \psi_{m,1}(\tau) d\tau,$

$$d_{2m+1} := \frac{1}{\Gamma(\alpha)} \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1} - \tau)^{\alpha-1} \psi_{m,2}(\tau) d\tau,$$

Functions $\phi_{k,i}(t) (i = 0, 1, 2)$ are quadratic Lagrange interpolating polynomials associated with points t_{2k}, t_{2k+1} and t_{2k+2} , precisely,

$$\phi_{k,0}(t) := \frac{(t - t_{2k+1})(t - t_{2k+2})}{2\tau^2}, \quad \phi_{k,1}(t) := \frac{(t - t_{2k})(t - t_{2k+2})}{\tau^2}, \quad \phi_{k,2}(t) := \frac{(t - t_{2k})(t - t_{2k+1})}{2\tau^2},$$

Similarly, $\psi_{k,i}(t) (i = 0, 1, 2)$ are Lagrange interpolating polynomials associated with points $t_{2k}, t_{2k+1/2}$ and t_{2k+1} . From Eq.(6)-(8), it is known that $W_{k,i}^{[2m+2]}$ and $W_{k,i}^{[2m+1]}$ can be explicitly calculated.

For simplicity, we reduce Eq. (4) and Eq.(5) to

$$y_{2m+2} = g_{2m+2} + \tau^\alpha \sum_{j=0}^{2m} \omega_{2m+2-j} f_j + \tau^\alpha \omega_1 f_{2m+1} + \tau^\alpha \omega_0 f_{2m+2},$$

$$y_{2m+1} = g_{2m+1} + \tau^\alpha \sum_{j=0}^{2m} \varpi_{2m+2-j} f_j + \tau^\alpha \varpi_1 f_{2m+1} + \tau^\alpha \varpi_0 f_{2m+2},$$

where

$$\omega_0 := W_{m,2}^{[2m+2]} / \tau^\alpha, \omega_{2k+1} := W_{m-k,1}^{[2m+2]} / \tau^\alpha, \quad k = 0, 1, \dots, m; \\ \omega_{2k} := W_{m-k,2}^{[2m+2]} / \tau^\alpha + W_{m-k+1,0}^{[2m+2]} / \tau^\alpha, \quad k = 1, 2, \dots, m; \quad \omega_{2m+2} := W_{0,0}^{[2m+2]} / \tau^\alpha. \quad (9)$$

and ϖ_j is defined similarly just by replacing the $W_{k,i}^{[2m+2]}$ in the definition of ω_j with $W_{k,i}^{[2m+1]} (j = 0, 1, \dots, 2m+2)$.

3. Difference format based on block-by-block method

Consider the following equations

$${}_0^c D_t^\alpha u(x, t) = c \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad 0 \leq x \leq L, 0 < t \leq T, 0 < \alpha < 1. \quad (10)$$

$$u(x, 0) = \phi(x), 0 \leq x \leq L. \quad (11)$$

$$u(0, t) = u(L, t) = 0, t > 0. \quad (12)$$

where c is a constant diffusion-wave coefficient, $f(x, t)$ is the source term.

From the section 2, we know that equation (10) is equivalent to the following equation

$$u(x, t) = \phi(x) + \frac{c}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial^2 u(x, s)}{\partial x^2} ds + F(x, t) \quad (13)$$

where $F(x, t) = J_t^\alpha f(x, t)$.

Firstly, discrete the temporal direction of Eq.(13) by the block-by-block method.

Assume that when $t = t_n$, $u^n(x) = u(x, t_n)$, where $t_n = n\tau$, $\tau = T/2N$. Eq.(13) can be described by the block-by-block method as follow

$$u^{2i+2}(x) \approx \phi(x) + c\tau^\alpha \sum_{k=0}^{2i} \omega_{2i+2-k} \frac{\partial^2 u^k(x)}{\partial x^2} + c\tau^\alpha \omega_1 \frac{\partial^2 u^{2i+1}(x)}{\partial x^2} + c\tau^\alpha \omega_0 \frac{\partial^2 u^{2i+2}(x)}{\partial x^2} + F^{2i+2}(x) \quad (14)$$

$$u^{2i+1}(x) \approx \phi(x) + c\tau^\alpha \sum_{k=0}^{2i} \varpi_{2i+2-k} \frac{\partial^2 u^k(x)}{\partial x^2} + c\tau^\alpha \varpi_1 \frac{\partial^2 u^{2i+1}(x)}{\partial x^2} + c\tau^\alpha \varpi_0 \frac{\partial^2 u^{2i+2}(x)}{\partial x^2} + F^{2i+1}(x) \quad (15)$$

where $i = 0, 1, \dots, N-1$, $F^n(x) = F(x, t_n)$.

Second, discrete the spatial direction of Eq.(14) and Eq.(15). The second derivative by using second-order centered finite difference method discretize its spatial direction,

$$\frac{\partial^2 u^n(x_j)}{\partial x^2} = \frac{u^n(x_{j-1}) - 2u^n(x_j) + u^n(x_{j+1}))}{h^2} + O(h^2) \quad (16)$$

here, $h = L/M$, M is a positive integer, $x_j = jh$.

Put Eq. (16) to Eq.(14) and Eq.(15), omit the truncation errors, and set $u^n(x_j) \approx u_j^n$, have

$$u_j^{2i+2} \approx \phi_j + \frac{c\tau^\alpha}{h^2} \sum_{k=0}^{2i} \omega_{2i+2-k} (u_{j-1}^k - 2u_j^k + u_{j+1}^k) + \frac{c\tau^\alpha}{h^2} \omega_1 (u_{j-1}^{2i+1} - 2u_j^{2i+1} + u_{j+1}^{2i+1}) + \frac{c\tau^\alpha}{h^2} \omega_0 (u_{j-1}^{2i+2} - 2u_j^{2i+2} + u_{j+1}^{2i+2}) + F_j^{2i+2} \quad (17)$$

$$u_j^{2i+1} \approx \phi_j + \frac{c\tau^\alpha}{h^2} \sum_{k=0}^{2i} \varpi_{2i+2-k} (u_{j-1}^k - 2u_j^k + u_{j+1}^k) + \frac{c\tau^\alpha}{h^2} \varpi_1 (u_{j-1}^{2i+1} - 2u_j^{2i+1} + u_{j+1}^{2i+1}) + \frac{c\tau^\alpha}{h^2} \varpi_0 (u_{j-1}^{2i+2} - 2u_j^{2i+2} + u_{j+1}^{2i+2}) + F_j^{2i+1} \quad (18)$$

where, $\phi_j = \phi(x_j)$, $F_j^n = F^n(x_j) = F(x_j, t_n)$.

Make $\gamma = c\tau^\alpha/h^2$, we have

$$u_j^{2i+2} - \gamma \omega_1 (u_{j-1}^{2i+1} - 2u_j^{2i+1} + u_{j+1}^{2i+1}) - \gamma \omega_0 (u_{j-1}^{2i+2} - 2u_j^{2i+2} + u_{j+1}^{2i+2}) = \phi_j + \gamma \sum_{k=0}^{2i} \omega_{2i+2-k} (u_{j-1}^k - 2u_j^k + u_{j+1}^k) + F_j^{2i+2} \quad (19)$$

$$u_j^{2i+1} - \gamma \varpi_1 (u_{j-1}^{2i+1} - 2u_j^{2i+1} + u_{j+1}^{2i+1}) - \gamma \varpi_0 (u_{j-1}^{2i+2} - 2u_j^{2i+2} + u_{j+1}^{2i+2}) = \phi_j + \gamma \sum_{k=0}^{2i} \varpi_{2i+2-k} (u_{j-1}^k - 2u_j^k + u_{j+1}^k) + F_j^{2i+1} \quad (20)$$

Introduction of symbols: $A = I - \gamma \omega_0 B$, $\tilde{A} = I - \gamma \varpi_0 B$, where I is identity matrix with $(M-1) \times (M-1)$,

$$B = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \dots & \dots & \dots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

Then we have

$$A \begin{pmatrix} u_1^{2i+2} \\ u_2^{2i+2} \\ \dots \\ u_{M-1}^{2i+2} \end{pmatrix} - \gamma \omega_1 B \begin{pmatrix} u_1^{2i+1} \\ u_2^{2i+1} \\ \dots \\ u_{M-1}^{2i+1} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_{M-1} \end{pmatrix} + \begin{pmatrix} F_1^{2i+2} \\ F_2^{2i+2} \\ \dots \\ F_{M-1}^{2i+2} \end{pmatrix} + \gamma \sum_{k=0}^{2i} \omega_{2i+2-k} B \begin{pmatrix} u_1^k \\ u_2^k \\ \dots \\ u_{M-1}^k \end{pmatrix} \quad (21)$$

$$-\gamma \varpi_0 B \begin{pmatrix} u_1^{2i+2} \\ u_2^{2i+2} \\ \dots \\ u_{M-2}^{2i+2} \end{pmatrix} + \tilde{A} \begin{pmatrix} u_1^{2i+1} \\ u_2^{2i+1} \\ \dots \\ u_{M-1}^{2i+1} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_{M-1} \end{pmatrix} + \begin{pmatrix} F_1^{2i+1} \\ F_2^{2i+1} \\ \dots \\ F_{M-1}^{2i+1} \end{pmatrix} + \gamma \sum_{k=0}^{2i} \varpi_{2i+2-k} B \begin{pmatrix} u_1^k \\ u_2^k \\ \dots \\ u_{M-1}^k \end{pmatrix} \quad (22)$$

Comprehensive above two mathematical expressions, the algorithm can be expressed as the

following matrix form

$$\begin{pmatrix} -\gamma \omega_1 B & A \\ \tilde{A} & -\gamma \varpi_0 B \end{pmatrix} \begin{pmatrix} u_1^{2i+1} \\ \dots \\ u_{M-1}^{2i+1} \\ u_1^{2i+2} \\ \dots \\ u_{M-1}^{2i+2} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \dots \\ \phi_{M-1} \\ \phi_1 \\ \dots \\ \phi_{M-1} \end{pmatrix} + \begin{pmatrix} F_1^{2i+1} \\ \dots \\ F_{M-1}^{2i+1} \\ F_1^{2i+2} \\ \dots \\ F_{M-1}^{2i+2} \end{pmatrix} + \gamma \sum_{k=0}^{2i} \begin{pmatrix} \omega_{2i+2-k} B \\ \varpi_{2i+2-k} B \end{pmatrix} \begin{pmatrix} u_1^k \\ u_2^k \\ \dots \\ u_{M-1}^k \end{pmatrix}$$

4. Numerical experiment

In order to verify the analysis of schemes, consider the following equation

$${}_0^c D_t^\alpha u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), 0 < x \leq 1, 0 < \alpha < 1$$

$$u(x, 0) = 0, u_t(x, 0) = 0, 0 \leq x \leq 1; u(0, t) = u(1, t) = 0, 0 < t \leq 1$$

The exact solution of this fractional differential equation is $u(x, t) = 0.5t^{3+\alpha} \sin(\pi x)$.

Fig. 1: The comparison of numerical solution

and exact solution for $\alpha = 0.9, h = \tau = 0.05$

Fig. 2: The comparison of numerical solution

and exact solution for $\alpha = 0.8, h = 0.05$ and $\tau = 0.0125$

The comparisons of numerical solution and exact solution for $\alpha = 0.8$ are shown in fig.1 and fig.2. For this case, take stepsize $h = \tau = 0.05$ and $h = 0.05, \tau = 0.0125$. It can be seen that numerical results are in excellent agreement with the exact solution. From table 1 and table 2, as the stepsize decreasing, the error is reduced. Thus the numerical results are consistent with the theoretical analysis in Section 3.

Table 1: The errors for different stepsize $h = \tau$

$h = \tau$	$\alpha = 0.5$	$\alpha = 0.8$
	$\sqrt{\Sigma(h * E^2)}$	$\sqrt{\Sigma(h * E^2)}$
1/10	6.6000×10^{-3}	4.9000×10^{-3}
1/20	4.7000×10^{-3}	1.8000×10^{-3}
1/40	2.4000×10^{-3}	5.1578×10^{-4}
1/80	1.1000×10^{-3}	1.3968×10^{-4}
1/160	4.5362×10^{-4}	3.8280×10^{-5}

Table 2: The errors for different stepsize h and $\alpha = 0.5, \alpha = 0.8$ and $\alpha = 0.5, \alpha = 0.8$

h	$\alpha = 0.5$	$\alpha = 0.8$
	$\sqrt{\Sigma(h * E^2)}$	$\sqrt{\Sigma(h * E^2)}$
1/10	8.5000×10^{-3}	2.8000×10^{-3}
1/20	4.7000×10^{-3}	1.8000×10^{-3}
1/40	2.4000×10^{-3}	9.5835×10^{-4}
1/80	1.2000×10^{-3}	4.8561×10^{-4}
1/160	6.0864×10^{-4}	2.4361×10^{-4}

5. Concluding remarks

In this paper we have constructed one finite difference scheme for time fractional diffusion-wave equation based on its equivalent partial integro- differential equation. The numerical examples have verified the theoretical results. It is an effective and convergent numerical scheme in solving a variety of FDEs.

References

- [1] R.Linz and F.Liu, Fractional high order methods for the nonlinear fractional ordinary differential equation, *Nonlinear Dynam.* 66 (2007) ,pp.856-869.
- [2] Ch.Lubich, Discretized fractional calculus, *SIAM J, Math. Anal.* vol .17,no.3,(1986), pp.704-719.
- [3] P.Kumar and O.P.Agrawal, An approximate method for numerical solution of fractional differential equations, *Singal Process.* 86(2006), pp.2602-2610.
- [4] K.Diethelm, N.J.Ford, and A.D.Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynam.*29(2002),pp.3-22.
- [5] R.Linz, An method for nonlinear solving Volterra integral equations of the second kind, *Math.Comput.*vol.23,no.107,(1969),pp.595-599.
- [6] K.Diethelm, *The Analysis of Fractional Differential Equations.* Springer, Berlin, 2010