Solution of a Class of High Order Differential Equations

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Abstract—In order to solve physical problems, we must establish mathematical models for the problems. Mathematical models often are differential equations relating an unknown function and one or more of its derivatives. In this paper, in order to solve high order differential equations, we first proved the expression for n repeated definite integrals by mathematical induction, Integration by parts and binomial formula. Then we obtained the solution of a class of high order differential equations by integral technique and the formula for n repeated definite integrals. Our results can be used to study the properties of high order differential equations, and then our results can be used to investigate physical or “real life” problems.

Keywords—Higher Order Differential Equation; Integration by Parts; Binomial Formula; Induction; Solution

I. INTRODUCTION

In order to solve physical problems, we must establish mathematical models for the problems. Mathematical models often are differential equations relating an unknown function and one or more of its derivatives. Wide applications of differential equations have attracted great interests of many mathematicians. Some mathematicians investigated existence of solutions for differential equations in [7-11]. Some mathematicians studied the solutions of high order differential equations and the theory on differential equations in [7-11].

II. MAIN RESULTS

In this section, we use the following lemmas to prove our main results.

Lemma 1. (see [12]) Suppose that \( f(t) \) is a continuous function on \([a, \infty)\). Then the function

\[
y(t) = \int_a^t \int_a^s \cdots \int_a^{s_{i-1}} f(\tau_i) d\tau_i \cdots d\tau_2 d\tau_1 ds
\]

\[
= \int_a^t (t-s)^n f(s) ds,
\]

is a solution of higher order differential equation

\[
\begin{align*}
y^{(n+1)}(t) &= f(t), t \in [a, \infty), \\
y^{(i)}(a) &= 0, i = 0, 1, \ldots, n.
\end{align*}
\]

Proof. For the reader’s convenience, we prove this Lemma by induction. Firstly, we consider the case of \( n = 1 \). The solution of differential equation (2) is

\[
y(t) = \int_a^t f(\tau) d\tau ds
\]

\[
= s \int_a^t f(\tau) d\tau_1 \bigg|_a^s - \int_a^s f(s) ds
\]

\[
= \int_a^t f(t-s) f(s) ds.
\]

(3)

So (1) is true for \( n = 1 \). Letting this be our basis step, suppose that (1) is true up to some \( k > 1 \), i.e. the solution of differential equation (2) is

\[
y(t) = \int_a^t \int_a^s \cdots \int_a^{s_{i-1}} f(\tau_k) (d\tau_k \cdots d\tau_2 d\tau_1 ds)
\]

\[
= \int_a^t (t-s)^k f(s) ds.
\]

(4)

If \( n = k + 1 \), then the induction hypothesis implies that the solution of differential equation (2) is

\[
y(t) = \int_a^t \int_a^s \cdots \int_a^{s_{i-1}} f(\tau_k) (d\tau_k \cdots d\tau_2 d\tau_1 ds)
\]

\[
= \int_a^t (t-s)^k f(s) ds - \int_a^s f(s) ds.
\]

(5)
We consider the integration
\[
\int_{a}^{b} \frac{(s - \tau_{1})^{k-2}}{(k-2)!} f(\tau_{1}) d\tau_{1} ds.
\]
When \(k = 2\), we have
\[
\int_{a}^{b} \frac{(s - \tau_{1})^{2}}{2!} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{s}{3!} \int_{a}^{b} (s - \tau_{1}) f(\tau_{1}) d\tau_{1} ds - \frac{s}{5!} \int_{a}^{b} f(s) ds
\]
\[
= \frac{t^{3}}{3!} \int_{a}^{b} (t - s)^{2} f(s) ds
\]
\[
- \frac{s^{4}}{4!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds - \frac{s^{4}}{4!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{3}}{3!} \int_{a}^{b} (t - s)^{2} f(s) ds
\]
\[
= \frac{t^{5}}{5!} \int_{a}^{b} f(s) ds - \frac{s^{5}}{5!} \int_{a}^{b} f(s) ds
\]
\[
= \frac{s^{5}}{5!} \int_{a}^{b} f(s) ds
\]
When \(k = 2\), we get
\[
\int_{a}^{b} \frac{(s - \tau_{1})^{3}}{3!} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{s^{3}}{3!} \int_{a}^{b} (s - \tau_{1})^{3} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{3}}{3!} \int_{a}^{b} (t - s)^{3} f(s) ds
\]
\[
= \frac{-s^{4}}{4!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds - \frac{s^{4}}{4!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{3}}{3!} \int_{a}^{b} (t - s)^{3} f(s) ds
\]
\[
= \frac{-s^{5}}{5!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds - \frac{s^{5}}{5!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{5}}{5!} \int_{a}^{b} f(s) ds - \frac{s^{5}}{5!} \int_{a}^{b} f(s) ds
\]
\[
= \frac{s^{5}}{5!} \int_{a}^{b} f(s) ds
\]
When \(k = 3\), we get
\[
\int_{a}^{b} \frac{(s - \tau_{1})^{3}}{3!} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{s^{3}}{3!} \int_{a}^{b} (s - \tau_{1})^{3} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{3}}{3!} \int_{a}^{b} (t - s)^{3} f(s) ds
\]
\[
= \frac{-s^{4}}{4!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds - \frac{s^{4}}{4!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{3}}{3!} \int_{a}^{b} (t - s)^{3} f(s) ds
\]
\[
= \frac{-s^{5}}{5!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds - \frac{s^{5}}{5!} \int_{a}^{b} f(\tau_{1}) d\tau_{1} ds
\]
\[
= \frac{t^{5}}{5!} \int_{a}^{b} f(s) ds - \frac{s^{5}}{5!} \int_{a}^{b} f(s) ds
\]
\[
= \frac{s^{5}}{5!} \int_{a}^{b} f(s) ds
\]
Taking into account (5) and (10), we obtain

\[ y(t) = \int_{a}^{t} f(s) ds = \int_{a}^{t} \left( \frac{t-\tau}{2} \right) f(s) ds \]

Therefore, by induction, (17) is true for all \( n > 1 \).

**Theorem 1.** Suppose that \( f(t) \) is a continuous function on \([a, \infty)\), and \( p(t) > 0 \) is an \( n \)th continuous differentiable function on \([a, \infty)\). Then the function

\[
y(t) = \int_{a}^{t} (t-s)^{m-1} \left( \frac{1}{p(s)} \int_{a}^{s} (s-\tau)^{n-1} f(\tau) d\tau ds\right)
\]

is a solutions of higher order differential equation

\[
d^{n} \frac{d}{dt^{m}} y(t) = f(t), t \in [a, \infty), \quad y^{(i)}(a) = 0, i = 0, 1, \ldots, n + m - 1.
\]

**Proof.** Carrying out integrations on both sides of (13) \( n \) times, using (1), we obtain

\[
p(t) \frac{d}{dt} y(t) = \int_{a}^{t} f(t-s) \left( \frac{1}{p(t)} \int_{a}^{t} (t-s)^{n-1} f(t) d\tau ds\right)
\]

Dividing both sides of (14) by \( p(t) \), we have

\[
\frac{d}{dt} y(t) = \frac{1}{p(t)} \int_{a}^{t} (t-s)^{n-1} f(t) d\tau ds.
\]

From (15), we get

\[
y(t) = \int_{a}^{t} \left( \frac{1}{p(t)} \int_{a}^{t} (t-s)^{n-1} f(t) d\tau ds\right)
\]

(16) is a solution of higher order differential equation (13).

**Theorem 2.** Suppose that \( c_1, c_2 \) are positive constants. Then the function

\[
y(t) = c_1 (t-a)^{m+n} + c_2 (m+n)!
\]

is a solutions of higher order differential equation

\[
d^{n} \frac{d}{dt^{m}} y(t) = c_1, t \in [a, \infty), \quad y^{(i)}(a) = 0, i = 0, 1, \ldots, n + m - 1.
\]

**Proof.** By Theorem 1, we see that the function

\[
y(t) = c_1 \int_{a}^{t} (t-s)^{m-1} \left( \frac{1}{p(s)} \int_{a}^{s} (s-\tau)^{n-1} d\tau d\tau ds\right)
\]

is a solutions of higher order differential equation

\[
\frac{d}{dt} y(t) = c_1, t \in [a, \infty), \quad y^{(i)}(a) = 0, i = 0, 1, \ldots, n + m - 1.
\]

From (14), we have

\[
y(t) = c_1 \int_{a}^{t} (t-s)^{m-1} \left( \frac{1}{p(s)} \int_{a}^{s} (s-\tau)^{n-1} d\tau d\tau ds\right)
\]
\[ y(t) = \frac{c_1 (t-a)^{m+n}}{c_2 (m+n)!} \int_a^t (t-s)^{m-1} (s-a)^n \frac{ds}{(m-1)!} \tag{21} \]

Let \( s-a = x(t-a) \), then \( t-s = t-a-(s-a) \).

From (21), we have

\[ y(t) = \frac{c_1 (t-a)^{m+n}}{(m-1)!n!} \int_0^t (1-v)^{m-1} v^n dv \]

\[ = \frac{c_1 (t-a)^{m+n}}{(m-1)!n!} B(m,n+1) \]

\[ = \frac{c_1 (t-a)^{m+n}}{(m-1)!n!(m+n)!} c_2 \]

\[ = \frac{c_1 (t-a)^{m+n}}{c_2 (m+n)!} . \]

III. CONCLUSIONS

In this paper, firstly, we proved the function

\[ y(t) = \int_a^t \int_a^{\tau_1} \ldots \int_a^{\tau_{n-1}} f(\tau_n)d\tau_n \int_a^t \ldots \int_a^{\tau_2} f(s)ds \]

be a solution of higher order differential equation

\[ y^{(n+1)}(t) = f(t), t \in [a, \infty), \]

\[ y^{(i)}(a) = 0, i = 0,1, \ldots, n. \]

Then we proved the function

\[ y(t) = \frac{\int_a^t (t-s)^{m-1} (s-a)^n f(\tau)d\tau ds}{\int_a^t (t-s)^{m-1} (s-a)^n (n-1)!} \]

be a solutions of higher order differential equation

\[ \frac{d^n}{dt^n} (p(t) \frac{d^m}{dt^m} y(t)) = f(t), t \in [a, \infty), \]

\[ y^{(i)}(a) = 0, i = 0,1, \ldots, n+m-1. \]

Finally, we proved the function

\[ y(t) = \frac{c_1 (t-a)^{m+n}}{c_2 (m+n)!} \]

to be a solutions of higher order differential equation

\[ \begin{cases} \frac{d^n}{dt^n} \left( c_2 \frac{d^m}{dt^m} y(t) \right) = c_1, t \in [a, \infty) \\ y^{(i)}(a) = 0, i = 0,1, \ldots, n+m-1. \end{cases} \]

ACKNOWLEDGMENT

This research was supported by Guangxi Natural Science Foundation (No. 2012GXNSFAA053009) and Scientific Research Foundation of the Education Department of Guangxi Autonomous Region of China (No. KY2015ZD103, KY2015YB257), and the high school specialty and curriculum integration project of Guangxi Zhuang Autonomous Region (No. GXTSZY2220).

REFERENCES


